Communications in Combinatorics and Optimization
Vol. ??, No. ?? (????), pp. 1-15
DOI:10.22049/CCO.2023.28236.1485

CCO
Commun. Comb. Optim.

Research Article

Some new families of generalized k-Leonardo and Gaussian Leonardo Numbers

Kalika Prasad[†], **Ritanjali Mohanty**[*], **Munesh Kumari**[‡], **Hrishikesh Mahato**[§]

Department of Mathematics, Central University of Jharkhand, India, 835205
[†]klkaprsd@gmail.com
[*]ritanjalipoonam6@gmail.com
[‡]muneshnasir94@gmail.com
[§]hrishikesh.mahato@cuj.ac.in

Received: 9 January 2023; Accepted: 13 April 2023
Published Online: 18 April 2023

Abstract: In this paper, we introduce a new family of the generalized k-Leonardo numbers and study their properties. We investigate the Gaussian Leonardo numbers and associated new families of these Gaussian forms. We obtain combinatorial identities like Binet formula, Cassini's identity, partial sum, etc. in the closed form. Moreover, we give various generating and exponential generating functions.

Keywords: k-Leonardo numbers; Gaussian Leonardo numbers; k-Gaussian Leonardo numbers; Binet Formula; Generating functions; Partial sum.

AMS Subject classification: 11B39, 11B37

1. Introduction

In a recent study, Mikkawy and Sogabe [4] introduced a new family of k-Fibonacci numbers $F_n^{(k)}$ where n is of the kind $sk + r$. They presented several interesting properties of this new sequence and shown a relation with the classic Fibonacci numbers. In [8], Özkan et al. added some more results to this newly introduced sequence and extended the study to a new family of k-Lucas numbers. In [9], Özkan et al. studied a new family of Gauss k-Lucas numbers and associated polynomials. A study on new families of Jacobsthal and the Jacobsthal-Lucas numbers is presented by Catarino et al. [2]. Recently, Kumari et al. [7] extended the study to Mersenne numbers and investigated some new families of k-Mersenne and generalized k-Gaussian Mersenne

* *Corresponding author*

numbers and associated polynomials. Some recent work in this direction can be seen in [5, 10–12, 18–20].

Motivated by these works on a new family of Fibonacci and Fibonacci-like sequences, in this paper, we introduce and investigate a new family of k-Leonardo numbers and k-Gaussian Leonardo numbers.

Leonardo numbers $\{\mathcal{L}_n\}_{n\geq 0}$ were recently investigated by Catarino and Borges [3] which have a close connection with the Fibonacci and Lucas numbers. They presented some interesting properties of Leonardo numbers. For $n \geq 0$, it is defined recurrently as

$$\mathcal{L}_{n+2} = \mathcal{L}_{n+1} + \mathcal{L}_n + 1, \quad \text{with} \quad \mathcal{L}_0 = \mathcal{L}_1 = 1. \tag{1}$$

First few Leonardo numbers are $1, 1, 3, 5, 9, 15, 25, 41, \ldots$.

Alp and Koçer [1] studied these Leonardo numbers, defined the matrix representation, and obtained some remarkable properties of them. A generalization of Leonardo numbers was studied by Kuhapatanakul and Chobsorn [6] where they presented some special properties of their generalized version of Leonardo numbers, for their generalization they considered the recurrence relation $\mathfrak{L}_{n+2} = \mathfrak{L}_{n+1} + \mathfrak{L}_n + N$, where N is any positive integer and initial values are same. For a recent study on Leonardo numbers, their generalization, and properties, one can refer to [13–17].

The relation of the Leonardo numbers with the Fibonacci numbers is investigated as $\mathcal{L}_n = 2F_{n+1} - 1$.

In homogeneous form, recurrence relation (1) can be written as

$$\mathcal{L}_{n+1} = 2\mathcal{L}_n - \mathcal{L}_{n-2}. \tag{2}$$

It is easy to see that the characteristic equation for the recurrence relation (2) is $x^3 - 2x^2 + 1 = 0$ and its roots are

$$\alpha = \frac{1+\sqrt{5}}{2}, \quad \beta = \frac{1-\sqrt{5}}{2} \quad \text{and} \quad \gamma = 1.$$

For $n \geq 0$, the Binet formula for the Leonardo numbers $\mathcal{L}_n$ is given by

$$\mathcal{L}_n = 2\Big(\frac{\alpha^{n+1} - \beta^{n+1}}{\alpha - \beta}\Big) - 1 \quad \text{or} \quad \mathcal{L}_n = \frac{1}{\sqrt{5}}\big[2(\alpha^{n+1} - \beta^{n+1}) - \sqrt{5}\big]. \tag{3}$$

For Leonardo numbers, the generating function is

$$G(t) = \frac{1 - t + t^2}{1 - 2t + t^3}.$$

For $n > r$, $r \geq 1$, the Catalan's identity is given as

$$\mathcal{L}_n^2 - \mathcal{L}_{n-r}\mathcal{L}_{n+r} = \mathcal{L}_{n-r} + \mathcal{L}_{n+r} - 2\mathcal{L}_n - (-1)^{n-r}(\mathcal{L}_{r-1} + 1)^2.$$

Setting $r = 1$ in the Catalan's identity, we have the Cassini's identity as follows:

$$\mathcal{L}_n^2 - \mathcal{L}_{n-1}\mathcal{L}_{n+1} = \mathcal{L}_{n-1} - \mathcal{L}_{n-2} + 4(-1)^n.$$

2. Main work

Throughout the paper, we adopt the symbol $\mathbb{N}_0$ for $\mathbb{N} \cup \{0\}$. We start with defining the generalized k-Leonardo numbers.

2.1. Generalized k-Leonardo numbers

Definition 1. Let k be a natural number and $n \in \mathbb{N}_0$, then $\exists$ unique $s, r \in \mathbb{N}_0$ such that $n = sk + r$, where $0 \leq r < k$. Then the generalized k-Leonardo numbers $\mathcal{L}_n^{(k)}$ is defined by

$$\mathcal{L}_n^{(k)} = \frac{1}{(\sqrt{5})^k}\left[2(\alpha^{s+1} - \beta^{s+1}) - \sqrt{5}\right]^{k-r}\left[2(\alpha^{s+2} - \beta^{s+2}) - \sqrt{5}\right]^{r}, \tag{4}$$

where $\alpha = (1+\sqrt{5})/2$ and $\beta = (1-\sqrt{5})/2$.

From Eqn.(3) and Definition 1, the generalized k-Leonardo numbers and Leonardo numbers are related as

$$\mathcal{L}_{sk+r}^{(k)} = \mathcal{L}_s^{k-r}\mathcal{L}_{s+1}^{r}. \tag{5}$$

In terms of Fibonacci Numbers, it is given as

$$\mathcal{L}_n^{(k)} = (2F_{s+1} - 1)^{k-r}(2F_{s+2} - 1)^{r}.$$

If $k = 1$ then $r = 0$ and hence $n = s$. So, from Eqn. (5), we have $\mathcal{L}_n^{(1)} = \mathcal{L}_n$.
For the case $k = 2$, we have $r = 0, 1$. Hence, the following relations are obtained for $k = 2$.

$$\text{If } r = 0, \quad \mathcal{L}_{2n}^{(2)} = \frac{1}{(\sqrt{5})^2}\left[2(\alpha^{n+1} - \beta^{n+1}) - \sqrt{5}\right]^2 = \mathcal{L}_n^2.$$

$$\begin{aligned}\text{If } r = 1, \quad \mathcal{L}_{2n+1}^{(2)} &= \frac{1}{(\sqrt{5})^2}\left[\left(2(\alpha^{n+1} - \beta^{n+1}) - \sqrt{5}\right)\left(2(\alpha^{n+2} - \beta^{n+2}) - \sqrt{5}\right)\right] \\ &= \mathcal{L}_n\mathcal{L}_{n+1}.\end{aligned}$$

Theorem 1. *The 2-Leonardo sequence $\{\mathcal{L}_{2n+1}^{(2)}\}$ satisfy the identity*

$$\mathcal{L}_{2n+1}^{(2)} = 2\mathcal{L}_{2n}^{(2)} - \mathcal{L}_n\mathcal{L}_{n-2}.$$

Proof. Form (5) and (2), we write

$$\mathcal{L}_{2n+1}^{(2)} = \mathcal{L}_n\mathcal{L}_{n+1} = \mathcal{L}_n(2\mathcal{L}_n - \mathcal{L}_{n-2}) = 2(\mathcal{L}_n)^2 - \mathcal{L}_n\mathcal{L}_{n-2} = 2\mathcal{L}_{2n}^{(2)} - \mathcal{L}_n\mathcal{L}_{n-2}. \tag{6}$$

Hence the result is obtained. □

For the case $k=3$, we have $r=0,1,2$. Thus, form (5)

$$\mathcal{L}^{(3)}_{3n+r} = \begin{cases} \mathcal{L}^3_n & : r=0, \\ \mathcal{L}^2_n \mathcal{L}_{n+1} & : r=1, \\ \mathcal{L}_n \mathcal{L}^2_{n+1} & : r=2. \end{cases} \tag{7}$$

Theorem 2. *The 3-Leonardo sequence $\{\mathcal{L}^{(3)}_{3n+1}\}$ satisfy the identity*

$$\mathcal{L}^{(3)}_{3n+1} = 2\mathcal{L}^{(3)}_{3n} - \mathcal{L}^2_n \mathcal{L}_{n-2}.$$

Proof. We have

$$\mathcal{L}^{(3)}_{3n+1} = \mathcal{L}^2_n \mathcal{L}_{n+1} = \mathcal{L}^2_n (2\mathcal{L}_n - \mathcal{L}_{n-2}) = 2\mathcal{L}^3_n - \mathcal{L}^2_n \mathcal{L}_{n-2} = 2\mathcal{L}^{(3)}_{3n} - \mathcal{L}^2_n \mathcal{L}_{n-2}.$$

□

For $k=1,2,3,4,5$, a list of the first few numbers of the generalized k-Leonardo sequence is displayed in the following table.

$n \downarrow$	$\mathcal{L}^{(k)}_n$	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$
0	$\mathcal{L}^{(k)}_0$	1	1	1	1	1	1
1	$\mathcal{L}^{(k)}_1$	1	1	1	1	1	1
2	$\mathcal{L}^{(k)}_2$	3	1	1	1	1	1
3	$\mathcal{L}^{(k)}_3$	5	3	1	1	1	1
4	$\mathcal{L}^{(k)}_4$	9	9	3	1	1	1
5	$\mathcal{L}^{(k)}_5$	15	15	9	3	1	1
6	$\mathcal{L}^{(k)}_6$	25	25	27	9	3	1
7	$\mathcal{L}^{(k)}_7$	41	45	45	27	9	3
8	$\mathcal{L}^{(k)}_8$	67	81	75	81	27	9
9	$\mathcal{L}^{(k)}_9$	109	135	125	135	81	27
10	$\mathcal{L}^{(k)}_{10}$	177	225	225	225	243	81

Table 1. Generalized k-Leonardo numbers.

Theorem 3. *Let $s \in \mathbb{N}_0$, for positive integer k, we have*

$$\mathcal{L}^{(k)}_{sk} = \mathcal{L}^k_s \quad \text{and} \quad \mathcal{L}^{(k)}_{sk+1} = 2\mathcal{L}^{(k)}_{sk} - \mathcal{L}^{k-1}_s \mathcal{L}_{s-2}.$$

Proof. For the first identity, if $n = sk$ then $r=0$, hence the result follows from (5). For the second identity, from (5), we write $\mathcal{L}^{(k)}_{sk+1} = \mathcal{L}^{k-1}_s \mathcal{L}_{s+1}$ and using recurrence (2), we have

$$\begin{aligned} \mathcal{L}^{(k)}_{sk+1} &= \mathcal{L}^{k-1}_s (2\mathcal{L}_s - \mathcal{L}_{s-2}) \\ &= 2\mathcal{L}^k_s - \mathcal{L}^{k-1}_s \mathcal{L}_{s-2} \\ &= 2\mathcal{L}^{(k)}_{sk} - \mathcal{L}^{k-1}_s \mathcal{L}_{s-2}. \end{aligned}$$

□

For $k,\ s \in \mathbb{N}$, the relation $\mathcal{L}_{sk+k}^{(k)} - \mathcal{L}_{sk}^{(k)} = \mathcal{L}_{s+1}^{k} - \mathcal{L}_{s}^{k}$ is verified from (5).

Theorem 4. *(Cassini's identity) For $n,\ k \geq 2$, the Cassini's identity for the generalized k-Leonardo numbers $\mathcal{L}_n^{(k)}$ is given by*

$$(\mathcal{L}_{sk+a-1}^{(k)})^2 - \mathcal{L}_{sk+a}^{(k)}\mathcal{L}_{sk+a-2}^{(k)} = \begin{cases} 0, & a \neq 1, \\ \mathcal{L}_s^{2k-2}(\mathcal{L}_{s-3} + 1 + 4(-1)^s), & a = 1. \end{cases}$$

Proof. We have

$$\begin{aligned}(\mathcal{L}_{sk+a-1}^{(k)})^2 - \mathcal{L}_{sk+a}^{(k)}\mathcal{L}_{sk+a-2}^{(k)} &= (\mathcal{L}_s^{k-a+1}\mathcal{L}_{s+1}^{a-1})^2 - (\mathcal{L}_s^{k-a}\mathcal{L}_{s+1}^{a})(\mathcal{L}_s^{k-a+2}\mathcal{L}_{s+1}^{a-2}) \\ &= 0, \quad \text{if} \quad a \neq 1,\end{aligned}$$

and for the case $a = 1$, we have

$$\begin{aligned}(\mathcal{L}_{sk}^{(k)})^2 - \mathcal{L}_{sk+1}^{(k)}\mathcal{L}_{sk-1}^{(k)} &= (\mathcal{L}_s^{k})^2 - (\mathcal{L}_s^{k-1}\mathcal{L}_{s+1}^{1})(\mathcal{L}_{s-1}^{1}\mathcal{L}_s^{k-1}) \\ &= \mathcal{L}_s^{2k-2}(\mathcal{L}_s^2 - \mathcal{L}_{s+1}\mathcal{L}_{s-1}) \\ &= \mathcal{L}_s^{2k-2}(\mathcal{L}_{s-1} - \mathcal{L}_{s-2} + 4(-1)^s) \\ &= \mathcal{L}_s^{2k-2}(\mathcal{L}_{s-3} + 1 + 4(-1)^s) \quad \text{(using (1))}.\end{aligned}$$

□

Theorem 5. *For the generalized k-Leonardo numbers $\{\mathcal{L}_n^{(k)}\}$ where n is of kind $sk + r$, we have*
(1). $\sum_{a=0}^{k-1}\binom{k-1}{a}\mathcal{L}_{sk+a}^{(k)} = \mathcal{L}_s(\mathcal{L}_{s+2} - 1)^{k-1}$,
(2). $\sum_{a=0}^{k-1}(-1)^a\binom{k-1}{a}\mathcal{L}_{sk+a}^{(k)} = (-1)^{k-1}\mathcal{L}_s(2F_s)^{k-1}$.

Proof. *(1).* We have

$$\begin{aligned}\sum_{a=0}^{k-1}\binom{k-1}{a}\mathcal{L}_{sk+a}^{(k)} &= \sum_{a=0}^{k-1}\binom{k-1}{a}\mathcal{L}_s^{k-a}\mathcal{L}_{s+1}^{a} \\ &= \mathcal{L}_s\sum_{a=0}^{k-1}\binom{k-1}{a}\mathcal{L}_{s+1}^{a}\mathcal{L}_s^{k-a-1} \\ &= \mathcal{L}_s(\mathcal{L}_s + \mathcal{L}_{s+1})^{k-1} \quad \text{(using binomial theorem)} \\ &= \mathcal{L}_s(\mathcal{L}_{s+2} - 1)^{k-1} \quad \text{(using (1))}.\end{aligned}$$

(2). By a similar argument, we write

$$\begin{aligned}\sum_{a=0}^{k-1}(-1)^a\binom{k-1}{a}\mathcal{L}_{sk+a}^{(k)} &= \sum_{a=0}^{k-1}(-1)^a\binom{k-1}{a}\mathcal{L}_s^{k-a}\mathcal{L}_{s+1}^a \\ &= \mathcal{L}_s\sum_{a=0}^{k-1}\binom{k-1}{a}(-\mathcal{L}_{s+1})^a\mathcal{L}_s^{k-a-1} \\ &= \mathcal{L}_s[\mathcal{L}_s-\mathcal{L}_{s+1}]^{k-1} \quad \text{(using binomial theorem)} \\ &= (-2)^{k-1}\mathcal{L}_s(F_s)^{k-1} \quad \text{(using } \mathcal{L}_s = 2F_{s+1}-1).\end{aligned}$$

□

Theorem 6. *For $k \geq 1$, sum of generalized k-Leonardo numbers is given by*

$$\sum_{a=0}^{k-1}\mathcal{L}_{sk+a}^{(k)} = \frac{\mathcal{L}_s((\mathcal{L}_{s+1})^k-(\mathcal{L}_s)^k)}{2F_s}. \tag{8}$$

Proof. We have

$$\begin{aligned}\sum_{a=0}^{k-1}\mathcal{L}_{sk+a}^{(k)} &= \sum_{a=0}^{k-1}\mathcal{L}_s^{k-a}\mathcal{L}_{s+1}^a = \mathcal{L}_s^k\sum_{a=0}^{k-1}\Big(\frac{\mathcal{L}_{s+1}}{\mathcal{L}_s}\Big)^a \\ &= \mathcal{L}_s^k\Big(\frac{(\mathcal{L}_{s+1}/\mathcal{L}_s)^k-1}{\mathcal{L}_{s+1}/\mathcal{L}_s-1}\Big) \\ &= \mathcal{L}_s^k\Big(\frac{(\mathcal{L}_{s+1})^k-(\mathcal{L}_s)^k}{(\mathcal{L}_s)^k}\frac{\mathcal{L}_s}{\mathcal{L}_{s+1}-\mathcal{L}_s}\Big) \\ &= \frac{\mathcal{L}_s((\mathcal{L}_{s+1})^k-(\mathcal{L}_s)^k)}{\mathcal{L}_{s+1}-\mathcal{L}_s}.\end{aligned}$$

Using the fact $\mathcal{L}_s = 2F_{s+1}-1$, the proof is completed. □

3. Gaussian Leonardo numbers

First, we define and investigate the Gaussian Leonardo numbers and then we introduce a new family of k-Gaussian Leonardo numbers.

Definition 2. The Gaussian Leonardo sequence $\{\mathcal{GL}_n\}_{n\geq 0}$ is defined by the recurrence relation

$$\mathcal{GL}_n = 2\mathcal{GL}_{n-1}-\mathcal{GL}_{n-3}, \quad n \geq 3, \tag{9}$$

with initials $\mathcal{GL}_0 = 1-i$, $\mathcal{GL}_1 = 1+i$ and $\mathcal{GL}_2 = 3+i$.

We should note that the Gaussian Leonardo numbers and Leonardo numbers are related as

$$\mathcal{GL}_n = \mathcal{L}_n + i\mathcal{L}_{n-1}. \tag{10}$$

The first few Gaussian Leonardo numbers are $1-i,\ 1+i,\ 3+i,\ 5+3i,\ \ldots.$
The Binet type formula for the Gaussian Leonardo numbers is given by

$$\mathcal{GL}_n = \Big[2\Big(\frac{\alpha^{n+1}-\beta^{n+1}}{\alpha-\beta}\Big)-1\Big] + i\Big[2\Big(\frac{\alpha^{n}-\beta^{n}}{\alpha-\beta}\Big)-1\Big], \tag{11}$$

where $\alpha = \dfrac{1+\sqrt{5}}{2}$ and $\beta = \dfrac{1-\sqrt{5}}{2}$ are roots of the characteristic equation $x^3 - 2x^2 + 1 = 0$.

Theorem 7 (Partial sum). *For Gaussian Leonardo numbers, we have*

1. $$\sum_{a=0}^{n-1} \mathcal{GL}_a = \mathcal{GL}_{n+1} - (1+n)(1+i), \tag{12}$$
2. $$\sum_{a=0}^{n-1} \mathcal{GL}_{2a} = \mathcal{GL}_{2n-1} + (1-n) - i(1+n), \tag{13}$$
3. $$\sum_{a=0}^{n-1} \mathcal{GL}_{2a+1} = \mathcal{GL}_{2n} - (1+n) + i(1-n). \tag{14}$$

Proof. To prove the results, we use the fact that $\mathcal{L}_n = 2F_{n+1} - 1$ and Eqn. (10). Thus, we have

$$\begin{aligned}
\sum_{a=0}^{n-1} \mathcal{GL}_a &= \sum_{a=0}^{n-1}[(2F_{a+1}-1) + i(2F_a - 1)] \\
&= 2\Big(\sum_{a=0}^{n-1} F_{a+1} + i\sum_{a=0}^{n-1} F_a\Big) - n - in \\
&= 2\Big(\sum_{a=0}^{n} F_a + i\sum_{a=0}^{n-1} F_a\Big) - n(1+i) \\
&= 2(F_{n+2} - 1 + i(F_{n+1} - 1)) - n(1+i) \\
&= (2F_{n+2} - 1) + i(2F_{n+1} - 1) - 1 - i - n(1+i) \\
&= \mathcal{L}_{n+1} + i\mathcal{L}_n - (1+n)(1+i) \\
&= \mathcal{GL}_{n+1} - (1+n)(1+i).
\end{aligned}$$

For the second and third identities, a similar argument holds. □

Theorem 8 (Casssini's identity). *For $a \geq 1$,*

$$\mathcal{GL}_a^2 - \mathcal{GL}_{a+1}\mathcal{GL}_{a-1} = \mathcal{L}_{a-5} + 1 + 8(-1)^a + i(\mathcal{L}_{a-2} + 1 + 4(-1)^{a-1}). \tag{15}$$

Proof. We have

$$\begin{aligned}
&\mathcal{GL}_a^2 - \mathcal{GL}_{a+1}\mathcal{GL}_{a-1} \\
&\quad = (\mathcal{L}_a + i\mathcal{L}_{a-1})^2 - (\mathcal{L}_{a+1} + i\mathcal{L}_a)(\mathcal{L}_{a-1} + i\mathcal{L}_{a-2}) \\
&\quad = (\mathcal{L}_a)^2 - (\mathcal{L}_{a-1})^2 + 2i\mathcal{L}_a\mathcal{L}_{a-1} - \mathcal{L}_{a+1}\mathcal{L}_{a-1} - i\mathcal{L}_{a+1}\mathcal{L}_{a-2} - i\mathcal{L}_a\mathcal{L}_{a-1} \\
&\qquad\quad + \mathcal{L}_a\mathcal{L}_{a-2} \\
&\quad = [(\mathcal{L}_a)^2 - \mathcal{L}_{a-1}\mathcal{L}_{a+1}] - [(\mathcal{L}_{a-1})^2 - \mathcal{L}_a\mathcal{L}_{a-2}] + i[\mathcal{L}_a\mathcal{L}_{a-1} - \mathcal{L}_{a+1}\mathcal{L}_{a-2}] \\
&\quad = (\mathcal{L}_{a-1} - \mathcal{L}_{a-2} + 4(-1)^a) - (\mathcal{L}_{a-2} - \mathcal{L}_{a-3} + 4(-1)^{a-1}) \\
&\qquad\quad + i(\mathcal{L}_a\mathcal{L}_{a-1} - \mathcal{L}_{a+1}\mathcal{L}_{a-2}).
\end{aligned}$$

Using the Binet formula of Leonardo numbers and after some elementary calculation, we write
$\mathcal{L}_a\mathcal{L}_{a-1} - \mathcal{L}_{a+1}\mathcal{L}_{a-2} = \mathcal{L}_{a-2} + 1 + 4(-1)^{a-1}$ and from Eqn. (1), we have

$$\begin{aligned}
\mathcal{L}_{a-1} - \mathcal{L}_{a-2} + 4(-1)^a - \mathcal{L}_{a-2} + \mathcal{L}_{a-3} - 4(-1)^{a-1} &= \mathcal{L}_{a-1} - 2\mathcal{L}_{a-2} + \mathcal{L}_{a-3} + 8(-1)^a \\
&= \mathcal{L}_{a-5} + 1 + 8(-1)^a.
\end{aligned}$$

Thus, $\mathcal{GL}_a^2 - \mathcal{GL}_{a+1}\mathcal{GL}_{a-1} = \mathcal{L}_{a-5} + 1 + 8(-1)^a + i(\mathcal{L}_{a-2} + 1 + 4(-1)^{a-1})$, as required. □

In next theorems, we discuss various generating functions for the Gaussian Leonardo numbers.

Theorem 9 (Generating function). *The generating function for Gaussian Leonardo numbers is*

$$\mathcal{GL}(t) = \frac{(1-i) - (1-3i)t + (1-i)t^2}{(1 - 2t + t^3)}.$$

Proof. Let $\mathcal{GL}(t)$ be a generating function for the sequence $\{\mathcal{GL}_n\}$. We start with the formal power series representation for a sequence as

$$\mathcal{GL}(t) = \mathcal{GL}_0 + \mathcal{GL}_1 t + \mathcal{GL}_2 t^2 + \cdots + \mathcal{GL}_n t^n + \ldots. \tag{16}$$

Here,

$$2t\mathcal{GL}(t) = 2\mathcal{GL}_0 t + 2\mathcal{GL}_1 t^2 + 2\mathcal{GL}_2 t^3 + \cdots + 2\mathcal{GL}_{n+1} t^{n+1} + \cdots, \tag{17}$$

$$t^3\mathcal{GL}(t) = \mathcal{GL}_0 t^3 + \mathcal{GL}_1 t^4 + \mathcal{GL}_2 t^5 + \cdots + \mathcal{GL}_{n+3} t^{n+3} + \cdots. \tag{18}$$

From (16), (17) and (18), we find that

$$
\begin{aligned}
(1-2t+t^3)\mathcal{GL}(t) &= \mathcal{GL}_0 + (\mathcal{GL}_1 - 2\mathcal{GL}_0)t + (\mathcal{GL}_2 - 2\mathcal{GL}_1)t^2 \\
&\quad + (\mathcal{GL}_3 - 2\mathcal{GL}_2 + \mathcal{GL}_0)t^3 + \cdots, \\
(1-2t+t^3)\mathcal{GL}(t) &= \mathcal{GL}_0 + (\mathcal{GL}_1 - 2\mathcal{GL}_0)t + (\mathcal{GL}_2 - 2\mathcal{GL}_1)t^2 \quad \text{(using relation (9))}.
\end{aligned}
$$

Thus, using values of $\mathcal{GL}_0, \mathcal{GL}_1, \mathcal{GL}_2$ in the above equation, we get

$$
\mathcal{GL}(t) = \frac{(1-i) - (1-3i)t + (1-i)t^2}{(1-2t+t^3)}.
$$

□

Theorem 10 (Exponential generating function ($E(t)$)). *For Gaussian Leonardo numbers, we have*

$$
E(t) = \frac{2}{\sqrt{5}}\Big[(\alpha+i)e^{\alpha t} - (\beta+i)e^{\beta t}\Big] - e^t(1+i).
$$

Proof. Using the Binet formula (11) of Gaussian Leonardo numbers, we have

$$
\begin{aligned}
E(t) &= \sum_{n=0}^{\infty} \mathcal{GL}_n \frac{t^n}{n!} \\
&= \sum_{n=0}^{\infty} \Big[\Big(2\Big(\frac{\alpha^{n+1}-\beta^{n+1}}{\alpha-\beta}\Big) - 1\Big) + i\Big(2\Big(\frac{\alpha^n-\beta^n}{\alpha-\beta}\Big) - 1\Big)\Big]\frac{t^n}{n!} \\
&= \frac{2}{\alpha-\beta}\Big(\alpha\sum_{n=0}^{\infty}\alpha^n\frac{t^n}{n!} - \beta\sum_{n=0}^{\infty}\beta^n\frac{t^n}{n!}\Big) - \sum_{n=0}^{\infty}\frac{t^n}{n!} \\
&\quad + i\Big[\frac{2}{\alpha-\beta}\Big(\sum_{n=0}^{\infty}\alpha^n\frac{t^n}{n!} - \sum_{n=0}^{\infty}\beta^n\frac{t^n}{n!}\Big) - \sum_{n=0}^{\infty}\frac{t^n}{n!}\Big] \\
&= \Big[\frac{2}{\alpha-\beta}(\alpha e^{\alpha t} - \beta e^{\beta t}) - e^t\Big] + i\Big[\frac{2}{\alpha-\beta}(e^{\alpha t} - e^{\beta t}) - e^t\Big] \\
&= \frac{2}{\sqrt{5}}(\alpha e^{\alpha t} - \beta e^{\beta t}) + i\frac{2}{\sqrt{5}}(e^{\alpha t} - e^{\beta t}) - e^t(1+i) \\
&= \frac{2}{\sqrt{5}}\Big[(\alpha+i)e^{\alpha t} - (\beta+i)e^{\beta t}\Big] - e^t(1+i).
\end{aligned}
$$

□

In order to obtain the exponential generating functions with even and odd-indexed terms of a sequence $\{a_n\}$, it is worth to note the following identity. Let $E(t) = \sum_{n=0}^{\infty} a_n \frac{t^n}{n!}$ be the exponential generating function for the sequence $\{a_n\}_{n\geq 0}$. Then the exponential generating functions for even and odd-indexed sequences $\{a_{2n}\}_{n\geq 0}$ and $\{a_{2n+1}\}_{n\geq 0}$, respectively, are

$$E_{a_{2n}}(t)=\frac{E(\sqrt{t})+E(-\sqrt{t})}{2} \quad \text{and} \quad E_{a_{2n+1}}(t)=\frac{E(\sqrt{t})-E(-\sqrt{t})}{2\sqrt{t}}. \tag{19}$$

Theorem 11. *For the Gaussian Leonardo sequence, the exponential generating functions for even and odd-indexed sequences $\mathcal{GL}_{2n}$ and $\mathcal{GL}_{2n+1}$, respectively, are*

$$E_{\mathcal{GL}_{2n}}(t)=\frac{2}{\sqrt{5}}\Big((\alpha+i)\cosh\alpha\sqrt{t}-(\beta+i)\cosh\beta\sqrt{t}\Big)-\cosh\sqrt{t}(1+i)$$

$$\textit{and}\quad E_{\mathcal{GL}_{2n+1}}(t)=\frac{2}{\sqrt{5t}}\Big((\alpha+i)\sinh\alpha\sqrt{t}-(\beta+i)\sinh\beta\sqrt{t}\Big)-\frac{(1+i)\sinh\sqrt{t}}{\sqrt{t}}.$$

Proof. Since we have

$$\begin{aligned}
E(t)+E(-t)&=\Big[\frac{2}{\sqrt{5}}(\alpha e^{\alpha t}-\beta e^{\beta t})-e^{t}\Big]+i\Big[\frac{2}{\sqrt{5}}(e^{\alpha t}-e^{\beta t})-e^{t}\Big]\\
&\quad+\Big[\frac{2}{\sqrt{5}}(\alpha e^{-\alpha t}-\beta e^{-\beta t})-e^{-t}\Big]+i\Big[\frac{2}{\sqrt{5}}(e^{-\alpha t}-e^{-\beta t})-e^{-t}\Big]\\
&=\Big[\frac{2}{\sqrt{5}}(2\alpha\cosh\alpha t-2\beta\cosh\beta t)-2\cosh t\Big]\\
&\quad+i\Big[\frac{2}{\sqrt{5}}(2\cosh\alpha t-2\cosh\beta t)-2\cosh t\Big]\\
&=\Big[\frac{4}{\sqrt{5}}(\alpha\cosh\alpha t-\beta\cosh\beta t)-2\cosh t\Big]\\
&\quad+i\Big[\frac{4}{\sqrt{5}}(\cosh\alpha t-\cosh\beta t)-2\cosh t\Big]\\
&=\frac{4}{\sqrt{5}}\Big((\alpha+i)\cosh\alpha t-(\beta+i)\cosh\beta t\Big)-2\cosh t(1+i).
\end{aligned}$$

Thus, from Eqn. (19), we get the desired result.
Similarly, evaluating $(E(t)-E(-t))/2t$ and simplifying by replacing t by $\sqrt{t}$ according to Eqn. (19), gives the exponential generating function for odd-indexed sequences $\{\mathcal{GL}_{2n}\}_{n\geq 0}$ as follow:

$$E_{\mathcal{GL}_{2n+1}}(t)=\frac{2}{\sqrt{5t}}\Big((\alpha+i)\sinh\alpha\sqrt{t}-(\beta+i)\sinh\beta\sqrt{t}\Big)-\frac{(1+i)\sinh\sqrt{t}}{\sqrt{t}}.$$

□

Theorem 12 (Catalan's identity). *For the Gaussian Leonardo numbers $\{\mathcal{GL}_n\}$, we have*

$$\begin{aligned}
\mathcal{GL}_a^2-\mathcal{GL}_{a-b}\mathcal{GL}_{a+b}&=\mathcal{L}_{a-b-2}+\mathcal{L}_{a+b-2}-2\mathcal{L}_{a-2}+(-1)^{a-b-1}2(\mathcal{L}_{b-1}+1)^2\\
&\quad+i(2\mathcal{L}_a\mathcal{L}_{a-1}-\mathcal{L}_{a-b}\mathcal{L}_{a+b-1}-\mathcal{L}_{a-b-1}\mathcal{L}_{a+b}).
\end{aligned}$$

Proof. From Eqn. (10), we have

$$\begin{aligned}
\mathcal{GL}_a^2 - \mathcal{GL}_{a-b}\mathcal{GL}_{a+b} &= (\mathcal{L}_a + i\mathcal{L}_{a-1})^2 - (\mathcal{L}_{a-b} + i\mathcal{L}_{a-b-1})(\mathcal{L}_{a+b} + i\mathcal{L}_{a+b-1}) \\
&= \mathcal{L}_a^2 - \mathcal{L}_{a-1}^2 + 2i\mathcal{L}_a\mathcal{L}_{a-1} - \mathcal{L}_{a-b}\mathcal{L}_{a+b} - i\mathcal{L}_{a-b}\mathcal{L}_{a+b-1} \\
&\quad - i\mathcal{L}_{a-b-1}\mathcal{L}_{a+b} + \mathcal{L}_{a-b-1}\mathcal{L}_{a+b-1} \\
&= (\mathcal{L}_a^2 - \mathcal{L}_{a-b}\mathcal{L}_{a+b}) - (\mathcal{L}_{a-1}^2 - \mathcal{L}_{a-b-1}\mathcal{L}_{a+b-1}) \\
&\quad + i(2\mathcal{L}_a\mathcal{L}_{a-1} - \mathcal{L}_{a-b}\mathcal{L}_{a+b-1} - \mathcal{L}_{a-b-1}\mathcal{L}_{a+b}) \\
&= [\mathcal{L}_{a-b} + \mathcal{L}_{a+b} - 2\mathcal{L}_a - (-1)^{a-b}(\mathcal{L}_{b-1} + 1)^2] \\
&\quad - [\mathcal{L}_{a-b-1} + \mathcal{L}_{a+b-1} - 2\mathcal{L}_{a-1} - (-1)^{a-b-1}(\mathcal{L}_{b-1} + 1)^2] \\
&\quad + i(2\mathcal{L}_a\mathcal{L}_{a-1} - \mathcal{L}_{a-b}\mathcal{L}_{a+b-1} - \mathcal{L}_{a-b-1}\mathcal{L}_{a+b}).
\end{aligned}$$

From Eqn. (1), we write $\mathcal{L}_a - \mathcal{L}_{a-1} = \mathcal{L}_{a-2} + 1$. Thus, the above equation simplified as

$$\begin{aligned}
\mathcal{GL}_a^2 - \mathcal{GL}_{a-b}\mathcal{GL}_{a+b} = \mathcal{L}_{a-b-2} + \mathcal{L}_{a+b-2} - 2\mathcal{L}_{a-2} + (-1)^{a-b-1}2(\mathcal{L}_{b-1} + 1)^2 \\
+ i(2\mathcal{L}_a\mathcal{L}_{a-1} - \mathcal{L}_{a-b}\mathcal{L}_{a+b-1} - \mathcal{L}_{a-b-1}\mathcal{L}_{a+b}).
\end{aligned}$$

Thus, the required result is obtained. □

Theorem 13 (d'Ocagne's identity). *For $a, b \geq 0$, we have*

$$\mathcal{GL}_{a+1}\mathcal{GL}_b - \mathcal{GL}_a\mathcal{GL}_{b+1} = 8(-1)^{a+1}F_{b-a} + \mathcal{L}_{b-3} - \mathcal{L}_{a-3} + i(2(-1)^a\mathcal{L}_{b-a-2} + \mathcal{L}_b - \mathcal{L}_a).$$

Proof. We have

$$\begin{aligned}
&\mathcal{GL}_{a+1}\mathcal{GL}_b - \mathcal{GL}_a\mathcal{GL}_{b+1} \\
&\quad = (\mathcal{L}_{a+1} + i\mathcal{L}_a)(\mathcal{L}_b + i\mathcal{L}_{b-1}) - (\mathcal{L}_a + i\mathcal{L}_{a-1})(\mathcal{L}_{b+1} + i\mathcal{L}_b) \\
&\quad = (\mathcal{L}_{a+1}\mathcal{L}_b - \mathcal{L}_a\mathcal{L}_{b+1}) - (\mathcal{L}_a\mathcal{L}_{b-1} - \mathcal{L}_{a-1}\mathcal{L}_b) + i(\mathcal{L}_{a+1}\mathcal{L}_{b-1} - \mathcal{L}_{a-1}\mathcal{L}_{b+1}) \\
&\quad = 2(-1)^{a+1}(\mathcal{L}_{b-a-1} + 1) + \mathcal{L}_{b-1} - \mathcal{L}_{a-1} - (2(-1)^a(\mathcal{L}_{b-a-1} + 1) + \mathcal{L}_{b-2} \\
&\qquad - \mathcal{L}_{a-2}) + i(\mathcal{L}_{a+1}\mathcal{L}_{b-1} - \mathcal{L}_{a-1}\mathcal{L}_{b+1}) \\
&\quad = 4(-1)^{a+1}(\mathcal{L}_{b-a-1} + 1) + (\mathcal{L}_{b-1} - \mathcal{L}_{b-2}) - (\mathcal{L}_{a-1} - \mathcal{L}_{a-2}) \\
&\qquad + i(\mathcal{L}_{a+1}\mathcal{L}_{b-1} - \mathcal{L}_{a-1}\mathcal{L}_{b+1}) \\
&\quad = 8(-1)^{a+1}F_{b-a} + \mathcal{L}_{b-3} - \mathcal{L}_{a-3} + i(\mathcal{L}_{a+1}\mathcal{L}_{b-1} - \mathcal{L}_{a-1}\mathcal{L}_{b+1}).
\end{aligned}$$

By using the d'Ocagne identity of Leonardo numbers, we write

$$\mathcal{L}_{a+1}\mathcal{L}_{b-1} - \mathcal{L}_{a-1}\mathcal{L}_{b+1} = 2(-1)^a\mathcal{L}_{b-a-2} + \mathcal{L}_b - \mathcal{L}_a.$$

Thus, the result is obtained. □

3.1. k-Gaussian Leonardo numbers

Definition 3. Let $k \in \mathbb{N}$ and $n \in \mathbb{N} \cup \{0\}$ then $\exists\ s, r$ such that $n = sk + r$, $0 \leq r < k$. The k-Gaussian Leonardo numbers $\{\mathcal{GL}_n^{(k)}\}$ are defined by

$$\begin{aligned}\mathcal{GL}_n^{(k)} &= \Big[\Big\{2\Big(\frac{\alpha^{s+1}-\beta^{s+1}}{\alpha-\beta}\Big)-1\Big\}+i\Big\{2\Big(\frac{\alpha^{s}-\beta^{s}}{\alpha-\beta}\Big)-1\Big\}\Big]^{k-r} \\ &\quad \Big[\Big\{2\Big(\frac{\alpha^{s+2}-\beta^{s+2}}{\alpha-\beta}\Big)-1\Big\}+i\Big\{2\Big(\frac{\alpha^{s+1}-\beta^{s+1}}{\alpha-\beta}\Big)-1\Big\}\Big]^{r}. \end{aligned} \tag{20}$$

By the above definition, the relation between k-Gaussian Leonardo numbers and Gaussian Leonardo numbers are observed as

$$\mathcal{GL}_{sk+r}^{(k)} = \mathcal{GL}_s^{k-r}\mathcal{GL}_{s+1}^{r}. \tag{21}$$

Note that if $k = 1$ then $r = 0$ and $n = s$, hence from the above equation we get $\mathcal{GL}_s^{(1)} = \mathcal{GL}_s$.
Similarly, if $k = 2$ then $r = 0, 1$ and $k = 3$ then $r = 0, 1, 2$. Thus the following relations obtained.

$$\text{For } k=2, \qquad \mathcal{GL}_{2s}^{(2)} = \mathcal{GL}_s^2 \quad \text{and} \quad \mathcal{GL}_{2s+1}^{(2)} = \mathcal{GL}_s\mathcal{GL}_{s+1}.$$
$$\text{For } k=3, \qquad \mathcal{GL}_{3s}^{(3)} = \mathcal{GL}_s^3, \quad \mathcal{GL}_{3s+1}^{(3)} = \mathcal{GL}_s^2\mathcal{GL}_{s+1} \quad \text{and} \quad \mathcal{GL}_{3s+2}^{(3)} = \mathcal{GL}_s\mathcal{GL}_{s+1}^2.$$

Also, the following identity satisfied.

$$\mathcal{GL}_{2s+1}^{(2)} = 2\mathcal{GL}_{2s}^{(2)} - \mathcal{GL}_s\mathcal{GL}_{s-2} \quad \text{and} \quad \mathcal{GL}_{3s+1}^{(3)} = 2\mathcal{GL}_{3s}^{(3)} - \mathcal{GL}_s^2\mathcal{GL}_{s-2}.$$

Since,

$$\begin{aligned}&\mathcal{GL}_{2s+1}^{(2)} = \mathcal{GL}_s\mathcal{GL}_{s+1} = \mathcal{GL}_s(2\mathcal{GL}_s - \mathcal{GL}_{s-2}) = 2(\mathcal{GL}_s)^2 - \mathcal{GL}_s\mathcal{GL}_{s-2} \\ \text{and} \quad &\mathcal{GL}_{3s+1}^{(3)} = \mathcal{GL}_s^2\mathcal{GL}_{s+1} = \mathcal{GL}_s^2(2\mathcal{GL}_s - \mathcal{GL}_{s-2}) = 2\mathcal{GL}_s^3 - \mathcal{GL}_s^2\mathcal{GL}_{s-2}.\end{aligned}$$

By a similar argument, we deduce the following general relation

$$\mathcal{GL}_{sk+1}^{(k)} = 2\mathcal{GL}_{sk}^{(k)} - \mathcal{GL}_s^{k-1}\mathcal{GL}_{s-2}.$$

Theorem 14. *Let $k,\ s \in \mathbb{N}$ then we have $\mathcal{GL}_{sk}^{(k)} = \mathcal{GL}_s^k$.*

Proof. For $r = 0$ we get $n = sk$, thus using Eqn. (21), result obtained. □

For $k = 1, 2, 3, 4, 5$, first few numbers of the k-Gaussian Leonardo sequence are displayed in the following table.

$n\downarrow$	$\mathcal{GL}_n^{(k)}$	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$
0	$\mathcal{GL}_0^{(k)}$	$1-i$	$-2i$	$-2-2i$	-4	$-4+4i$
1	$\mathcal{GL}_1^{(k)}$	$1+i$	2	$2-2i$	$-4i$	$-4-4i$
2	$\mathcal{GL}_2^{(k)}$	$3+i$	$2i$	$2+2i$	4	$4-4i$
3	$\mathcal{GL}_3^{(k)}$	$5+3i$	$2+4i$	$-2+2i$	$4i$	$4+4i$
4	$\mathcal{GL}_4^{(k)}$	$9+5i$	$8+6i$	$-2+6i$	-4	$-4+4i$
5	$\mathcal{GL}_5^{(k)}$	$15+9i$	$12+14i$	$2+14i$	$-8+4i$	$-4-4i$
6	$\mathcal{GL}_6^{(k)}$	$25+15i$	$16+30i$	$18+26i$	$-12+16i$	$-12-4i$
7	$\mathcal{GL}_7^{(k)}$	$41+25i$	$30+52i$	$22+54i$	$-8+44i$	$-28+4i$
8	$\mathcal{GL}_8^{(k)}$	$67+41i$	$56+90i$	$18+106i$	$28+96i$	$-52+36i$
9	$\mathcal{GL}_9^{(k)}$	$109+67i$	$90+156i$	$-10+198i$	$12+184i$	$-68+124i$
10	$\mathcal{GL}_{10}^{(k)}$	$177+109i$	$144+270i$	$-6+350i$	$-52+336i$	$-12+316i$

Table 2. **k-Gaussian Leonardo numbers.**

Theorem 15. *For $k,\ s \in \mathbb{N}$, we have* $\mathcal{GL}_{sk+k}^{(k)} - \mathcal{GL}_{sk}^{(k)} = \mathcal{GL}_{s+1}^{k} - \mathcal{GL}_{s}^{k}$.

Proof. It can be easily proved using Eqn. (21). □

Theorem 16. *(Cassini's identity) The Cassini's identity for the k-Gaussian Leonardo numbers $\mathcal{GL}_n^{(k)}$ is given by*

$$(\mathcal{GL}_{sk+a-1}^{(k)})^2 - \mathcal{GL}_{sk+a}^{(k)}\mathcal{GL}_{sk+a-2}^{(k)} = \begin{cases} 0, & a \neq 1, \\ \mathcal{GL}_s^{2k-2}[\mathcal{L}_{s-5} + 1 + 8(-1)^s + i(\mathcal{L}_{s-2} + 1 + 4(-1)^{s-1})], & a = 1. \end{cases}$$

Proof. From (21),

$$\begin{aligned}(\mathcal{GL}_{sk+a-1}^{(k)})^2 - \mathcal{GL}_{sk+a}^{(k)}\mathcal{GL}_{sk+a-2}^{(k)} &= (\mathcal{GL}_s^{k-a+1}\mathcal{GL}_{s+1}^{a-1})^2 \\ &\quad - (\mathcal{GL}_s^{k-a}\mathcal{GL}_{s+1}^{a})(\mathcal{GL}_s^{k-a+2}\mathcal{GL}_{s+1}^{a-2}) \\ &= 0, \text{ for } a \neq 1,\end{aligned}$$

and for $a = 1$, we have

$$\begin{aligned}(\mathcal{GL}_{sk}^{(k)})^2 - \mathcal{GL}_{sk+1}^{(k)}\mathcal{GL}_{sk-1}^{(k)} &= (\mathcal{GL}_s^{k})^2 - (\mathcal{GL}_s^{k-1}\mathcal{GL}_{s+1}^{1})(\mathcal{GL}_{s-1}^{1}\mathcal{GL}_s^{k-1}) \\ &= \mathcal{GL}_s^{2k} - \mathcal{GL}_s^{2k-2}\mathcal{GL}_{s+1}\mathcal{GL}_{s-1} \\ &= \mathcal{GL}_s^{2k-2}(\mathcal{GL}_s^{2} - \mathcal{GL}_{s+1}\mathcal{GL}_{s-1}) \\ &= \mathcal{GL}_s^{2k-2}[\mathcal{L}_{s-5} + 1 + 8(-1)^s + i(\mathcal{L}_s\mathcal{L}_{s-1} - \mathcal{L}_{s+1}\mathcal{L}_{s-2})] \\ &= \mathcal{GL}_s^{2k-2}[\mathcal{L}_{s-5} + 1 + 8(-1)^s + i(\mathcal{L}_{s-2} + 1 + 4(-1)^{s-1})],\end{aligned}$$

as required. □

4. Conclusion

In summary, we presented a study on some new families of the k-Leonardo numbers and the Gaussian Leonardo numbers, where the subscript n is considered to be of the form $sk + r$ with $0 \leq r < k$. We discussed various combinatorial properties of these new families and obtained ordinary (and exponential) generating functions, partial sum, etc. of them in closed form.

Acknowledgment. The authors would like to thank the anonymous reviewers for carefully reading the manuscript and giving insightful comments. The first and third authors would like to thank the University Grant Commission (UGC), India for the financial support in the form of research fellowship.

References

[1] Y. Alp and E.G. Koçer, *Some properties of Leonardo numbers*, Konuralp J. Math. **9** (2021), no. 1, 183–189.

[2] P. Catarino, P. Vasco, H. Campos, A.P. Aires, and A. Borges, *New families of Jacobsthal and Jacobsthal-Lucas numbers*, Algebra Discrete Math. **20** (2015), no. 1, 40–54.

[3] P.M.M.C. Catarino and A. Borges, *On Leonardo numbers*, Acta Math. Univ. Comenian. **89** (2019), no. 1, 75–86.

[4] M. El-Mikkawy and T. Sogabe, *A new family of k-Fibonacci numbers*, Appl. Math. Comput. **215** (2010), no. 12, 4456–4461.

[5] T. Koshy, *Fibonacci and Lucas Numbers with Applications*, John Wiley & Sons, 2019.

[6] K. Kuhapatanakul and J. Chobsorn, *On the generalized Leonardo numbers*, Integer **22** (2022), Article number A48.

[7] M. Kumari, J. Tanti, and K. Prasad, *On some new families of k-Mersenne and generalized k-Gaussian Mersenne numbers and their polynomials*, arXiv preprint arXiv:2111.09592 (2021).

[8] E. Özkan, İ. Altun, and A.A. Göçer, *On relationship among a new family of k-Fibonacci, k-Lucas numbers, Fibonacci and Lucas numbers*, Chiang Mai J. Sci. **44** (2017), no. 4, 1744–1750.

[9] E. Özkan and M. Taştan, *On a new family of Gauss k-Lucas numbers and their polynomials*, Asian-Eur. J. Math. **14** (2021), no. 6, Article ID: 2150101.

[10] K. Prasad, H. Mahato, and M. Kumari, *On the generalized k-Horadam-like sequences*, Algebra, Analysis, and Associated Topics, Trends in Mathematics, Springer, 2022, pp. 11–26.

[11] P.K. Ray, *On the properties of k-balancing and k-Lucas-balancing numbers*, Acta Comment. Univ. Tartu. Math. **21** (2017), no. 2, 259–274.

[12] ———, *On the properties of k-balancing numbers*, Ain Shams Engineering J. **9** (2018), no. 3, 395–402.

[13] A.G. Shannon, *A note on generalized Leonardo numbers*, Notes Number Theory Discrete Math. **25** (2019), no. 3, 97–101.

[14] A.G. Shannon and Ö. Deveci, *A note on generalized and extended Leonardo sequences*, Notes Number Theory Discrete Math. **28** (2022), no. 1, 109–114.

[15] M. Shattuck, *Combinatorial proofs of identities for the generalized Leonardo numbers*, Notes Number Theory Discrete Math. **28** (2022), no. 4, 778–790.

[16] Y. Soykan, *Generalized Leonardo numbers*, J. Progressive Res. Math. **18** (2021), no. 4, 58–84.

[17] ———, *Special cases of generalized Leonardo numbers: Modified p-Leonardo, p-Leonardo-Lucas and p-Leonardo numbers*, Earthline J. Math. Sci. **11** (2023), no. 2, 317–342.

[18] M. Taştan, N. Yılmaz, and E. Özkan, *A new family of Gauss* (k, t)*-Horadam numbers*, Asian-Eur. J. Math. **15** (2022), no. 12, Article ID: 2250225.

[19] Y. Yazlik and N. Taskara, *A note on generalized* k*-Horadam sequence*, Compute. Math. Appl. **63** (2012), no. 1, 36–41.

[20] N. Yilmaz, M. Tastan, and E. Özkan, *A new family of Horadam numbers*, Electronic Journal of Mathematical Analysis and Applications **10** (2022), no. 1, 64–70.