Communications in Combinatorics and Optimization
Vol. 9, No. 3 (2024), pp. 509-526
https://doi.org/10.22049/cco.2023.28046.1426

CCO
Commun. Comb. Optim.

Research Article

On the vertex irregular reflexive labeling of generalized friendship graph and corona product of graphs

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Received: 17 October 2022; Accepted: 30 March 2023
Published Online: 27 April 2023

Abstract: For a graph G, we define a total k-labeling φ as a combination of an edge labeling $\varphi_e : E(G) \to \{1, 2, \ldots, k_e\}$ and a vertex labeling $\varphi_v : V(G) \to \{0, 2, \ldots, 2k_v\}$, where $k = \max\{k_e, 2k_v\}$. The total k-labeling φ is called a vertex irregular reflexive k-labeling of G if any pair of vertices u, u' have distinct vertex weights $wt_\varphi(u) \neq wt_\varphi(u')$, where $wt_\varphi(u) = \varphi(u) + \sum_{uu' \in E(G)} \varphi(uu')$ for any vertex $u \in V(G)$. The smallest value of k for which such a labeling exists is called the reflexive vertex strength of G, denoted by $\text{rvs}(G)$. In this paper, we present a new lower bound for the reflexive vertex strength of any graph. We investigate the exact values of the reflexive vertex strength of generalized friendship graphs, corona product of two paths, and corona product of a cycle with isolated vertices by referring to the lower bound. This study discovers some interesting open problems that are worth further exploration.

Keywords: Corona product, cycle, generalized friendship graph, isolated vertices, lower bound, path, reflexive vertex strength, vertex irregular reflexive labeling

AMS Subject classification: 05C78, 05C38

1. Introduction

A graph G consists of a finite nonempty set $V(G)$ of vertices and a set $E(G)$ of unordered pairs of vertices called edges. The symbol $|V(G)|$ is known as the order

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and $|E(G)|$ as the size of the graph G. The degree of a vertex v is the number of edges incident with the vertex, which is denoted as $\deg(v)$ for $v \in V(G)$. Additionally, the minimum degree among the vertices of the graph G is denoted by $\delta(G)$ and the maximum degree of the graph G is $\Delta(G)$. For other terminologies and notations that are not defined in this paper, we refer to [7].

All the graphs considered in this paper are simple and connected. It is known that a simple graph is impossible to become an irregular, which is to have distinct degrees of vertices, but it is logical in multigraphs. Therefore, Chartrand et al. [6] overturned this concept by introducing an irregular labeling, which consequently the simple graph becomes irregular by replacing the multiple edges incident on every vertex of a multigraph with a positive integer set on the edges of a simple graph of order at least 3. The minimum of maximal values is known as the irregularity strength of the graph G, denoted as $s(G)$.

Bača et al. [5] subsequently proposed irregular total labelings, i.e., a vertex irregular total labeling and an edge irregular total labeling, which consider the vertex weights and edge weights, respectively. In addition, we may refer to Gallian [9] for his comprehensive survey, which comprises the latest and most relevant articles on graph labelings.

Definition 1. [5] For a graph $G = (V, E)$, a labeling $\rho : V(G) \cup E(G) \to \{1, 2, \ldots, k\}$ is a total k-labeling. The total k-labeling is an edge irregular total k-labeling if every two different edges xy and $x'y'$ have distinct weights, $wt_\rho(xy) \neq wt_\rho(x'y')$, where

$$wt_\rho(xy) = \rho(x) + \rho(xy) + \rho(y).$$

Likewise, the total k-labeling is called a vertex irregular total k-labeling if every two different vertices x and y have the distinct weights, $wt_\rho(x) \neq wt_\rho(y)$, where

$$wt_\rho(x) = \rho(x) + \sum_{xy \in E(G)} \rho(xy).$$

The minimum k for which such labelings exist is called the total edge irregularity strength of G, denoted by $\mathrm{tes}(G)$ (resp. the total vertex irregularity strength of G, denoted by $\mathrm{tvs}(G)$).

Inspired by the concepts of irregular labeling and irregular total labelings, Ryan et al. [10] introduced the concept of vertex irregular reflexive labeling, in which the vertex labels are represented as the vertex degrees contributed by the loops. They made two observations, (a) the vertex labels are non-negative even integers, which represent the fact that each loop contributes twice to the vertex degree, and (b) the vertex label 0 is permissible to represent a loopless vertex.

Definition 2. [10] A total k-labeling φ is a combination of an edge labeling $\varphi_e : E(G) \to \{1, 2, \ldots, k_e\}$ and a vertex labeling $\varphi_v : V(G) \to \{0, 2, \ldots, 2k_v\}$, where $k = \max\{k_e, 2k_v\}$.

The labeling φ is called a *vertex irregular reflexive k-labeling* of G if any two different vertices u and u' have the distinct vertex weights, $wt_\varphi(u) \neq wt_\varphi(u')$, where

$$wt_\varphi(u) = \varphi(u) + \sum_{uu' \in E(G)} \varphi(uu').$$

The smallest value k for which such a labeling exists is called the reflexive vertex strength of G, and is denoted by $\mathrm{rvs}(G)$.

Furthermore, Tanna et al. [10] extended this study by proving the minimum value of the largest vertex weight among all vertices in a graph G. They also investigated the exact values of the reflexive vertex strength for several graphs, such as prisms, wheels, fan graphs, and baskets. Additionally, Alfarisi et al. [4] studied a vertex irregular reflexive labeling on the disjoint union of gear and book graphs in 2020. Agustin and her research team obtained the exact value of the reflexive vertex strength of certain graphs with pendant vertices in [3], some almost regular graphs in [2], and some regular and regular-like graphs in [1]. In 2022, Saleem et al. [12] obtained the precise value of the reflexive vertex strength of generalized Petersen graph.
In this paper, we present a new lower bound on the reflexive vertex strength of any graph. We then investigate the exact values of the reflexive vertex strength of generalized friendship graphs, corona product of two paths, and corona product of a cycle with isolated vertices by using the lower bound. Some interesting open problems are included in this study for further research.

2. New lower bound of the reflexive vertex strength

Tanna et al. [13] derived a lower bound for the reflexive vertex strength of an r-regular graph of order p as shown in Corollary 1. They verified this bound by prism D_n, which is a 3-regular graph of order $2n$ for $n \geq 3$.

Corollary 1. [13] *Let G be an r-regular graph of order p. Then*

$$\mathrm{rvs}(G) \geq \begin{cases} \left\lceil \frac{p+r-1}{r+1} \right\rceil, & \text{if } p \equiv 0, 1 \pmod 4, \\ \left\lceil \frac{p+r}{r+1} \right\rceil, & \text{if } p \equiv 2, 3 \pmod 4. \end{cases}$$

If a graph G has a vertex irregular reflexive labeling, then the vertices must have distinct vertex weights.

Lemma 1. [13] *The largest vertex weight of a graph G of order p and minimum degree δ under a vertex irregular reflexive labeling is at least*

1. *$p + \delta - 1$ if $p \equiv 0 \pmod 4$ or $p \equiv 1 \pmod 4$ and $\delta \equiv 0 \pmod 2$ or $p \equiv 3 \pmod 4$ and $\delta \equiv 1 \pmod 2$,*

2. $p+\delta$ otherwise.

Whereafter, Agustin et al. [3] posed a lower bound on the reflexive vertex strength of any graph with pendant vertices, such as the sunlet graph, helm graph, subdivided star graph, and broom graph.

Lemma 2. [3] *For any graph G of order p with minimum degree δ and maximum degree Δ,*

$$\operatorname{rvs}(G) \geq \left\lceil \frac{p+\delta-1}{\Delta+1} \right\rceil.$$

Recently, Agustin et al. [1] introduced a lower bound for the reflexive vertex strength of any graph as follows.

Lemma 3. [1] *If any graph G of order p with miminum degree δ and maximum degree Δ, then*

$$\operatorname{rvs}(G) \geq \begin{cases} \left\lceil \frac{p+\delta-1}{\Delta+1} \right\rceil, & \textit{if } p \equiv 0,1 \pmod 4,\ \delta \equiv 0 \pmod 2 \textit{ or} \\ & \quad p \equiv 3 \pmod 4,\ \delta \equiv 1 \pmod 2, \\ \left\lceil \frac{p+\delta}{\Delta+1} \right\rceil, & \textit{otherwise.} \end{cases}$$

We have observed that the exact value of the reflexive vertex strength of the sunlet graph S_n, $n \equiv 0 \pmod 4$, as given in [3], contradicts Lemma 3. We provide a counterexample below.

Example 1. By referring to Lemma 3, $\operatorname{rvs}(S_4) \geq 3$. But, the exact value of $\operatorname{rvs}(S_4)$ is 2 as illustrated in Figure 1.

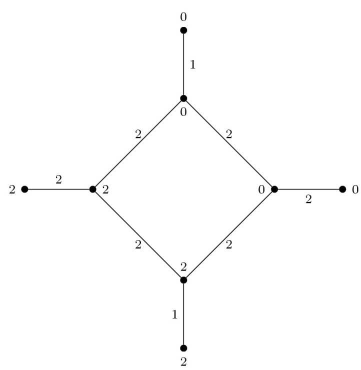


Figure 1. **A vertex irregular reflexive 2-labeling of the sunlet graph S_4.**

Therefore, in the following theorem, we show a new lower bound for the reflexive vertex strength of any graph.

Theorem 1. *For a graph G with the minimum degree δ and maximum degree Δ,*

$$\mathrm{rvs}(G) \geq \max_i \left\{ \left\lceil \frac{(\sum_{d=\delta}^{i} v_d) + \delta - 1}{i+1} \right\rceil + \beta_i \right\},$$

where v_d represents the number of vertices of degree d as $\delta \leq d \leq i \leq \Delta$, and

$$\beta_i = \begin{cases} 1, & \text{if } \frac{(\sum_{d=\delta}^{i} v_d)+\delta-1}{i+1} \equiv 1 \pmod 2, \\ 0, & \text{otherwise.} \end{cases}$$

Proof. Let G be a graph with the minimum degree δ and the maximum degree Δ, and let v_d represent the number of vertices of degree d as $\delta \leq d \leq i \leq \Delta$. Assume that $r = \max_i \left\{ \left\lceil \frac{(\sum_{d=\delta}^{i} v_d)+\delta-1}{i+1} \right\rceil + \beta_i \right\}$, where $\beta_i = 1$ if $\frac{(\sum_{d=\delta}^{i} v_d)+\delta-1}{i+1} \equiv 1 \pmod 2$, otherwise $\beta_i = 0$. For a vertex irregular reflexive k-labeling on G, we note that the smallest vertex weight among all the vertices of degree i is at least δ, while the largest vertex weight is at least $(\sum_{d=\delta}^{i} v_d) + \delta - 1$. Additionally, the largest vertex weight has at least one label of $\left\lceil \frac{(\sum_{d=\delta}^{i} v_d)+\delta-1}{i+1} \right\rceil + 1$ for $\frac{(\sum_{d=\delta}^{i} v_d)+\delta-1}{i+1} \equiv 1 \pmod 2$ or $\left\lceil \frac{(\sum_{d=\delta}^{i} v_d)+\delta-1}{i+1} \right\rceil$ for $\frac{(\sum_{d=\delta}^{i} v_d)+\delta-1}{i+1} \not\equiv 1 \pmod 2$, where the vertex weight is defined as the sum of a vertex label and its i incident edge labels. Therefore, the value k is the minimum of the maximal edge labels or vertex labels of the graph G. It is also called the reflexive vertex strength of G and is at least r. This gives $\mathrm{rvs}(G) \geq \max_i \left\{ \left\lceil \frac{(\sum_{d=\delta}^{i} v_d)+\delta-1}{i+1} \right\rceil + \beta_i \right\}$. This completes the proof. □

Remark 1. For a graph G with the minimum degree δ and maximum degree Δ, the sum of the number of vertices of degree d as d goes from δ to Δ, $\sum_{d=\delta}^{\Delta} v_d$, is equal to the order of the graph G, $|V(G)|$.

In the following examples, we demonstrate the lower bound of the reflexive vertex strength of prism D_4 and sunlet graph S_{10} by referring to Theorem 1. In particular, Example 3 clearly shows that Theorem 1 provides a better lower bound compared to Lemma 2.

Example 2. Tanna et al. [13] proved that $\mathrm{rvs}(D_4) \geq 3$. Similarly, by using Theorem 1, we also obtain $\mathrm{rvs}(D_4) \geq 3$ as shown below.
Let D_4 has $\delta = \Delta = 3$ and $3 \leq i \leq 3$. Since $\frac{(\sum_{d=3}^{3} v_d)+3-1}{4} = \frac{8+3-1}{4} \not\equiv 1 \pmod 2$, then $\beta_3 = 0$. Hence, $\mathrm{rvs}(D_4) \geq \max_i \left\{ \left\lceil \frac{8+3-1}{4} \right\rceil + 0 \right\} \geq 3$.

Example 3. According to Lemma 2, $\mathrm{rvs}(S_{10}) \geq 5$. However, we obtain $\mathrm{rvs}(S_{10}) \geq 6$ by referring to Theorem 1, which shows that Theorem 1 is a better lower bound.
Let S_{10} has $\delta = 1$, $\Delta = 3$, and $1 \leq i \leq 3$. We note that:

$$\text{if } \left\lceil \frac{(\sum_{d=1}^{1} v_d) + 1 - 1}{2} \right\rceil = \left\lceil \frac{10}{2} \right\rceil \equiv 1 \pmod 2, \text{then } \beta_1 = 1,$$

if $\left\lceil \frac{(\sum_{d=1}^{2} v_d)+1-1}{3} \right\rceil = \left\lceil \frac{10}{3} \right\rceil \not\equiv 1 \pmod 2, \text{then } \beta_2 = 0,$

and if $\left\lceil \frac{(\sum_{d=1}^{3} v_d)+1-1}{4} \right\rceil = \left\lceil \frac{20}{4} \right\rceil \equiv 1 \pmod 2, \text{then } \beta_3 = 1.$

Hence, $\operatorname{rvs}(S_{10}) \geq \max_i \left\{ \lceil \frac{10}{2} \rceil + 1, \lceil \frac{10}{3} \rceil + 0, \lceil \frac{20}{4} \rceil + 1 \right\} \geq 6$.

3. Generalized friendship graph

The generalized friendship graph $f_{n,m}$ is a collection of m copies of cycles, each cycle of order n, joined together with a common vertex. Fernau et al. [8] stated that the friendship graph f_n, $n \geq 2$, is a collection of n copies of triangles (or cycles of order 3) with joining a common vertex, which is a special case of $f_{n,m}$. The $f_{n,m}$ is also known as a flower graph, where the cycles of $f_{n,m}$ are called "petals" of the flower graph, see Ryjáček and Schiermeyer [11]. Additionally, the vertex set and edge set of $f_{n,m}$ are defined as $V(f_{n,m}) = \{x, x_i^j | 1 \leq i \leq n-1, 1 \leq j \leq m\}$ and $E(f_{n,m}) = \{xx_1^j, xx_{n-1}^j | 1 \leq j \leq m\} \cup \{x_i^j x_{i+1}^j | 1 \leq i \leq n-2, 1 \leq j \leq m\}$, respectively.

Theorem 2. *For $n \geq 3$ and $m \geq 2$,*

$$\operatorname{rvs}(f_{n,m}) = \begin{cases} \frac{m(n-1)+1}{3} + 1, & \text{if } m(n-1)+1 \equiv 3 \pmod 6, \\ \left\lceil \frac{m(n-1)+1}{3} \right\rceil, & \text{if } m(n-1)+1 \not\equiv 3 \pmod 6. \end{cases}$$

Proof. Since $f_{n,m}$ has $m(n-1)+1$ vertices and nm edges, by Theorem 1, we obtain

$$\operatorname{rvs}(f_{n,m}) \geq k = \begin{cases} \frac{m(n-1)+1}{3} + 1, & \text{if } m(n-1)+1 \equiv 3 \pmod 6, \\ \left\lceil \frac{m(n-1)+1}{3} \right\rceil, & \text{if } m(n-1)+1 \not\equiv 3 \pmod 6. \end{cases}$$

Therefore, we prove that k is also an upper bound for $\operatorname{rvs}(f_{n,m})$, where $n \geq 3$ and $m \geq 2$. We define a total k-labeling φ of $f_{n,m}$ as follows.

Case 1. $m = 2$.

$$\varphi(x) = \begin{cases} k, & \text{if } k \text{ is even}, \\ k-1, & \text{if } k \text{ is odd}. \end{cases}$$

$$\varphi(x_i^1) = 0, \ \text{for } 1 \leq i \leq n-1.$$

$$\varphi(x_i^2) = \begin{cases} k, & \text{if } 1 \leq i \leq n-1, \ k \text{ is even}, \\ k-1, & \text{if } 1 \leq i \leq n-1, \ k \text{ is odd}. \end{cases}$$

$$\varphi(xx_i^1) = \begin{cases} 1, & \text{if } i = 1, \\ \left\lceil \frac{n}{2} \right\rceil, & \text{if } i = n-1. \end{cases}$$

$$\varphi(xx_i^2) = \begin{cases} \left\lfloor \frac{n+1-k}{2} \right\rfloor, & \text{if } i = 1,\ k \text{ is even}, \\ \left\lfloor \frac{n+2-k}{2} \right\rfloor, & \text{if } i = 1,\ k \text{ is odd}, \\ \left\lceil \frac{2n-1-k}{2} \right\rceil, & \text{if } i = n-1,\ k \text{ is even}, \\ \left\lceil \frac{2n-k}{2} \right\rceil, & \text{if } i = n-1,\ k \text{ is odd}. \end{cases}$$

$$\varphi(x_i^j x_{i+1}^j) = \begin{cases} \left\lceil \frac{i+1}{2} \right\rceil, & \text{if } 1 \le i \le n-2,\ j = 1, \\ \left\lceil \frac{n+i-k}{2} \right\rceil, & \text{if } 1 \le i \le n-2,\ j = 2,\ k \text{ is even}, \\ \left\lceil \frac{n+i-k+1}{2} \right\rceil, & \text{if } 1 \le i \le n-2,\ j = 2,\ k \text{ is odd}. \end{cases}$$

Case 2. $m \ge 3$.

$\varphi(x) = 0.$

$$\varphi(x_i^j) = \begin{cases} 0, & \text{if } 1 \le i \le n-1,\ 1 \le j \le \left\lceil \frac{2k}{n-1} \right\rceil - 1, \\ k, & \text{if } 1 \le i \le n-1,\ \left\lceil \frac{2k}{n-1} \right\rceil \le j \le m,\ k \text{ is even}, \\ k-1, & \text{if } 1 \le i \le n-1,\ \left\lceil \frac{2k}{n-1} \right\rceil \le j \le m,\ k \text{ is odd}. \end{cases}$$

$$\varphi(xx_1^j) = \begin{cases} \left\lfloor \frac{(n-1)(j-1)+2}{2} \right\rfloor, & \text{if } 1 \le j \le \left\lceil \frac{2k}{n-1} \right\rceil - 1, \\ \left\lfloor \frac{(n-1)(j-1)+2-k}{2} \right\rfloor, & \text{if } \left\lceil \frac{2k}{n-1} \right\rceil \le j \le m,\ k \text{ is even}, \\ \left\lfloor \frac{(n-1)(j-1)+3-k}{2} \right\rfloor, & \text{if } \left\lceil \frac{2k}{n-1} \right\rceil \le j \le m,\ k \text{ is odd}. \end{cases}$$

$$\varphi(xx_{n-1}^j) = \begin{cases} \left\lceil \frac{j(n-1)+1}{2} \right\rceil, & \text{if } 1 \le j \le \left\lceil \frac{2k}{n-1} \right\rceil - 1, \\ \left\lceil \frac{j(n-1)+1-k}{2} \right\rceil, & \text{if } \left\lceil \frac{2k}{n-1} \right\rceil \le j \le m,\ k \text{ is even}, \\ \left\lceil \frac{j(n-1)+2-k}{2} \right\rceil, & \text{if } \left\lceil \frac{2k}{n-1} \right\rceil \le j \le m,\ k \text{ is odd}. \end{cases}$$

$$\varphi(x_i^j x_{i+1}^j) = \begin{cases} \left\lceil \frac{(n-1)(j-1)+1+i}{2} \right\rceil, & \text{if } 1 \le i \le n-2,\ 1 \le j \le \left\lceil \frac{2k}{n-1} \right\rceil - 1, \\ \left\lceil \frac{(n-1)(j-1)+1+i-k}{2} \right\rceil, & \text{if } 1 \le i \le n-2,\ \left\lceil \frac{2k}{n-1} \right\rceil \le j \le m,\ k \text{ is even}, \\ \left\lceil \frac{(n-1)(j-1)+2+i-k}{2} \right\rceil, & \text{if } 1 \le i \le n-2,\ \left\lceil \frac{2k}{n-1} \right\rceil \le j \le m,\ k \text{ is odd}. \end{cases}$$

Evidently, k is the maximum value of $\mathrm{rvs}(f_{n,m})$. We then prove that every vertex of $f_{n,m}$ has a distinct vertex weight under the total k-labeling of φ, as shown in Tables

1 and 2. Specifically, the vertex weights are a set of positive integers $\{2, 3, \ldots, m(n-1)+1\} \cup \left\{ \sum_{j=1}^{m} \varphi(xx_1^j) + \sum_{j=1}^{m} \varphi(xx_{n-1}^j) + \varphi(x) \right\}$. Thus, the total k-labeling of φ is a vertex irregular reflexive labeling of $f_{n,m}$ and k is the exact value of $\mathrm{rvs}(f_{n,m})$, where $n \geq 3$ and $m \geq 2$. The theorem holds.

Table 1. **A summary of the vertex weights of $f_{n,2}$ for $n \geq 3$.**

				$m = 2$
$wt_\varphi(x)$		$1 \leq j \leq m$	even k	$k + 1 + \left\lfloor \frac{n+1-k}{2} \right\rfloor + \left\lceil \frac{n}{2} \right\rceil + \left\lceil \frac{2n-1-k}{2} \right\rceil$
			odd k	$k + \left\lfloor \frac{n+2-k}{2} \right\rfloor + \left\lceil \frac{n}{2} \right\rceil + \left\lceil \frac{2n-k}{2} \right\rceil$
$wt_\varphi(x_i^j)$	$i = 1$	$j = 1$		2
		$j = 2$	even k	$k + \left\lceil \frac{n+1-k}{2} \right\rceil + \left\lfloor \frac{n+1-k}{2} \right\rfloor$
			odd k	$k - 1 + \left\lceil \frac{n+2-k}{2} \right\rceil + \left\lfloor \frac{n+2-k}{2} \right\rfloor$
	$2 \leq i \leq n-2$	$j = 1$		$i + 1$
		$j = 2$	even k	$k + \left\lceil \frac{n+i-k}{2} \right\rceil + \left\lceil \frac{n+i-1-k}{2} \right\rceil$
			odd k	$k - 1 + \left\lceil \frac{n+i-k+1}{2} \right\rceil + \left\lceil \frac{n+i-k}{2} \right\rceil$
	$i = n-1$	$j = 1$		n
		$j = 2$	even k	$k + \left\lceil \frac{2n-1-k}{2} \right\rceil + \left\lceil \frac{2n-2-k}{n} \right\rceil$
			odd k	$k - 1 + \left\lceil \frac{2n-k}{2} \right\rceil + \left\lceil \frac{2n-1-k}{n} \right\rceil$

Table 2. **A summary of the vertex weights of $f_{n,m}$ for $n \geq 3$ and $m \geq 3$.**

				$m \geq 3$
$wt_\varphi(x)$		$1 \leq j \leq m$	even k	$\sum_{j=1}^{\lceil \frac{2k}{n-1} \rceil - 1} \left(\left\lfloor \frac{(n-1)(j-1)+2}{2} \right\rfloor + \left\lceil \frac{j(n-1)+1}{2} \right\rceil \right) + \sum_{j=\lceil \frac{2k}{n-1} \rceil}^{m} \left(\left\lfloor \frac{(n-1)(j-1)+2-k}{2} \right\rfloor + \left\lceil \frac{j(n-1)+1-k}{2} \right\rceil \right)$
			odd k	$\sum_{j=1}^{\lceil \frac{2k}{n-1} \rceil - 1} \left(\left\lfloor \frac{(n-1)(j-1)+2}{2} \right\rfloor + \left\lceil \frac{j(n-1)+1}{2} \right\rceil \right) + \sum_{j=\lceil \frac{2k}{n-1} \rceil}^{m} \left(\left\lfloor \frac{(n-1)(j-1)+3-k}{2} \right\rfloor + \left\lceil \frac{j(n-1)+2-k}{2} \right\rceil \right)$
$wt_\varphi(x_i^j)$	$i = 1$	$1 \leq j \leq \left\lceil \frac{2k}{n-1} \right\rceil - 1$		$\left\lfloor \frac{(n-1)(j-1)+2}{2} \right\rfloor + \left\lceil \frac{(j-1)(n-1)+2}{2} \right\rceil$
		$\left\lceil \frac{2k}{n-1} \right\rceil \leq j \leq m$	even k	$k + \left\lceil \frac{(j-1)(n-1)+2-k}{2} \right\rceil + \left\lfloor \frac{(j-1)(n-1)+2-k}{2} \right\rfloor$
			odd k	$k - 1 + \left\lceil \frac{(j-1)(n-1)+3-k}{2} \right\rceil + \left\lfloor \frac{(j-1)(n-1)+3-k}{2} \right\rfloor$
	$2 \leq i \leq n-2$	$1 \leq j \leq \left\lceil \frac{2k}{n-1} \right\rceil - 1$		$\left\lceil \frac{(j-1)(n-1)+1+i}{2} \right\rceil + \left\lceil \frac{(j-1)(n-1)+i}{2} \right\rceil$
		$\left\lceil \frac{2k}{n-1} \right\rceil \leq j \leq m$	even k	$k + \left\lceil \frac{(j-1)(n-1)+1+i-k}{2} \right\rceil + \left\lceil \frac{(j-1)(n-1)+i-k}{2} \right\rceil$
			odd k	$k - 1 + \left\lceil \frac{(j-1)(n-1)+2+i-k}{2} \right\rceil + \left\lceil \frac{(j-1)(n-1)+1+i-k}{2} \right\rceil$
	$i = n-1$	$1 \leq j \leq \left\lceil \frac{2k}{n-1} \right\rceil - 1$		$\left\lceil \frac{j(n-1)+1}{2} \right\rceil + \left\lceil \frac{(j-1)(n-1)+n-1}{2} \right\rceil$
		$\left\lceil \frac{2k}{n-1} \right\rceil \leq j \leq m$	even k	$k + \left\lceil \frac{j(n-1)+1-k}{2} \right\rceil + \left\lceil \frac{(j-1)(n-1)+n-1-k}{2} \right\rceil$
			odd k	$k - 1 + \left\lceil \frac{j(n-1)+2-k}{2} \right\rceil + \left\lceil \frac{(j-1)(n-1)+n-k}{2} \right\rceil$

$\square$

Example 4. Figure 2 illustrates the corresponding vertex irregular reflexive k-labelings of $f_{4,2}$ and $f_{5,6}$.

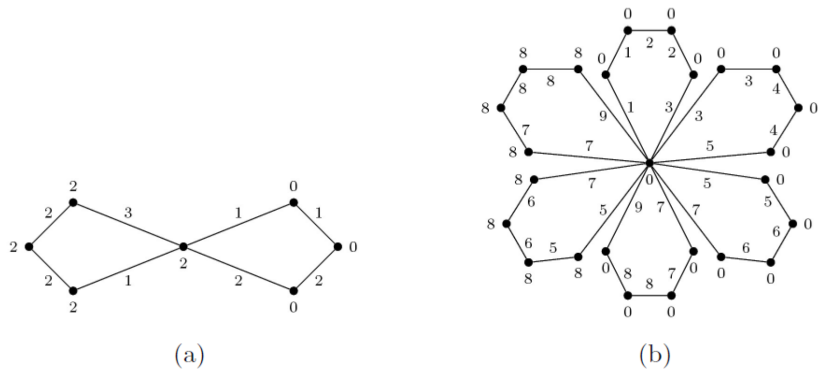


Figure 2. (a) A vertex irregular reflexive 3-labeling of $f_{4,2}$, and (b) A vertex irregular reflexive 9-labeling of $f_{5,6}$.

4. Corona product of graphs

In this section, we study the reflexive vertex strength of two families of corona product of graphs, which are corona product of two paths and corona product of a cycle with isolated vertices. In general, corona product of graphs G and H, denoted by $G \odot H$, is a graph obtained by taking a copy of G (with n vertices) and n copies of H, and joining the i-th vertex of G to every vertex in the i-th copy of H.

4.1. Corona product of two paths

We can refer to the definition, vertex set, and edge set of the corona product of two paths, denoted as $P_n \odot P_m$, through Yoong et al. [14]. Since the following study only considers $P_n \odot P_2$ for $n \geq 2$, the vertex set and edge set of $P_n \odot P_2$ are defined as $V(P_n \odot P_2) = \{x_i, y_i^j | 1 \leq i \leq n, 1 \leq j \leq 2\}$ and $E(P_n \odot P_2) = \{x_i y_i^j | 1 \leq i \leq n, 1 \leq j \leq 2\} \cup \{y_i^j y_i^{j+1} | 1 \leq i \leq n, j = 1\} \cup \{x_i x_{i+1} | 1 \leq i \leq n-1\}$, respectively.

Theorem 3. *For $n \geq 2$,*

$$\operatorname{rvs}(P_n \odot P_2) = \begin{cases} \frac{2n+1}{3} + 1, & \text{if } n \equiv 1 \pmod 3, \\ \left\lceil \frac{2n+1}{3} \right\rceil, & \text{if } n \not\equiv 1 \pmod 3. \end{cases}$$

Proof. Let $P_n \odot P_2$ have $3n$ vertices and $4n-1$ edges. We can obtain a lower bound for $\operatorname{rvs}(P_n \odot P_2)$ using Theorem 1.

$$\operatorname{rvs}(P_n \odot P_2) \geq k = \begin{cases} \frac{2n+1}{3} + 1, & \text{if } n \equiv 1 \pmod 3, \\ \left\lceil \frac{2n+1}{3} \right\rceil, & \text{if } n \not\equiv 1 \pmod 3. \end{cases}$$

Furthermore, we show k is an upper bound of $\mathrm{rvs}(P_n \odot P_2)$ under a total k-labeling φ by distinguishing the following cases.

Case 1. $n = 2$.

$\varphi(x_i) = 2(i-1), \ \text{for } 1 \le i \le 2.$

$$\varphi(y_i^j) = \begin{cases} 0, & \text{if } i = 1,\ 1 \le j \le 2, \\ 2(j-1), & \text{if } i = 2,\ 1 \le j \le 2. \end{cases}$$

$$\varphi(x_i y_i^j) = \begin{cases} j, & \text{if } i = 1,\ 1 \le j \le 2, \\ 2, & \text{if } i = 2,\ 1 \le j \le 2. \end{cases}$$

$\varphi(y_i^1 y_i^2) = i, \ \text{for } 1 \le i \le 2.$

$\varphi(x_1 x_2) = 2.$

Case 2. $n \ge 3$.

$$\varphi(x_i) = \begin{cases} k-1, & \text{if } n \equiv 0 \pmod 3, \\ k, & \text{if } n \equiv 1, 2 \pmod 3. \end{cases}$$

$$\varphi(y_i^j) = \begin{cases} 0, & \text{if } 1 \le i \le k-1,\ 1 \le j \le 2, \\ k-1, & \text{if } k \le i \le n,\ 1 \le j \le 2,\ n \equiv 0 \pmod 3, \\ k, & \text{if } k \le i \le n,\ 1 \le j \le 2,\ n \equiv 1, 2 \pmod 3. \end{cases}$$

$$\varphi(x_i y_i^j) = \begin{cases} i-1+j, & \text{if } 1 \le i \le k-1,\ 1 \le j \le 2, \\ k-1, & \text{if } k \le i \le n,\ j = 1, \\ k, & \text{if } k \le i \le n,\ j = 2. \end{cases}$$

$$\varphi(y_i^1 y_i^2) = \begin{cases} i, & \text{if } 1 \le i \le k-1, \\ 2(i-k)+3, & \text{if } k \le i \le n,\ n \equiv 0 \pmod 3, \\ 2(i+1-k), & \text{if } k \le i \le n,\ n \equiv 1, 2 \pmod 3. \end{cases}$$

$$\varphi(x_i x_{i+1}) = \begin{cases} k-2-\frac{i-1}{2}, & \text{if } i \text{ is odd},\ 1 \le i \le k-2,\ n \equiv 0 \pmod 3, \\ k-3-\frac{i-1}{2}, & \text{if } i \text{ is odd},\ 1 \le i \le k-3,\ n \equiv 1, 2 \pmod 3, \\ k+1-\frac{i}{2}, & \text{if } i \text{ is even},\ 2 \le i \le k-1,\ n \equiv 0 \pmod 3, \text{ or} \\ & \quad i \text{ is even},\ 2 \le i \le k-2,\ n \equiv 1, 2 \pmod 3, \\ i+k-n+2, & \text{if } k \le i \le n-2,\ n \equiv 0 \pmod 3, \text{ or} \\ & \quad k-1 \le i \le n-2,\ n \equiv 1, 2 \pmod 3, \\ 3, & \text{if } i = n-1,\ n \equiv 0 \pmod 3, \\ 2, & \text{if } i = n-1,\ n \equiv 1, 2 \pmod 3. \end{cases}$$

Through the total k-labeling φ of $P_n \odot P_2$, we obtain that k is the maximum value of $\mathrm{rvs}(P_n \odot P_2)$. In the following, we create Table 3 to show that all vertices have distinct vertex weights, i.e., $\{2, 3, \ldots, 2n+2\} \cup \big\{\varphi(x_n) + \sum_{j=1}^{2} \varphi(x_n y_n^j) + \varphi(x_{n-1}x_n)\big\} \cup \big\{\varphi(x_i) + \sum_{j=1}^{2} \varphi(x_i y_i^j) + \varphi(x_i x_{i+1}) + \varphi(x_{i-1}x_i) \big| 2 \le i \le n-1\big\}$. Thus, the total k-labeling φ is a vertex irregular reflexive k-labeling of $P_n \odot P_2$ and $k = \mathrm{rvs}(P_n \odot P_2)$, where $n \ge 2$. This completes the proof.

Table 3. **A summary of distinct vertex weights of $P_n \odot P_2$ for $n \ge 2$.**

			$n=2$	$n \ge 3$	
				$n \equiv 0 \pmod 3$	$n \equiv 1, 2 \pmod 3$
$wt_\varphi(x_i)$	$i=1$	$1 \le j \le 2$	$2+\sum_{j=1}^{2} j$	$2k$	
	$i=2$		$4+\sum_{j=1}^{2} 2$		
	$2 \le i \le k-1$			$3k+i$	
	$2 \le i \le k-2$				$3k+i$
	$i=k$			$6k-n+1-\frac{k-1}{2}$	
	$i=k-1$				$6k-n+1-\frac{k-2}{2}$
	$k+1 \le i \le n-2$			$5k+2i-2n+1$	
	$k \le i \le n-2$				$5k+2i-2n+2$
	$i=n-1$			$4k+1$	
	$i=n$			$3k+1$	
$wt_\varphi(y_i^j)$	$i=1$	$1 \le j \le 2$	$j+1$		
	$i=2$		$2(j+1)$		
	$1 \le i \le k-1$			$2i-1+j$	
	$k \le i \le n$	$j=1$		$2i+1$	
		$j=2$		$2i+2$	

□

Example 5. Figure 3 shows the exact values of the reflexive vertex strength of $P_2 \odot P_2$ and $P_5 \odot P_2$.

The study of $\mathrm{rvs}(P_n \odot P_m)$ for $n \ge 2$ and $m \ge 3$ is still open for further extension of work.

Problem 1. Determine the exact value of $\mathrm{rvs}(P_n \odot P_m)$ for $n \ge 2$ and $m \ge 3$, while the lower bound of $\mathrm{rvs}(P_n \odot P_m)$ is shown below.

$$\mathrm{rvs}(P_n \odot P_m) \ge \begin{cases} \frac{nm+1}{4} + 1, & \text{if } n \equiv 1 \pmod 8,\ m \equiv 3 \pmod 8 \text{ or} \\ & n \equiv 3 \pmod 8,\ m \equiv 1 \pmod 8 \text{ or} \\ & n \equiv 5 \pmod 8,\ m \equiv 7 \pmod 8 \text{ or} \\ & n \equiv 7 \pmod 8,\ m \equiv 5 \pmod 8, \\ \left\lceil \frac{nm+1}{4} \right\rceil, & \text{otherwise.} \end{cases}$$

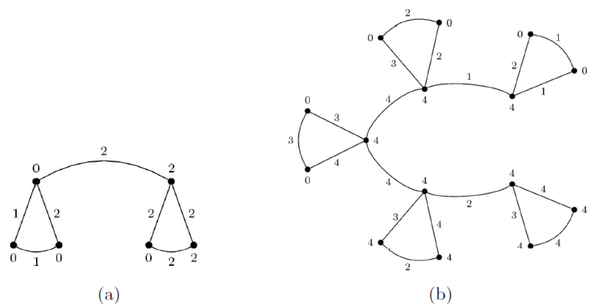


Figure 3. (a) A vertex irregular reflexive 2-labeling of $P_2 \odot P_2$, and (b) A vertex irregular reflexive 4-labeling of $P_5 \odot P_2$.

4.2. Corona product of a cycle with isolated vertices

The corona product of a cycle with m copies of isolated vertices is denoted by $C_n \odot mK_1$, where $n \geq 3$ and $m \geq 1$. When $m = 1$, the resulting graph is known as the sunlet graph S_n, which has previously been studied by Agustin et al. [3]. In the following study, we provide a proof for $\mathrm{rvs}(C_n \odot mK_1)$ for all $n \geq 3$ and $m \geq 1$. Additionally, the vertex set $V(C_n \odot mK_1) = \{x_i, y_i^j \mid 1 \leq i \leq n, 1 \leq j \leq m\}$ and an edge set $E(C_n \odot mK_1) = \{x_i y_i^j \mid 1 \leq i \leq n, 1 \leq j \leq m\} \cup \{x_i x_{i+1} \mid 1 \leq i \leq n-1\} \cup \{x_1 x_n\}$.

Theorem 4. *For $n \geq 3$ and m is any positive integer,*

$$\mathrm{rvs}(C_n \odot mK_1) = \begin{cases} \frac{nm}{2} + 1, & \textit{if } nm \equiv 2 \pmod 4 \\ \lceil \frac{nm}{2} \rceil, & \textit{if } nm \not\equiv 2 \pmod 4. \end{cases}$$

Proof. Let $C_n \odot mK_1$ have $n(1+m)$ vertices or edges. By Theorem 1,

$$\mathrm{rvs}(C_n \odot mK_1) \geq k = \begin{cases} \frac{nm}{2} + 1, & \text{if } nm \equiv 2 \pmod 4 \\ \lceil \frac{nm}{2} \rceil, & \text{if } nm \not\equiv 2 \pmod 4. \end{cases}$$

Therefore, we prove that k is also an upper bound for $\mathrm{rvs}(C_n \odot mK_1)$ under a total k-labeling φ by dividing m into even and odd cases.

Case 1. m is even.

$$\varphi(x_i) = \varphi(y_i^j) = \begin{cases} m(i-1), & \text{if } 1 \leq i \leq \lceil \frac{n}{2} \rceil,\ 1 \leq j \leq m, \\ k, & \text{if } \lceil \frac{n}{2} \rceil + 1 \leq i \leq n,\ 1 \leq j \leq m. \end{cases}$$

$$\varphi(x_i y_i^j) = \begin{cases} j, & \text{if } 1 \leq i \leq \lceil \frac{n}{2} \rceil,\ 1 \leq j \leq m, \\ m(i-1) - k + j, & \text{if } \lceil \frac{n}{2} \rceil + 1 \leq i \leq n,\ 1 \leq j \leq m. \end{cases}$$

$\varphi(x_1x_n) = k.$

$\varphi(x_ix_{i+1}) = k,\ \text{for } 1 \leq i \leq n-1.$

Evidently, k is the maximum value of rvs$(C_n \odot mK_1)$ under a total k-labeling φ. The vertex weights of all vertices in $C_n \odot mK_1$ for even m are summarized as follows. For $1 \leq j \leq m$,

(a) $wt_\varphi(x_i) = \{m(i-1) + \sum_{j=1}^{m-1}(j) + 2k \big| 1 \leq i \leq \lceil \frac{n}{2} \rceil\}$, and
$wt_\varphi(x_i) = \{3k + \sum_{j=1}^{m}[m(i-1) - k + j] \big| \lceil \frac{n}{2} \rceil + 1 \leq i \leq n\}$.

(b) $wt_\varphi(y_i^j) = \{m(i-1) + j | 1 \leq i \leq n\}$.

It is clear that all vertex weights of $C_n \odot mK_1$ are distinct integers in $\{1, 2, \ldots, nm\} \cup \{\sum_{j=1}^{m} \varphi(x_1y_1^j) + 2k\} \cup \{\varphi(x_i) + \sum_{j=1}^{m} \varphi(x_iy_i^j) + 2k \big| 2 \leq i \leq n-1\} \cup \{3k + \sum_{j=1}^{m} \varphi(x_ny_n^j)\}$.

Case 2. m is odd.
We distinguish the following situations.

Subcase 2.1. $n, m \equiv 3 \pmod 4$ or $n, m \equiv 1 \pmod 4$.

(a) $n \not\equiv 1 \pmod 4$ and $m = 1$.

$$\varphi(x_i) = \varphi(y_i^1) = \begin{cases} i-1, & \text{if } i \text{ is odd, } 1 \leq i \leq k, \\ i-2, & \text{if } i \text{ is even, } 1 \leq i \leq k, \\ k, & \text{if } k+1 \leq i \leq n. \end{cases}$$

$$\varphi(x_iy_i^1) = \begin{cases} 1, & \text{if } i \text{ is odd, } 1 \leq i \leq k, \\ 2, & \text{if } i \text{ is even, } 1 \leq i \leq k, \\ i-k, & \text{if } k+1 \leq i \leq n. \end{cases}$$

$\varphi(x_1x_n) = k.$

$\varphi(x_ix_{i+1}) = k,\ \text{for } 1 \leq i \leq n-1.$

(b) $n \equiv 1 \pmod 4$ and $m = 1$.

$$\varphi(x_i) = \varphi(y_i^1) = \begin{cases} i-1, & \text{if } i \text{ is odd, } 1 \leq i \leq k-1, \\ i-2, & \text{if } i \text{ is even, } 1 \leq i \leq k-1, \\ k-1, & \text{if } k \leq i \leq n. \end{cases}$$

$$\varphi(x_iy_i^1) = \begin{cases} 1, & \text{if } i \text{ is odd, } 1 \le i \le k-1, \\ 2, & \text{if } i \text{ is even, } 1 \le i \le k-1, \\ i-k+1, & \text{if } k \le i \le n. \end{cases}$$

$$\varphi(x_1x_n) = k.$$

$$\varphi(x_ix_{i+1}) = k, \text{ for } 1 \le i \le n-1.$$

(c) $n, m = 3$.

$$\varphi(x_i) = 0, \text{ for } 1 \le i \le 3.$$

$$\varphi(y_i^j) = 2(i-1), \text{ for } 1 \le i \le 3,\ 1 \le j \le 3.$$

$$\varphi(x_iy_i^j) = i-1+j, \text{ for } 1 \le i \le 3,\ 1 \le j \le 3.$$

$$\varphi(x_1x_3) = 3.$$

$$\varphi(x_ix_{i+1}) = 3, \text{ for } 1 \le i \le 2.$$

(d) otherwise.

$$\varphi(x_i) = 0, \text{ for } 1 \le i \le n.$$

$$\varphi(y_i^j) = \begin{cases} (i-1)(m-1), & \text{if } 1 \le i \le \lceil \frac{k}{m-1} \rceil,\ 1 \le j \le m, \\ k-1, & \text{if } \lceil \frac{k}{m-1} \rceil + 1 \le i \le n,\ 1 \le j \le m. \end{cases}$$

$$\varphi(x_iy_i^j) = \begin{cases} i-1+j, & \text{if } 1 \le i \le \lceil \frac{k}{m-1} \rceil,\ 1 \le j \le m, \\ m(i-1)-k+1+j, & \text{if } \lceil \frac{k}{m-1} \rceil + 1 \le i \le n,\ 1 \le j \le m. \end{cases}$$

$$\varphi(x_1x_n) = k-1.$$

$$\varphi(x_ix_{i+1}) = k-1, \text{ for } 1 \le i \le n-1.$$

Subcase 2.2. $n \not\equiv 3 \pmod 4$, $m \equiv 3 \pmod 4$ or $n \not\equiv 1 \pmod 4$, $m \equiv 5 \pmod 4$.

$\varphi(x_i) = 0$, for $1 \le i \le n$.

$$\varphi(y_i^j) = \begin{cases} (i-1)(m-1), & \text{if } 1 \le i \le \lceil \frac{k}{m-1} \rceil,\ 1 \le j \le m, \\ k, & \text{if } \lceil \frac{k}{m-1} \rceil + 1 \le i \le n,\ 1 \le j \le m. \end{cases}$$

$$\varphi(x_i y_i^j) = \begin{cases} i-1+j, & \text{if } 1 \le i \le \lceil \frac{k}{m-1} \rceil,\ 1 \le j \le m, \\ m(i-1)-k+j, & \text{if } \lceil \frac{k}{m-1} \rceil + 1 \le i \le n,\ 1 \le j \le m. \end{cases}$$

$\varphi(x_1 x_n) = k$.

$\varphi(x_i x_{i+1}) = k$, for $1 \le i \le n-1$.

We definitely prove that k is the maximum value of $\text{rvs}(C_n \odot mK_1)$ under the total k-labeling φ, where m is odd. Additionally, the following summarizes the distinct vertex weights of all the vertices.

(a) For $n, m \equiv 1 \pmod 4$ or $n, m \equiv 3 \pmod 4$.

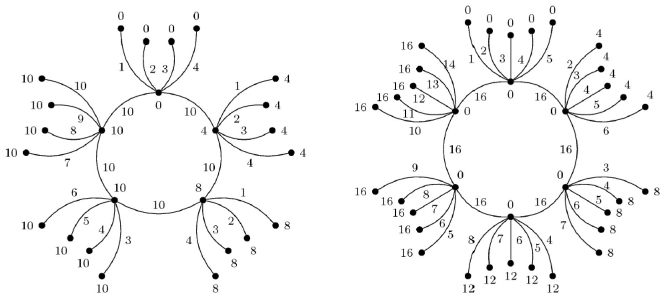
- (i) If $n \equiv 1, 3 \pmod 4$ and $m = 1$, $wt_\varphi(x_i) = \{i + 2k | 1 \le i \le n\}$. If $n, m = 3$, $wt_\varphi(x_i) = \big\{ \sum_{j=1}^{3}(i-1+j) + 8 \big| 1 \le i \le n, 1 \le j \le m \big\}$. If $n \ne 3$ and $m \ne 1, 3$, $wt_\varphi(x_i) = \big\{2(k-1) + \sum_{j=1}^{m}(i-1+j) \big| 1 \le i \le \lceil \frac{k}{m-1} \rceil, 1 \le j \le m \big\}$ and $wt_\varphi(x_i) = \big\{2(k-1) + \sum_{j=1}^{m}[m(i-1)-k+1+j] \big| \lceil \frac{k}{m-1} \rceil + 1 \le i \le n, 1 \le j \le m \big\}$.
- (ii) If $n \equiv 1, 3 \pmod 4$ and $m = 1$, $wt_\varphi(y_i^1) = \{i | 1 \le i \le n\}$. Otherwise, $wt_\varphi(y_i^j) = \{m(i-1) + j | 1 \le i \le n, 1 \le j \le m\}$.

(b) For $n \not\equiv 1 \pmod 4$, $m \equiv 5 \pmod 4$, or $n \not\equiv 3 \pmod 4$, $m \equiv 3 \pmod 4$.

- (i) $wt_\varphi(x_i) = \big\{2k + \sum_{j=1}^{m}(i-1+j) \big| 1 \le i \le \lceil \frac{k}{m-1} \rceil, 1 \le j \le m \big\}$ and $wt_\varphi(x_i) = \big\{2k + \sum_{j=1}^{m}[m(i-1)-k+j] \big| \lceil \frac{k}{m-1} \rceil + 1 \le i \le n, 1 \le j \le m \big\}$.
- (ii) $wt_\varphi(y_i^j) = \{m(i-1) + j | 1 \le i \le n, 1 \le j \le m\}$.

It clearly shows that all vertices have different vertex weights, which is $\{1, 2, \ldots, nm\} \cup \big\{ \sum_{j=1}^{m} \varphi(x_1 y_1^j) + \varphi(x_1 x_2) + \varphi(x_n x_1) \big\} \cup \big\{ \varphi(x_i) + \sum_{j=1}^{m} \varphi(x_i y_i^j) + \varphi(x_i x_{i+1}) + \varphi(x_{i-1} x_i) \big| 2 \le i \le n-1 \big\} \cup \big\{ \varphi(x_n) + \sum_{j=1}^{m} \varphi(x_n y_n^j) + \varphi(x_n x_1) + \varphi(x_{n-1} x_n) \big\}$. Thus, the total k-labeling φ is a vertex irregular reflexive k-labeling of $C_n \odot mK_1$ and k is the exact value of $\text{rvs}(C_n \odot mK_1)$, where $n \ge 3$ and $m \ge 1$. This theorem is proven. □

Example 6. Figure 4 depicts the exact values of the reflexive vertex strength of $C_5 \odot 4K_1$ and $C_6 \odot 5K_1$.



(a) (b)

Figure 4. (a) A vertex irregular reflexive 10-labeling of $C_5 \odot 4K_1$, and (b) A vertex irregular reflexive 16-labeling of $C_6 \odot 5K_1$.

5. Conclusion

In this paper, we proved a lower bound on the reflexive vertex strength of any graph. This bound is optimal when compared to previous lower bounds, as discussed in Section 2. Using this lower bound, we successfully determined $\mathrm{rvs}(f_{n,m})$ for $n \geq 3$ and $m \geq 2$, $\mathrm{rvs}(P_n \odot P_2)$ for $n \geq 2$, and $\mathrm{rvs}(C_n \odot mK_1)$ for $n \geq 3$ and $m \geq 1$. Moreover, this paper leads to some interesting open problems of a vertex irregular reflexive labeling on other graph products, such as the join graph of K_1 with star (denoted as $K_1 + K_{1,m}$), Cartesian product of two cycles (denoted as $C_n \square C_m$), and rooted product of two cycles (denoted as $C_n \circ C_m$). The lower bounds of the reflexive vertex strength of these graphs are given by referring to Theorem 1.

Problem 2. Determine the exact value of $\mathrm{rvs}(K_1 + K_{1,m})$ for $m \geq 3$, where

$$\mathrm{rvs}(K_1 + K_{1,m}) \geq \begin{cases} \frac{m+1}{3} + 1, & \text{if } m \equiv 2 \pmod 6, \\ \left\lceil \frac{m+1}{3} \right\rceil, & \text{if } m \not\equiv 2 \pmod 6. \end{cases}$$

Problem 3. Investigate the exact value of $\mathrm{rvs}(C_n \square C_m)$ for $n \geq 3$ and

(i) m is odd.

$$\mathrm{rvs}(C_n \square C_m) \geq \begin{cases} \frac{nm+3}{5} + 1, & \text{if } n \equiv 2 \pmod{10},\ m \equiv 1 \pmod{10} \text{ or} \\ & \quad n \equiv 4 \pmod{10},\ m \equiv 3 \pmod{10} \text{ or} \\ & \quad n \equiv 6 \pmod{10},\ m \equiv 7 \pmod{10} \text{ or} \\ & \quad n \equiv 8 \pmod{10},\ m \equiv 9 \pmod{10}, \\ \left\lceil \frac{nm+3}{5} \right\rceil, & \text{otherwise.} \end{cases}$$

(ii) m is even.

$$\mathrm{rvs}(C_n \square C_m) \geq \begin{cases} \frac{nm+3}{5}+1, & \text{if } n \equiv 1 \pmod 5,\ m \equiv 2 \pmod{10} \text{ or} \\ & n \equiv 2 \pmod 5,\ m \equiv 6 \pmod{10} \text{ or} \\ & n \equiv 3 \pmod 5,\ m \equiv 4 \pmod{10} \text{ or} \\ & n \equiv 4 \pmod 5,\ m \equiv 8 \pmod{10}, \\ \left\lceil \frac{nm+3}{5} \right\rceil, & \text{otherwise.} \end{cases}$$

Problem 4. Study the exact value of $\mathrm{rvs}(C_n \circ C_m)$ for $n,m \geq 3$, where

$$\mathrm{rvs}(C_n \circ C_m) \geq \begin{cases} \frac{n(m-1)+1}{3}+1, & \text{if } n \equiv 1 \pmod 3,\ m \equiv 3 \pmod 6 \text{ or} \\ & n \equiv 2 \pmod 3,\ m \equiv 5 \pmod 6 \text{ or} \\ & n \equiv 4 \pmod 6,\ m \equiv 0 \pmod 6 \text{ or} \\ & n, m \equiv 2 \pmod 6, \\ \left\lceil \frac{n(m-1)+1}{3} \right\rceil, & \text{otherwise.} \end{cases}$$

Acknowledgement: This research is supported by the Fundamental Research Grant Scheme (FRGS), Ministry of Higher Education, Malaysia with Reference Number FRGS/1/2020/STG06/UMT/02/1 (Grant Vot. 59609). The authors wish to thank the referee for his/her valuable comments which improved the paper.

Conflict of Interest: The authors declare that they have no conflict of interest.

Data Availability Statement: Data sharing is not applicable to this article as no datasets were generated or analyzed during the current study.

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