Communications in Combinatorics and Optimization
Vol. 9, No. 3 (2024), pp. 595-605
https://doi.org/10.22049/cco.2023.28145.1456

CCO
Commun. Comb. Optim.

Research Article

On the Roman domination polynomials

Seyed Hosein Mirhoseini† **and Nader Jafari Rad***

Department of Mathematics, Shahed University, Tehran, Iran
†seyedhosein.mirhoseini@shahed.ac.ir
*n.jafarirad@shahed.ac.ir

Received: 7 December 2022; Accepted: 7 May 2023
Published Online: 20 May 2023

Abstract: A Roman dominating function (RDF) on a graph G is a function $f : V(G) \to \{0, 1, 2\}$ satisfying the condition that every vertex u with $f(u) = 0$ is adjacent to at least one vertex v for which $f(v) = 2$. The weight of an RDF f is the sum of the weights of the vertices under f. The Roman domination number, $\gamma_R(G)$ of G is the minimum weight of an RDF in G. The Roman domination polynomial of a graph G of order n is the polynomial $RD(G, x) = \sum_{i=\gamma_R(G)}^{2n} d_R(G, i)x^i$, where $d_R(G, i)$ is the number of RDFs of G with weight i. In this paper we prove properties of Roman domination polynomials and determine $RD(G, x)$ in several classes of graphs G by new approaches. We also present bounds on the number of all Roman domination polynomials in a graph.

Keywords: Roman domination polynomial, Roman dominating function, Roman domination number

AMS Subject classification: 05C69

1. Introduction

For notations and definitions not given here we refer to [13]. We consider simple and finite graphs $G = (V, E)$, where $V = V(G)$ is the vertex set and $E = E(G)$ is the edge set. The *order* of G, denoted $|V(G)| = n$, is the number of vertices in G and the size of G, denoted $|E(G)| = m$, is the number of edges in G. For any two vertices $x, y \in V(G)$, x and y are *adjacent* if the edge $xy \in E(G)$. The *degree* of a vertex v, denoted by $\deg(v)$ (or $\deg_G(v)$), is the number of vertices adjacent to v. A vertex of degree zero is called an *isolated vertex*. We denote by Δ and δ, respectively, the *maximum degree* and *minimum degree* among the vertices of G. An *induced subgraph*

* *Corresponding Author*

of a graph G is a graph formed from a subset D of vertices of G and all of the edges in G connecting pairs of vertices in that subset, denoted by $\langle D\rangle$. An *independent set* is a set of vertices any two of which are not adjacent. A graph G is *bipartite* if $V(G)$ can be partitioned into two independent sets called *partite sets*. The *join* of two graphs G_1 and G_2, denoted by $G_1 \vee G_2$ is a graph with vertex set $V(G_1) \cup V(G_2)$ and edge set $E(G_1) \cup E(G_2) \cup \{uv | u \in V(G_1) \textit{ and } v \in V(G_2)\}$.

A *dominating set* of a graph G is a subset D of vertices such that every vertex outside D has a neighbor in D. The *domination number* of G, denoted by $\gamma(G)$, is the minimum cardinality amongst all dominating sets of G. Cockayne et al. [9] introduced the mathematical definition of Roman domination. This concept was subsequently developed very vastly, and to see the latest progress until 2020 we refer to [6–8]. A function $f : V \longrightarrow \{0, 1, 2\}$ is called a *Roman dominating function* or just an RDF for G if for every vertex $v \in V$ with $f(v) = 0$ there exists a vertex $u \in N(v)$ such that $f(u) = 2$. The *weight* of an RDF f is the sum $f(V) = \sum_{v\in V} f(v)$. The minimum weight of an RDF on G is called the *Roman domination number* of G and is denoted by $\gamma_R(G)$.

Graph polynomials play an important role in studying the structure of a graph, and there are some polynomials associated to graphs such as Chromatic polynomial, clique polynomial, characteristic polynomial and Tutte polynomial. Alikhani and Peng [4] introduced the concept of domination polynomials in graphs. This concept was further studied in [1, 3] and has been considered for some other types of dominating sets, for example, for total dominating sets ([2]), connected dominating sets ([14]) and hope dominating sets ([15]).

Gangabylaiah et al. [12] introduced the concept of Roman domination polynomial of a graph. For a graph G of order n with Roman domination number $\gamma_R(G)$, the Roman domination polynomial of a graph G, denoted $RD(G, x)$, is defined as follows

$$RD(G, x) = \sum_{i=\gamma_R(G)}^{2n} d_R(G, i)x^i,$$

where, $d_R(G, i)$ is the number of all Roman dominating functions on the graph G with weight i. They presented several basic properties and exact values of the Roman domination polynomial of a graph. This concept was further studied by Deepak et al. [10, 11].

In this paper we prove some further properties of Roman domination polynomial in graphs. We prove some previous results given in [11, 12] by new and easier approach. We also present bounds for the number of all RDFs of graph G.

We recall that the number of solutions of the equation $x_1 + x_2 + \cdots + x_n = r$, $x_i \in \mathbb{Z}^+$, is

$$\binom{r+n-1}{r} = \binom{r+n-1}{n-1}$$

(see e.g. [5]), and thus we have the following proposition:

Proposition 1. *The number of integral solutions of* $x_1+x_2+...+x_n = r$, $a \leq x_i \leq b$, *is*

$$\binom{r-na+n-1}{n-1} + \sum_{k=1}^{n}(-1)^k \binom{n}{k}\binom{r-na-k(b-a+1)+n-1}{n-1}.$$

We also make use of the following.

Proposition 2 ([9]). *For a path* P_n, $\gamma_R(P_n) = \lceil \frac{2n}{3} \rceil$.

2. Roman domination polynomial in join of graphs

Roman domination polynomial in join of two graphs was studied in [12]. In this section, we determine the Roman domination polynomial in join of two graphs by a new approach and then using it we determine the Roman domination polynomial in the complete and complete bipartite graphs. For this purpose, we first introduce some notations. For a graph G of order n, let:

- $D_R(G,k)$ stands for the set of all RDFs on the graph G with weight k, and let $d_R(G,k) = |D_R(G,k)|$.
- $D_{nR}(G,k)$ stands for the set of all functions $f : V(G) \to \{0,1,2\}$ on the graph G with weight k such that f is not an RDF, and let $d_{nR}(G,k) = |D_{nR}(G,k)|$.
- $D(G,k)$ stands for the set of all functions $f : V(G) \to \{0,1,2\}$ on the graph G with weight k, and let $d(G,k) = |D(G,k)|$.
- $P(G,x) = \sum_{i=0}^{2|V(G)|} d(G,i)x^i$.

Clearly, $d(G,k) = d_R(G,k) + d_{nR}(G,k)$. Furthermore, the following is easily verified.

Observation 1. If G_1 and G_2 are two graphs of order n_1 and n_2, respectively, then

$$P(G_1 \vee G_2, x) = P(G_1,x)P(G_2,x).$$

We now determine the Roman domination polynomial in join of two graphs.

Theorem 2. *If* G_1 *and* G_2 *are two connected graphs of order* n_1 *and* n_2, *respectively, then*

$$\begin{aligned} RD(G_1 \vee G_2, x) &= \sum_{p=1}^{n_1}\sum_{r=0}^{n_1-p}\sum_{q=1}^{n_2}\sum_{s=0}^{n_2-q}\binom{n_1}{p}\binom{n_1-p}{r}\binom{n_2}{q}\binom{n_2-q}{s}x^{2p+r+2q+s} \\ &+ RD(G_1,x)\sum_{i=0}^{n_2}\binom{n_2}{i}x^i - x^{n_1}\sum_{i=0}^{n_2-1}\binom{n_2}{i}x^i \\ &+ RD(G_2,x)\sum_{i=0}^{n_1}\binom{n_1}{i}x^i - x^{n_2}\sum_{i=0}^{n_1-1}\binom{n_1}{i}x^i - x^{n_1+n_2}. \end{aligned}$$

Proof. For an RDF f in a graph G, we denote by V_i the set of all vertices of G with label i under f. Thus an RDF f can be represented by a triplet (V_0, V_1, V_2), and we use the notation $f = (V_0, V_1, V_2)$. In order to enumerate the RDFs of the graph $G_1 \vee G_2$, for any RDF $f : V(G_1 \vee G_2) \to \{0, 1, 2\}$ put $p = |\{v : v \in V(G_1),\ f(v) = 2\}|$ and $q = |\{v : v \in V(G_2),\ f(v) = 2\}|$. Now we enumerate all RDFs on $G_1 \vee G_2$ by dividing them into the following types:

Type-1: RDFs $f = (V_0, V_1, V_2)$, where $V_2 = \emptyset$.
Note that there is only one Type-1 RDF assigning 1 to every vertex of $G_1 \vee G_2$. Thus we obtain the term $x^{n_1+n_2}$ of the Roman domination polynomial.

Type-2: RDFs $f = (V_0, V_1, V_2)$, where $V_2 \cap V(G_1) \neq \emptyset$ and $V_2 \cap V(G_2) = \emptyset$.
Observe that f is Type-2 RDF for $G_1 \vee G_2$ if and only if $f|_{V(G_1)}$ is an RDF for G_1. Note that a typical RDF of G_1 is a Type-2 RDF of $G_1 \vee G_2$ with exception that all the vertices of G_1 assigned value 1 and there is at least one vertex in G_2 with weight 0. Thus, we obtain the following terms of the Roman domination polynomial.

$$RD(G_1, x)\sum_{i=0}^{n_2}\binom{n_2}{i}x^i - x^{n_1}\sum_{i=0}^{n_2}\binom{n_2}{i}x^i = RD(G_1, x)\sum_{i=0}^{n_2}\binom{n_2}{i}x^i - x^{n_1}\sum_{i=0}^{n_2-1}\binom{n_2}{i}x^i - x^{n_1+n_2},$$

where i is the number of vertices of G_2 with weight one.

Type-3: RDFs $f = (V_0, V_1, V_2)$, where $V_2 \cap V(G_1) = \emptyset$ and $V_2 \cap V(G_2) \neq \emptyset$.
Similar to Type-2 RDFs, we find the following terms of the Roman domination polynomial.

$$RD(G_2, x)\sum_{i=0}^{n_1}\binom{n_1}{i}x^i - x^{n_2}\sum_{i=0}^{n_1}\binom{n_1}{i}x^i = RD(G_2, x)\sum_{i=0}^{n_1}\binom{n_1}{i}x^i - x^{n_2}\sum_{i=0}^{n_1-1}\binom{n_1}{i}x^i - x^{n_1+n_2},$$

where i is the number of vertices of G_1 with weight one.

Type-4: RDFs $f = (V_0, V_1, V_2)$, where $V_2 \cap V(G_1) \neq \emptyset$ and $V_2 \cap V(G_2) \neq \emptyset$.
We enumerate the number of Type-4 RDFs on $G_1 \vee G_2$ by summing all such RDFs that assign 2 to p vertices of G_1 and q vertices of G_2, where $1 \leq p \leq n_1$ and $1 \leq q \leq n_2$. For a fixed $p \in \{1, \ldots, n_1\}$ and fixed $q \in \{1, \ldots, n_2\}$, and a fixed Type-4 RDF f on $G_1 \vee G_2$, it may be possible that f assign 1 to some vertices of G_1 or G_2. We enumerate Type-4 RDFs on $G_1 \vee G_2$ assigning 2 to p vertices of G_1 and q vertices of G_2, by summing all such RDFs assigning 1 to r vertices of G_1 and s vertices of G_2, where $0 \leq r \leq n_1 - p$ and $0 \leq s \leq n_2 - q$.
There are $\binom{n_1}{p}\binom{n_1-p}{r}$ functions on G_1 such that p vertices are assigned 2 and r vertices are assigned 1. For each such choice, there are $\binom{n_2}{q}\binom{n_2-q}{s}$ functions on the graph G_2, such that q vertices are assigned 2 and s vertices are assigned 1. Thus we obtain the term

$$\sum_{p=1}^{n_1}\sum_{r=0}^{n_1-p}\sum_{q=1}^{n_2}\sum_{s=0}^{n_2-q}\binom{n_1}{p}\binom{n_1-p}{r}\binom{n_2}{q}\binom{n_2-q}{s}x^{2p+r+2q+s}.$$

Therefore

$$\begin{aligned} RD(G_1 \vee G_2, x) &= x^{n_1+n_2} + RD(G_1,x)\sum_{i=0}^{n_2}\binom{n_2}{i}x^i - x^{n_1}\sum_{i=0}^{n_2-1}\binom{n_2}{i}x^i - x^{n_1+n_2} \\ &+ RD(G_2,x)\sum_{i=0}^{n_1}\binom{n_1}{i}x^i - x^{n_2}\sum_{i=0}^{n_1-1}\binom{n_1}{i}x^i - x^{n_1+n_2} \\ &+ \sum_{p=1}^{n_1}\sum_{r=0}^{n_1-p}\sum_{q=1}^{n_2}\sum_{s=0}^{n_2-q}\binom{n_1}{p}\binom{n_1-p}{r}\binom{n_2}{q}\binom{n_2-q}{s}x^{2p+r+2q+s} \\ &= \sum_{p=1}^{n_1}\sum_{r=0}^{n_1-p}\sum_{q=1}^{n_2}\sum_{s=0}^{n_2-q}\binom{n_1}{p}\binom{n_1-p}{r}\binom{n_2}{q}\binom{n_2-q}{s}x^{2p+r+2q+s} \\ &+ RD(G_1,x)\sum_{i=0}^{n_2}\binom{n_2}{i}x^i - x^{n_1}\sum_{i=0}^{n_2-1}\binom{n_2}{i}x^i \\ &+ RD(G_2,x)\sum_{i=0}^{n_1}\binom{n_1}{i}x^i - x^{n_2}\sum_{i=0}^{n_1-1}\binom{n_1}{i}x^i - x^{n_1+n_2}. \end{aligned}$$

Thus the proof is complete. □

Since $RD(\overline{K_n}, x) = x^n(1+x)^n$ (see [12]), $K_n = \underbrace{K_1 \vee K_1 \vee \cdots \vee K_1}_{n\ times}$, and $K_{m,n} = \overline{K_m} \vee \overline{K_n}$, we obtain the following.

Corollary 1 ([12]).

$$\begin{aligned} 1.\ RD(K_n,x) &= (1+x+x^2)^n - (1+x)^n + x^n. \\ 2.\ RD(K_{m,n},x) &= x^{m+n} + (1+x)^{m+n}(x^m+x^n) - x^m(1+x)^n - x^n(1+x)^m \\ &+ \sum_{p=1}^{m}\sum_{r=0}^{m-p}\sum_{q=1}^{n}\sum_{s=0}^{n-q}\binom{m}{p}\binom{m-p}{r}\binom{n}{q}\binom{n-q}{s}x^{2p+r+2q+s}. \end{aligned}$$

3. Roman domination polynomial in Paths

The Roman domination polynomial for a path P_n is determined in [11] in which the authors employed the results of Alikhani and Peng [3] on the domination polynomials of graphs. In this section we determine the Roman domination polynomial for a path P_n without the need of domination polynomials. For this purpose, we need some notations. For any vertex v of a graph G, let:

- $d_k^0(G,v) = |\{f : V(G) \to \{0,1,2\} \mid f \textit{ is an RDF with } f(V) = k, f(v) = 0\}|$.
- $d_k^1(G,v) = |\{f : V(G) \to \{0,1,2\} \mid f \textit{ is an RDF with } f(V) = k, f(v) = 1\}|$.

- $d_k^2(G,v) = |\{f : V(G) \to \{0,1,2\} \mid f \textit{ is an RDF with } f(V) = k, f(v) = 2\}|$.

Then $d_k^0(G,v) + d_k^1(G,v) + d_k^2(G,v)$ is equal to the number of all RDFs with weight k. That is,

$$d_R(G,k) = d_k^0(G,v) + d_k^1(G,v) + d_k i^2(G,v).$$

So the coefficients of x^i in the polynomial $\sum_{i=\gamma_R(G)}^{2n} (d_i^0(G,v)+d_i^1(G,v)+d_i^2(G,v))x^i$, are independent of the choice of the vertex v, and so the Roman domination polynomial of graph G can be written as

$$RD(G,x) = \sum_{i=\gamma_R(G)}^{2n} (d_i^0(G,v) + d_i^1(G,v) + d_i^2(G,v))x^i,$$

where v is a vertex in $V(G)$.

Lemma 1. *Let $G = P_n$, $V(P_n) = \{v_1, v_2, \ldots, v_n\}$ and $E(P_n) = \{v_1v_2, v_2v_3, \ldots, v_{n-1}v_n\}$. Then*

$$\begin{aligned} d_i^0(P_n,v_1) &= d_i^2(P_{n-1},v_1), \\ d_i^1(P_n,v_1) &= d_{i-1}^0(P_{n-1},v_1) + d_{i-1}^1(P_{n-1},v_1) + d_{i-1}^2(P_{n-1},v_1), \\ d_i^2(P_n,v_1) &= d_{i-2}^0(P_{n-2},v_1) + d_{i-2}^1(P_{n-2},v_1) + d_{i-2}^2(P_{n-2},v_1) + d_{i-2}^1(P_{n-1},v_1) \\ &+ d_{i-2}^2(P_{n-1},v_1), \end{aligned}$$

where $P_{n-1} = G[\{v_1, v_2, \ldots, v_{n-1}\}]$ and $P_{n-2} = G[\{v_1, v_2, \ldots, v_{n-2}\}]$.

Proof. Let $Q_{n-1} = G[\{v_2, v_3, \ldots, v_n\}]$ and $R_{n-2} = G[\{v_3, v_4, \ldots, v_n\}]$. For any RDF f for P_n, if $f(v_1) = 0$ then $f(v_2) = 2$. So the number of RDFs on P_n with $f(v_1) = 0$ is equal to the number of RDFs on Q_{n-1} with $f(v_2) = 2$. Therefore $d_i^0(P_n,v_1) = d_i^2(Q_{n-1},v_2)$.
Assume that f is an RDF for P_n with weight i and $f(v_1) = 1$. Then $g = f|_{\{v_2,v_3,\ldots,v_n\}}$ is an RDF for Q_{n-1}, $g(V) = i-1$ and $g(v_2) \in \{0,1,2\}$. Conversely, suppose that g is an RDF for Q_{n-1} with weight $i-1$. Then the function $f : V(P_n) \to \{0,1,2\}$ defined by $f(v_1) = 1$ and $f(v_j) = g(v_j)$ for $2 \le j \le n$, is an RDF for P_n with weight i. Thus in this case, $d_i^1(P_n,v_1) = d_{i-1}^0(Q_{n-1},v_2) + d_{i-1}^1(Q_{n-1},v_2) + d_{i-1}^2(Q_{n-1},v_2)$.
Assume that f is an RDF for P_n with weight i and $f(v_1) = 2$. Assume that $f(v_2) = 0$. With a similar argument to that used for the calculation of $d_i^1(P_n,v_1)$, we obtain that $d_i^2(P_n,v_1) = d_{i-2}^0(R_{n-2},v_3) + d_{i-2}^1(R_{n-2},v_3) + d_{i-2}^2(R_{n-2},v_3)$. Next assume that $f(v_2) = 1$. It is evident that the restriction of g to $V(P_n) - \{v_1\}$ is an RDF for Q_{n-1}, $g(V) = i-2$ and $g(v_2) = 1$. Conversely, if g is an RDF for Q_{n-1} with weight $i-2$ and $g(v_2) = 1$, then $f(v_1) = 2$ and $f(v_j) = g(v_j)$ for $2 \le j \le n$, and so f is an RDF for P_n

with weight i. Thus in this case $d_i^2(P_n, v_1) = d_{i-2}^1(Q_{n-1}, v_2)$. Similarly, if $f(v_2) = 2$, then $d_i^2(P_n, v_1) = d_{i-2}^2(Q_{n-1}, v_2)$. We thus have the following:

$$\begin{aligned} d_i^2(P_n, v_1) &= d_{i-2}^0(R_{n-2}, v_3) + d_{i-2}^1(R_{n-2}, v_3) + d_{i-2}^2(R_{n-2}, v_3) \\ &+ d_{i-2}^1(Q_{n-1}, v_2) + d_{i-2}^2(Q_{n-1}, v_2). \end{aligned}$$

Now, let P_{n-1} and P_{n-2} be obtained by relabeling the vertices of Q_{n-1} and R_{n-2} as $V(P_{n-1}) = \{u_1 = v_2, u_2 = v_3, \ldots, u_{n-1} = v_n | v_i \in V(Q_{n-1}), i = 2, 3, \ldots, n\}$, and $V(P_{n-2}) = \{u_1 = v_3, u_2 = v_4, \ldots, u_{n-2} = v_n | v_i \in V(R_{n-2}), i = 3, 4, \ldots, n\}$. Then $d_i^0(P_n, v_1) = d_i^2(P_{n-1}, u_1)$, $d_i^1(P_n, v_1) = d_{i-1}^0(P_{n-1}, u_1) + d_{i-1}^1(P_{n-1}, u_1) + d_{i-1}^2(P_{n-1}, u_1)$ and $d_i^2(P_n, v_1) = d_{i-2}^0(P_{n-2}, u_1) + d_{i-2}^1(P_{n-2}, u_1) + d_{i-2}^2(P_{n-2}, u_1) + d_{i-2}^1(P_{n-1}, u_1) + d_{i-2}^2(P_{n-1}, u_1)$, as desired. □

Note that Lemma 1 expresses a recursive relation to obtain the values $d_i^0(P_n, v_1)$, $d_i^1(P_n, v_1)$ and $d_i^2(P_n, v_1)$. In a recursive relationship, it is essential to know the initial conditions as well as the end condition. By Proposition 2 we obtain the following.

Proposition 3. *If P_n is a path graph with n vertices then*

1) $d_{2n-2}^0(P_n, v_1) = 1$, $d_m^0(P_n, v_1) = 0$ for $m > 2n-2$ or $m < \lceil \frac{2n}{3} \rceil$.

2) $d_{2n-1}^1(P_n, v_1) = 1$, $d_m^1(P_n, v_1) = 0$ for $m > 2n-1$ or $m < \lceil \frac{2n}{3} \rceil$.

3) $d_{2n}^2(P_n, v_1) = 1$, $d_m^2(P_n, v_1) = 0$ for $m > 2n$ or $m < \lceil \frac{2n}{3} \rceil$.

Theorem 3. *For $n \geq 4$, let $G = P_n$ with $V(G) = \{v_1, v_2, \ldots, v_n\}$ and $E(G) = \{v_1v_2, v_2v_3, \ldots, v_{n-1}v_n\}$. Then,*

$$RD(P_n, x) = RD(P_1, x)RD(P_{n-1}, x) + x^2 RD(P_{n-2}, x) + xRD(P_1, x)RD(P_{n-3}, x),$$

with the initial values $RD(P_1, x) = x + x^2$, $RD(P_2, x) = 3x^2 + 2x^3 + x^4$, $RD(P_3, x) = x^2 + 5x^3 + 6x^4 + 3x^5 + x^6$.

Proof. It is evident that

$$\begin{aligned} RD(P_n, x) &= \sum_{i=\gamma_R(P_n)}^{2n} (d_i^0(P_n, v_1) + d_i^1(P_n, v_1) + d_i^2(P_n, v_1))x^i \\ &= \sum_{i=\lceil \frac{2n}{3} \rceil}^{2n} (d_i^0(P_n, v_1) + d_i^1(P_n, v_1) + d_i^2(P_n, v_1))x^i. \end{aligned}$$

Using Lemma 1, it can be written:

$$
\begin{aligned}
RD(P_n,x) &= \sum_{i=\lceil (2n)/3\rceil}^{2n} (d_i^2(P_{n-1},v_1)+d_{i-1}^0(P_{n-1},v_1)+d_{i-1}^1(P_{n-1},v_1)+d_{i-1}^2(P_{n-1},v_1)\\
&+ d_{i-2}^0(P_{n-2},v_1)+d_{i-2}^1(P_{n-2},v_1)+d_{i-2}^2(P_{n-2},v_1)+d_{i-2}^1(P_{n-1},v_1)\\
&+ d_{i-2}^2(P_{n-1},v_1))x^i\\
&= \sum_{i=\lceil (2n)/3\rceil}^{2n} (d_{i-1}^0(P_{n-1},v_1)+d_{i-1}^1(P_{n-1},v_1)+d_{i-1}^2(P_{n-1},v_1))x^i\\
&+ \sum_{i=\lceil (2n)/3\rceil}^{2n} (d_{i-2}^0(P_{n-2},v_1)+d_{i-2}^1(P_{n-2},v_1)+d_{i-2}^2(P_{n-2},v_1))x^i\\
&+ \sum_{i=\lceil (2n)/3\rceil}^{2n} (d_i^2(P_{n-1},v_1)+d_{i-2}^1(P_{n-1},v_1)+d_{i-2}^2(P_{n-1},v_1))x^i\\
&= xRD(P_{n-1},x)+x^2RD(P_{n-2},x)+\sum_{i=\lceil (2n)/3\rceil}^{2n} (d_i^2(P_{n-1},v_1)+d_{i-2}^1(P_{n-1},v_1)\\
&+ d_{i-2}^2(P_{n-1},v_1))x^i\\
&= xRD(P_{n-1},x)+x^2RD(P_{n-2},x)+\sum_{i=\lceil (2n)/3\rceil}^{2n} (d_i^2(P_{n-1},v_1)+d_{i-2}^1(P_{n-1},v_1)\\
&+ d_{i-2}^2(P_{n-1},v_1)+d_{i-2}^0(P_{n-1},v_1)-d_{i-2}^0(P_{n-1},v_1))x^i\\
&= xRD(P_{n-1},x)+x^2RD(P_{n-2},x)+x^2RD(P_{n-1},x)+\sum_{i=\lceil (2n)/3\rceil}^{2n} (d_i^2(P_{n-1},v_1)\\
&- d_{i-2}^0(P_{n-1},v_1))x^i\\
&= (x+x^2)RD(P_{n-1},x)+x^2RD(P_{n-2},x)+\sum_{i=\lceil (2n)/3\rceil}^{2n} (d_i^2(P_{n-1},v_1)\\
&- d_{i-2}^0(P_{n-1},v_1))x^i\\
&= RD(P_1,x)RD(P_{n-1},x)+x^2RD(P_{n-2},x)+\sum_{i=\lceil (2n)/3\rceil}^{2n} (d_{i-2}^0(P_{n-3},v_1)\\
&+ d_{i-2}^1(P_{n-3},v_1)+d_{i-2}^2(P_{n-3},v_1)+d_{i-2}^1(P_{n-2},v_1)+d_{i-2}^2(P_{n-2},v_1)\\
&- d_{i-2}^2(P_{n-2},v_1))x^i\\
&= RD(P_1,x)RD(P_{n-1},x)+x^2RD(P_{n-2},x)\\
&+ \sum_{i=\lceil (2n)/3\rceil}^{2n} (d_{i-2}^0(P_{n-3},v_1)+d_{i-2}^1(P_{n-3},v_1)+d_{i-2}^2(P_{n-3},v_1)+d_{i-2}^1(P_{n-2},v_1))x^i\\
&= RD(P_1,x)RD(P_{n-1},x)+x^2RD(P_{n-2},x)+x^2RD(P_{n-3},x)\\
&+ \sum_{i=\lceil \frac{2n}{3}\rceil}^{2n} (d_{i-3}^0(P_{n-3},v_1)+d_{i-3}^1(P_{n-3},v_1)+d_{i-3}^2(P_{n-3},v_1))x^i\\
&= RD(P_1,x)RD(P_{n-1},x)+x^2RD(P_{n-2},x)+x^2RD(P_{n-3},x)+x^3RD(P_{n-3},x)\\
&= RD(P_1,x)RD(P_{n-1},x)+x^2RD(P_{n-2},x)+(x^2+x^3)RD(P_{n-3},x)\\
&= RD(P_1,x)RD(P_{n-1},x)+x^2RD(P_{n-2},x)+xRD(P_1,x)RD(P_{n-3},x),
\end{aligned}
$$

as desired. □

4. A bound for the total number of Roman domination polynomial

In this section, we prove an upper bound as well as a lower bound for the number of all RDFs of a graph and characterize graphs achieving equality for the lower bound.

Theorem 4. *Let G be an arbitrary graph with n vertices. Then*

$$2^n \leq d_R(G) \leq 3^n - \sum_{k=0}^{n}(-1)^k \binom{n}{k}\binom{2n-2k-1}{n}.$$

Equality for the lower bound holds if and only if $G = \bar{K_n}$.

Proof. Any function that assigns the values 1 or 2 to the vertices of a graph is an RDF, and so the number of such functions is equal to 2^n. So $2^n \leq d_R(G)$. On the other hand, the total number of functions such as $f : V(G) \to \{0,1,2\}$ is equal to 3^n, as a result $d_R(G) \leq 3^n - |A|$, where $A = \{f : V(G) \to \{0,1,2\} | f \textit{ is not an RDF}\}$. It is enough to calculate the value of $|A|$. For every function $f : V(G) \to \{0,1,2\}$, let $x_1 = f(v_1), x_2 = f(v_2), \ldots, x_n = f(v_n)$. Then, $|A|$ is at least equal to the number of non-negative integer solutions of the inequality

$$x_1 + x_2 + \cdots + x_n \leq n-1,$$

where $0 \leq x_i \leq 1$, $0 \leq i \leq n$. The number of solutions of $x_1 + x_2 + \cdots + x_n \leq n-1$, $0 \leq x_i \leq 1$, $0 \leq i \leq n$ is

$$\binom{2n-1}{n} + \sum_{k=1}^{n}(-1)^k \binom{n}{k}\binom{2n-2k-1}{n} = \sum_{k=0}^{n}(-1)^k \binom{n}{k}\binom{2n-2k-1}{n}.$$

Thus $\sum_{k=0}^{n}(-1)^k \binom{n}{k}\binom{2n-2k-1}{n} \leq |A|$.
We next prove the equality part. Let $G = \bar{K_n}$ and f be an arbitrary RDF on G. Then $f(v) = 1$ or $f(v) = 2$ for each vertex v. The number of these functions is equal to 2^n. Conversely, suppose that G is an arbitrary graph with $d_R(G) = 2^n$. If $G \neq \bar{K_n}$, then G has at least one edge $v_t v_s$. Set $A = \{f : V(G) \to \{0,1,2\} \mid f(v_i) \geq 1,\ 1 \leq i \leq n\} \cup \{g : V(G) \to \{0,1,2\} \mid f(v_t) = 0, f(v_s) = 2, f(v_i) = 1, 1 \leq i \leq n, i \neq t, i \neq s\}$. Clearly, every member of A is an RDF for G and $|A| = 2^n + 1$. This contradicts the assumption $d_R(G) = 2^n$. This completes the proof. □

Note that the upper bound of Theorem 4 is sharp, as can be seen in the graph K_3.

Acknowledgements: We would like to thank the referees for their very careful evaluation and very useful comments.

Conflict of Interest: The authors declare that they have no conflict of interest.

Data Availability Statement: Data sharing is not applicable to this article as no datasets were generated or analyzed during the current study.

References

[1] S. Alikhani, *The domination polynomial of a graph at -1*, Graphs Comb. **29** (2013), no. 5, 1175–1181.
https://doi.org/10.1007/s00373-012-1211-x.

[2] S. Alikhani and N. Jafari, *Some new results on the total domination polynomial of a graph*, Ars Combin. **149** (2020), 185–197.

[3] S. Alikhani and Y.H. Peng, *Dominating sets and domination polynomials of paths*, Int. J. Math. Math. Sci. **2009** (2009), Article ID: 542040.
https://doi.org/10.1155/2009/542040.

[4] ———, *Introduction to domination polynomial of a graph*, Ars Combin. **114** (2014), 257–266.

[5] R.A. Brualdi, *Introductory Combinatorics*, Pearson India, 2019.

[6] M. Chellali, N. Jafari Rad, S.M. Sheikholeslami, and L. Volkmann, *Roman domination in graphs*, Topics in Domination in Graphs (T.W. Haynes, S.T. Hedetniemi, and M.A. Henning, eds.), Springer, Berlin/Heidelberg, 2020, p. 365–409.

[7] ———, *Varieties of Roman domination II*, AKCE Int. J. Graphs Comb. **17** (2020), no. 3, 966–984.
https://doi.org/10.1016/j.akcej.2019.12.001.

[8] ———, *Varieties of Roman domination*, Structures of Domination in Graphs (T.W. Haynes, S.T. Hedetniemi, and M.A. Henning, eds.), Springer, Berlin/Heidelberg, 2021, p. 273–307.

[9] E.J. Cockayne, P.A. Dreyer Jr, S.M. Hedetniemi, and S.T. Hedetniemi, *Roman domination in graphs*, Discrete Math. **278** (2004), no. 1-3, 11–22.
https://doi.org/10.1016/j.disc.2003.06.004.

[10] G. Deepak, A. Alwardi, and M.H. Indiramma, *Roman domination polynomial of cycles*, Int. J. Math. Combin. **3** (2021), 86–97.

[11] ———, *Roman domination polynomial of paths*, Palestine J. Math. **11** (2022), no. 2, 439–448.

[12] D. Gangabylaiah, M.H. Indiramma, N.D. Soner, and A. Alwardi, *On the Roman domination polynomial of graphs*, Bull. Int. Math. Virtual Inst. **11** (2021), no. 2, 355–365.

[13] T.W. Haynes, S.T. Hedetniemi, and P.J. Salter, *Fundamentals of Domination in*

Graphs, Marcel Dekker, New York, 1998.

[14] D.A. Mojdeh and A.S. Emadi, *Connected domination polynomial of graphs*, Fasc. Math, **60** (2018), no. 1, 103–121. https://doi.org/10.1515/fascmath-2018-0007.

[15] ______, *Hop domination polynomial of graphs*, J. Discrete Math. Sci. Crypt. **23** (2020), no. 4, 825–840. https://doi.org/10.1080/09720529.2019.1621493.