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Research Article

Independent Italian bondage of graphs

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Abstract: An independent Italian dominating function (IID-function) on a graph G is a function $f : V(G) \to \{0, 1, 2\}$ satisfying the conditions that (i) $\sum_{u \in N(v)} f(u) \geq 2$ when $f(v) = 0$, and (ii) the set of all vertices assigned non-zero values under f is independent. The weight of an IID-function is the sum of its function values over all vertices, and the independent Italian domination number $i_I(G)$ of G is the minimum weight of an IID-function on G. In this paper, we initiate the study of the independent Italian bondage number $b_{iI}(G)$ of a graph G having at least one component of order at least three, defined as the smallest size of a set of edges of G whose removal from G increases $i_I(G)$. We show that the decision problem associated with the independent Italian bondage problem is NP-hard for arbitrary graphs. Moreover, various upper bounds on $b_{iI}(G)$ are established as well as exact values on it for some special graphs. In particular, for trees T of order at least three, it is shown that $b_{iI}(T) \leq 2$.

Keywords: Independent Italian dominating function, independent Italian domination number, independent Italian bondage number

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1. Introduction

We consider simple graphs G with vertex set $V = V(G)$ and edge set $E = E(G)$. The *order* of G is $n = n(G) = |V|$. For a vertex x of V, let $N_G(x)$ denote the set of neighbors of x and let $N_G[x] = N_G(x) \cup \{x\}$. The *degree* of a vertex x is $d_G(x) = |N_G(x)|$. The *maximum degree* and *minimum degree* of G are denoted by $\Delta(G)$ and $\delta(G)$, respectively. When no confusion arises, we write N, d, δ and Δ instead of N_G, d_G, $\delta(G)$ and $\Delta(G)$, respectively. A *universal vertex* in a graph G is a vertex adjacent to all vertices of G. A *leaf* is a vertex of degree one while its neighbor is called a *support vertex*. If x is a support vertex, then we denote by $L(x)$ the set of leaves adjacent to x. For definitions and notations not given here we refer to [5].
As always, the *path* (*cycle*, *complete graph*, *complete bipartite graph*, respectively) of order n is denoted by P_n (C_n, K_n, $K_{p,q}$, respectively). A *tree* is a connected acyclic graph. A *star* of order n is the graph $K_{1,n-1}$. A tree T is a *double star* if it contains exactly two vertices that are not leaves. A double star with respectively p and q leaves attached at each support vertex is denoted by $DS_{p,q}$.
A set $S \subseteq V(G)$ is a *dominating set* if every vertex not in S has at least one neighbor in S. The *domination number* of G is the minimum cardinality of a dominating set of G. In 1990, Fink et al. [3] introduced the *bondage number* $b(G)$ to measure the vulnerability or the stability of the domination number in an interconnection network G under edge failure. The bondage number of a graph G has been defined in [3] as the minimum number of edges whose removal from G increases the domination number. Since then the concept of bondage has been widely studied for several graph parameters, for instance see [1, 6–8, 13].
The concept of Italian domination has been introduced in 2016 by Chellali et al. [2] as a new variation of Roman domination but called differently, namely Roman $\{2\}$-domination. An *Italian dominating function* (ID-function) on a graph G is a function $f : V \longrightarrow \{0, 1, 2\}$ having the property that $f(N[u]) \geq 2$ for each vertex u with $f(u) = 0$. The weight of an ID-function f is the sum $w(f) = \sum_{v \in V(G)} f(v)$, and the minimum weight of an ID-function of G is the *Italian domination number* $\gamma_I(G)$. Some variants of Italian domination have been studied, for instance see [9, 12].
An ID-function $f = (V_0, V_1, V_2)$ on a graph G is an *independent Italian dominating function* (IID-function) if the set $V_1 \cup V_2$ is independent, that is no two vertices in $V_1 \cup V_2$ are adjacent. The *independent Italian domination number* $i_I(G)$ is the minimum weight of an IID-function on G. Moreover, an IID-function of a graph G with minimum weight is called an $i_I(G)$-function. Independent Italian domination was first defined and studied in [11] by Rahmouni and Chellali.
In this paper, we initiate the study of the *independent Italian bondage number* $b_{iI}(G)$ of a graph G defined as the smallest set of edges $F \subseteq E(G)$ for which $i_I(G - F) > i_I(G)$. Note that since the independent Italian domination number of a connected graph of order two does not increase after the deletion of the unique edge, we will assume that $\Delta(G) \geq 2$. We also note that Moradi et al. [10] have initiated in 2020 the study of the Italian bondage number of a graph G denoted by $b_I(G)$.
We start our results by showing that the decision problem associated with the inde-

pendent Italian bondage number is NP-hard for general graphs. Then, we establish several upper bounds for $b_{iI}(G)$. In particular for trees T of order at least three it is shown that $b_{iI}(T) \leq 2$. Furthermore, exact values of the independent Italian bondage number are also given for some special graphs including paths, cycles and complete bipartite graphs.
We close this section by mentioning that every connected graph G of order at least two satisfies $i_I(G) \geq 2$. Extremal graphs attaining the bound are given by the following result whose proof is omitted because of its easiness.

Proposition 1. *Let G be a connected graph of order $n \geq 2$. Then $i_I(G) = \gamma_I(G) = 2$ if and only if $\Delta(G) = n-1$ or $\Delta(G) = n-2$ and there are two non-adjacent vertices of degree $n-2$.*

2. NP-hardness result

In this section, we will show that the problem of computing the independent Italian bondage number is NP-hard. We first state it as the following decision problem.

Independent Italian bondage number (IIB):

Instance: *A graph G and a positive integer k.*

Question: *Is* $b_{iI}(G) \leq k$?

We show that the NP-hardness of the IIB problem by transforming the 3-SAT problem to it in polynomial time. Recall that the 3-SAT problem specified below was proven to be NP-complete in [4].

3-satisfiability problem (3SAT):

Instance: A collection $\mathscr{C} = \{C_1, C_2, \ldots, C_m\}$ of clauses over a finite set U of variables such that $|Cj| = 3$ for $j = 1, 2, \ldots, m$.

Question: *Is there a truth assignment for U that satisfies all the clauses in $\mathscr{C}$?*

Theorem 1. *The IIB problem is NP-hard for general graphs.*

Proof. Let $U = \{u_1, u_2, \ldots, u_n\}$ and $\mathscr{C} = \{C_1, C_2, \ldots, C_m\}$ be an arbitrary instance of 3SAT. We will construct a graph G and choose a positive integer k such that $\mathscr{C}$ is satisfiable if and only if $b_{iI}(G) \leq k$. For each $i \in \{1, 2, \ldots, n\}$, we associate to each variable $u_i \in U$, a graph H_i obtained from a complete bipartite graph $K_{2,3}$ with bipartite sets $\{x_i, y_i, z_i\}$ and $\{u_i, \overline{u_i}\}$ by adding the edge $u_i\overline{u_i}$. For each $j \in$

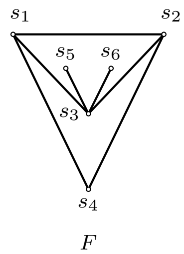

Figure 1. **Graph F**

$\{1, 2, \ldots, m\}$, we associate to the clause $C_j = \{p_j, q_j, r_j\} \in \mathscr{C}$, a single vertex c_j and we add the edge-set $E_j = \{c_j p_j, c_j q_j, c_j r_j\}$. Finally, add a graph F as depicted in Figure 1 by joining s_1 and s_2 to every vertex c_j. Clearly, G is a graph of order $5n + m + 6$, and therefore it can be constructed in polynomial time. An example of the constructed graph G when $U = \{u_1, u_2, u_3, u_4\}$ and $\mathscr{C} = \{C_1, C_2, C_3\}$, where $C_1 = \{u_1, u_2, \overline{u_3}\}, C_2 = \{\overline{u_1}, u_2, u_4\}, C_3 = \{\overline{u_2}, u_3, u_4\}$ is illustrated in Figure 2. Set $k = 1$, and let us show that $\mathscr{C}$ is satisfiable if and only if $b_{iI}(G) = 1$. For this aim, we need to show first the following claims.

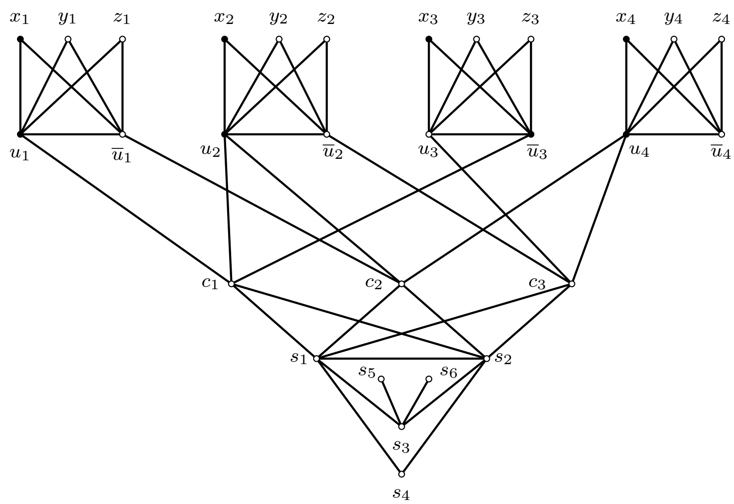

Figure 2. **NP-hardness for general graphs**

Claim 1. *$i_I(G) \geq 2n + 3$ and for any $i_I(G)$-function $f = (V_0, V_1, V_2)$, we have $f(V(H_i)) \geq 2$. Moreover, if $\gamma_{iI}(G) = 2n + 3$, then $f(V(H_i)) = 2, f(s_3) = 2, f(s_4) = 1$ and $f(s_1) = f(s_2) = f(s_5) = f(s_6) = 0$ and $|\{u_i, \overline{u_i}\} \cap V_2| = 1$ for each $i \in \{1, 2, \ldots, n\}$, while $\sum_{j=1}^{m} f(c_j) = 0$.*

Proof of Claim 1. Let $f = (V_0, V_1, V_2)$ be an $i_I(G)$-function. By the construction of G, we have $f(V(H_i)) \geq 2$ for each $i \in \{1, 2, \dots, n\}$. Moreover, one can easily see that $f(V(F)) + \sum_{j=1}^{m} f(c_j) \geq 3$, and therefore $i_I(G) \geq 2n + 3$.
Suppose that $i_I(G) = 2n+3$. Then $f(V(H_i)) = 2$ for each $i \in \{1, 2, \dots, n\}$. To Italian dominate the vertices x_i, y_i, z_i and noting that u_i and $\overline{u_i}$ are adjacent we must have $|\{u_i, \overline{u_i}\} \cap V_2| = 1$. Now, if $f(s_1) \geq 1$ (the case $f(s_2) \geq 1$ is similar), then to dominate other vertices in F we must have $f(V(F)) \geq 4$ which leads to the contradiction that $w(f) \geq 2n + 4$. Hence $f(s_1) = f(s_2) = 0$ and this implies that $f(s_4) \geq 1$ and $f(s_3) + f(s_5) + f(s_6) \geq 2$. Therefore $\sum_{j=1}^{m} f(c_j) = 0, f(s_3) = 2, f(s_4) = 1$ and $f(s_5) = f(s_6) = 0$. ♦

Claim 2. *$i_I(G) = 2n + 3$ if and only if $\mathscr{C}$ is satisfiable.*

Proof of Claim 2. Suppose that $i_I(G) = 2n + 3$ and let $f = (V_0, V_1, V_2)$ be an $i_I(G)$-function. By Claim 1, $|\{u_i, \overline{u_i}\} \cap V_2| = 1$ for each $i \in \{1, 2, \dots, n\}$. Also, $f(s_1) = f(s_2) = 0, \sum_{j=1}^{m} f(c_j) = 0, f(s_3) = 2, f(s_4) = 1$ and $f(s_5) = f(s_6) = 0$. Define a mapping $t : U \to \{T, F\}$ by

$$t(u_i) = \begin{cases} T & \text{if} \quad f(u_i) = 2, \\ F & \text{otherwise}, \end{cases} \tag{1}$$

for $i \in \{1, \dots, n\}$. We now show that t is a satisfying truth assignment for $\mathscr{C}$. It is sufficient to show that every clause in $\mathscr{C}$ is satisfied by t. Since the vertex c_j is not adjacent to any member of $\{s_3, s_4\} \cup \{x_i, y_i, z_i\}$, there exists some $i \in \{1, \dots, n\}$ such that $|N(c_j) \cap \{u_i, \overline{u_i}\}| = 1$ and one of u_i and $\overline{u_i}$ belongs to V_2. Now, if c_j is adjacent to u_i and $f(u_i) = 2$, then let $t(u_i) = T$, while if c_j is adjacent to $\overline{u_i}$ and $f(\overline{u_i}) = 2$, then $t(u_i) = F$ and so $t(\overline{u_i}) = T$ by (1). Hence, in either case the clause C_j is satisfied. The arbitrariness of $j \in \{1, \dots, m\}$ shows that all the clauses in $\mathscr{C}$ are satisfied by t, that is, $\mathscr{C}$ is satisfiable.
Conversely, suppose that $\mathscr{C}$ is satisfiable, and let $t : U \to \{T, F\}$ be a satisfying truth assignment for $\mathscr{C}$. We construct a subset D of vertices of G as follows. If $t(u_i) = T$, then put the vertex u_i in D; if $t(u_i) = F$, then put the vertex $\overline{u_i}$ in D. Hence $|D| = n$. Now define the function h by $h(x) = 2$ for every $x \in D$, $h(s_3) = 2, h(s_4) = 1$ and $h(y) = 0$ for any other vertex. Since t is a satisfying truth assignment for $\mathscr{C}$, the corresponding vertex c_j in G is adjacent to at least one vertex in D. One can easy check that h is IID-function of G of weight $2n + 3$ and so $i_I(G) \leq 2n + 3$. By Claim 1, $i_I(G) \geq 2n + 3$, and therefore $i_I(G) = 2n + 3$. ♦

Claim 3. *For any edge $e \in E(G), i_I(G - e) \leq 2n + 4$.*

Proof of Claim 3. Assume first that e is an edge belonging to $E(H) - \{u_i\overline{u_i}\}$, and since the edges of such a set play the same role, we take $e = x_iu_i$. Then the function

f defined by $f(\overline{u_i}) = 2$ for each $i \in \{1, \ldots, n\}$ and $f(s_2) = 2, f(s_5) = f(s_6) = 1$ and $f(y) = 0$ for the remaining vertices is an IID-function of $G - e$ of weight $2n + 4$. The same function f as defined previously remains valid when the edge e to be removed belongs to $\{s_1s_4, s_3s_5, s_3s_6, s_1s_3\}$ or e has an endvertex in $V(H_i) \cup \{s_1\}$ and the other endvertex some c_j. Assume now that $e = u_j\overline{u_j}$ for some j. Then the function f defined by $f(u_j) = f(\overline{u_j}) = 1$, $f(\overline{u_i}) = 2$ for each $i \in \{1, \ldots, n\} - \{j\}$, $f(s_2) = 2, f(s_5) = f(s_6) = 1$ and $f(y) = 0$ for the remaining vertices is an IID-function of $G-e$ of weight $2n+4$. If $e = s_1s_2$, then the function f defined by $f(\overline{u_i}) = 2$ for each i, and $f(s_1) = f(s_2) = f(s_5) = f(s_6) = 1$ and $f(y) = 0$ for any other vertex is an IID-function of $G - e$ of weight $2n + 4$. If $e = s_2s_4$ or $e = s_2c_j$ for some j, then the function f defined by $f(\overline{u_i}) = 2$ for each i, $f(s_1) = 2$ and $f(s_5) = f(s_6) = 1$ and $f(y) = 0$ for any other vertex is an IID-function of $G - e$ of weight $2n + 4$. In either case, we deduce that for every edge $e \in E(G), i_I(G - e) \leq 2n + 4$. ♦

Claim 4. *$i_I(G) = 2n + 3$ if and only if $b_{iI}(G) = 1$.*

Proof of Claim 4. Assume that $i_I(G) = 2n+3$ and take $e = s_3s_5$. Let $f = (V_0, V_1, V_2)$ be a $i_I(G - e)$-function. Clearly $f(s_5) = 1$, since the vertex s_5 is isolated. Also, $f(V(H_i)) \geq 2$ for each $i \in \{1, \ldots, n\}$, and thus the total weight for all H_i's is at least $2n$. Now, using the fact that $i_I(G) = 2n + 3$ and $f(s_5) = 1$ we deduce that the sum of the values assigned to the c_j's and s_i's except s_5 is 2, which is impossible. Therefore, we conclude that $i_I(G - e) > i_I(G)$, and thus $b_{iI}(G) = 1$.
Now assume that $b_{iI}(G) = 1$ and let e be an edge such that $i_I(G - e) > i_I(G)$. By Claim 1, we have $i_I(G) \geq 2n + 3$ while by Claim 3, we have $i_I(G - e) \leq 2n + 4$. Therefore $2n + 3 \leq i_I(G) < i_I(G - e) \leq 2n + 4$, which yields $2n + 3 = i_I(G)$. ♦
It follows from Claim 2 and Claim 4, that $b_{iI}(G) = 1$ if and only if $\mathscr{C}$ is satisfiable and the theorem follows. □

3. Exact values of $b_{iI}(G)$

In this section, we determine the independent Roman bondage number for some special graphs. We begin by recalling some useful results given in [10]. Moreover, we gather some results in the following proposition whose its proof is omitted.

Proposition 2. *1. $i_I(K_n) = \gamma_I(K_n) = 2$.*

2. $i_I(P_n) = \gamma_I(P_n) = \lceil \frac{n+1}{2} \rceil$.

3. For $n \geq 3$, $i_I(C_n) = \gamma_I(C_n) = \frac{n}{2}$ if n is even and $i_I(C_n) = \lceil \frac{n+2}{2} \rceil$ if n is odd.

4. If $G = K_{n_1,n_2,\ldots,n_t}$ is a complete t-partite graph with $t \geq 2$ such that $2 \leq n_1 < n_2 \leq n_3 \leq \ldots \leq n_t$, then $i_I(G) = n_1$.

5. If G is a connected graph of order at least three such that $i_I(G) = \gamma_I(G)$, then $b_{iI}(G) \leq b_I(G)$.

Proposition 3 ([10]). *Let G be a graph of order $n \geq 3$ with exactly t universal vertices and ℓ non-adjacent pair vertices of degree $n-2$ where $n > k + 2\ell$. Then*

$$b_I(G) \leq \begin{cases} \lfloor \frac{t}{2} \rfloor + \left\lfloor \frac{\lfloor \frac{t}{2} \rfloor + \ell}{2} \right\rfloor & \text{if both } t \text{ and } \lfloor \frac{t}{2} \rfloor + \ell \text{ are even,} \\ \lfloor \frac{t}{2} \rfloor + \left\lfloor \frac{\lfloor \frac{t}{2} \rfloor + \ell}{2} \right\rfloor + 1 & \text{otherwise.} \end{cases}$$

Proposition 4 ([10]). *For $n \geq 3$, $b_I(P_n) = 1$.*

Proposition 5 ([10]). *For $n \geq 3$, $b_I(C_n) = 1$ if $n \equiv 0 \pmod 2$.*

As an immediate consequence of Propositions 4 and 2-(5), we have the following.

Corollary 1. *For $n \geq 3$, $b_{iI}(P_n) = 1$.*

Proposition 6. *Let G be a graph of order $n \geq 3$ with exactly $t \geq 1$ universal vertices and ℓ non-adjacent pair vertices of degree $n-2$ where $n > k + 2\ell$ and $t \geq 1$ or $\ell \geq 2$. Then*

$$b_{iI}(G) \leq \begin{cases} \lfloor \frac{t}{2} \rfloor + \left\lfloor \frac{\lfloor \frac{t}{2} \rfloor + \ell}{2} \right\rfloor & \text{if both } t \text{ and } \lfloor \frac{t}{2} \rfloor + \ell \text{ are even,} \\ \lfloor \frac{t}{2} \rfloor + \left\lfloor \frac{\lfloor \frac{t}{2} \rfloor + \ell}{2} \right\rfloor + 1 & \text{otherwise.} \end{cases}$$

In particular, $b_{iI}(K_n) = \lceil \frac{n}{2} \rceil$ for $n \geq 3$.

Proof. Since $t \geq 1$ or $\ell \geq 2$, we have $i_I(G) = \gamma_I(G) = 2$, and thus by Propositions 2-(5) and 3 the desired bound follows. □

Proposition 7. *For $n \geq 3$,* $b_{iI}(C_n) = \begin{cases} 1 & \text{if } n \equiv 0 \pmod 2, \\ 2 & \text{if } n = 3, \\ 3 & \text{otherwise.} \end{cases}$

Proof. If n is even, then it follows from Propositions 2-(2,3,5) and 5 that $b_{iI}(C_n) = 1$. Hence we can assume that n is odd. Since the result is immediate for $n = 3$, suppose that $n \geq 5$. Let $C_n = v_1 v_2 \ldots v_n v_1$ and let G be obtained from C_n by deleting the edges $v_1 v_n, v_2 v_3, v_4 v_5$. Then $G = 2P_2 \cup P_{n-4}$ and we deduce from Propositions 2-(2) that $i_I(G) = 4 + \lceil \frac{n-3}{2} \rceil > i_I(C_n)$ and hence $b_{iI}(C_n) \leq 3$. To achieve the proof it is enough to show that $b_{iI}(C_n) \geq 3$ if n is odd and $n \geq 5$. Assume $C_n = v_1 v_2 \ldots v_n v_1$, and let e and e' be two arbitrary edges of C_n. Clearly, $C_n - \{e, e'\}$ is the union of two disjoint paths P and Q such that $n(P) + n(Q) = n$. Therefore $i_I(C_n - \{e, e'\}) = i_I(P) + i_I(Q)$. Without loss of generality, we may assume that $n(P)$ is even and $n(Q)$ is odd. It follows from Proposition 2-(2) that $i_I(C_n - \{e, e'\}) = i_I(P) + i_I(Q) = \lceil \frac{n(P)+1}{2} \rceil + \frac{n(Q)+1}{2} = \lceil \frac{n+2}{2} \rceil = i_I(C_n)$ which leads to $b_{iI}(C_n) \geq 3$, and hence $b_{iI}(C_n) = 3$. □

Proposition 8. *Let $G = K_{n_1,n_2,\ldots,n_t}$ be a complete t-partite graph with $t \geq 2$ such that $2 \leq n_1 < n_2 \leq n_3 \leq \ldots \leq n_t$. Then $b_{iI}(G) = n_1 - 1$.*

Proof. Let $X_1, X_2, \ldots, X_t$ be the partite sets of G with $|X_i| = n_i$ for each $i \in \{1, \ldots, t\}$, and let in particular $X_1 = \{u_1, u_2, \ldots, u_{n_1}\}$ and $X_2 = \{y_1, y_2, \ldots, y_{n_2}\}$. We note that the function h defined on $V(G)$ by $h(u_i) = 1$ for each $i \in \{1, 2, \ldots, n_1\}$ and $h(x) = 0$ for any other vertex of G is the unique $i_I(G)$-function. Let $F = \{u_i y_1 \mid 1 \leq i \leq n_1 - 1\}$, and let H be the spanning graph of G obtained from G by removing all edges of F. We claim that $i_I(H) = n_1 + 1 > i_I(G)$. To show this, let $f = (V_0, V_1, V_2)$ be an $i_I(H)$-function. We examine the possibilities according to whether $f(y_1) \in \{0, 1, 2\}$.
If $f(y_1) = 0$, then $f(u_{n_1}) = 2$ or $f(w) \geq 1$ for some vertex $w \in V(G) \setminus (X_1 \cup X_2)$. If $f(u_{n_1}) = 2$, then the condition that $V_1 \cup V_2$ is independent implies $V(G) - X_1 \subseteq V_0$ and so $\{u_1, u_2, \ldots, u_{n_1-1}\} \subseteq V_1 \cup V_2$ yielding $i_I(H) = \omega(f) \geq n_1 + 1 > i_I(G)$. Now assume that $f(w) \geq 1$ for some vertex $w \in V(G) \setminus (X_1 \cup X_2)$. If, without loss of generality, $w \in X_3$, then it follows that $f(x) \geq 1$ for every vertex $x \in X_3$ and thus $i_I(H) = \omega(f) = n_3 \geq n_1 + 1 > i_I(G)$.
If $f(y_1) = 1$, then $f(u_{n_1}) = 0$ and u_{n_1} needs another neighbor in $V_1 \cup V_2$, but since $V_1 \cup V_2$ is independent we deduce that $\{y_2, y_3, \ldots, y_{n_2}\} \subseteq V_1 \cup V_2$ and so $i_I(H) = \omega(f) \geq n_2 > i_I(G)$.
Finally, assume that $f(y_1) = 2$. Then $f(u_{n_1}) = 0$, and since $V_1 \cup V_2$ is independent, we must have either $\{y_2, y_3, \ldots, y_{n_2}\} \subseteq V_1 \cup V_2$ or $\{u_1, u_2 \ldots, u_{n_1-1}\} \subseteq V_1 \cup V_2$. In either case $i_I(H) = \omega(f) \geq n_1 + 1 > i_I(G)$. Therefore $b_{iI}(G) \leq n_1 - 1$.
To prove the inverse inequality, let $F \subseteq E(G)$ be an arbitrary subset of edges with $|F| < n_1 - 1$, and let H be the graph obtained from G by removing all edges of F. Then clearly $d_H(u_i) = d_H(u_j) = n_2 + n_3 + \ldots + n_t$ for some two distinct indices i and j, and hence the function g defined on $V(H)$ by $g(u_i) = 1$ for each $i \in \{1, \ldots, n_i\}$ and $g(x) = 0$ otherwise, is an IID-function of H of weight n_1, leading to $b_{iI}(G) \geq n_1 - 1$. Therefore $b_{iI}(G) = n_1 - 1$, and the proof is complete. □

4. Bounds on $b_{iI}(G)$

In this section, we first present an upper for the independent Italian bondage number for general graphs and then we show that the independent Italian bondage number of any tree with at least three vertices is at most two.

Theorem 2. Let G be a connected graph. If $x_1x_2x_3$ is a path of length 2 in G and G has no $i_I(G)$-function f assigning a 2 to some neighbor of x_i for each $i \in \{1, 2, 3\}$ simultaneously, then

$$b_{iI}(G) \leq d(x_1) + d(x_2) + d(x_3) - 3 - \ell(x_1, x_3),$$

where $\ell(x_1, x_3) = 1$ if $x_1x_3 \in E(G)$ and $\ell(x_1, x_3) = 0$ otherwise.

Proof. Let $E' \subseteq E(G)$ be the set of all edges incident with either x_1, x_2 or x_3 except the edge x_2x_3. Obviously, $|E'| = d(x_1) + d(x_2) + d(x_3) - 3$ when $x_1x_3 \notin E(G)$ and $|E'| = d(x_1) + d(x_2) + d(x_3) - 4$ when $x_1x_3 \in E(G)$. Let H be the graph obtained from G by removing all edges of E'. We claim that $i_I(H) > i_I(G)$, resulting in $b_{iI}(G) \leq |E'|$. Let $f = (V_0^f, V_1^f, V_2^f)$ be an $i_I(H)$-function. Since x_1 is isolated in H and x_2, x_3 induce a path on two vertices, we have $f(x_1) = 1$ and $f(x_2) = 2$ or $f(x_3) = 2$. Without loss of generality, assume that $f(x_2) = 2$ and $f(x_3) = 0$. If $d_G(x_2) = 2$ or $\sum_{x \in N_G(x_2)-\{x_1\}} f(x) = 0$, then the function h defined on $V(G)$ by $h(x_1) = h(x_3) = 0$ and $h(z) = f(z)$ for any other vertex z, is an IID-function of G of weight less than $i_I(H)$. Hence we assume $d_G(x_2) \geq 3$ and $\sum_{x \in N_G(x_2)-\{x_1\}} f(x) \geq 1$. We consider the following cases.

Case 1. $\sum_{x \in N_G(x_2)-\{x_1\}} f(x) \geq 2$.
If $\sum_{x \in N_G(x_1)-\{x_2\}} f(x) \geq 2$ and $\sum_{x \in N_G(x_3)-\{x_1,x_2\}} f(x) \geq 2$, then the function h defined by $f(x_1) = f(x_2) = f(x_3) = 0$ and $h(z) = f(z)$ for any other vertex z, is an IID-function of G of weight less than $i_I(H)$. If $\sum_{x \in N_G(x_1)-\{x_2\}} f(x) \geq 2$ and $\sum_{x \in N_G(x_3)-\{x_1,x_2\}} f(x) = 1$, then the function h defined by $h(x_1) = h(x_2) = h(x_3) = 0$, $h(w) = 2$ for some vertex $w \in V_1^f \cap (N_G(x_3) - \{x_1, x_2\})$ and $h(z) = f(z)$ for any other vertex z, is an IID-function of G of weight less than $i_I(H)$. If $\sum_{x \in N_G(x_1)-\{x_2\}} f(x) \geq 2$ and $\sum_{x \in N_G(x_3)-\{x_1,x_2\}} f(x) = 0$, then the function h defined by $h(x_1) = h(x_2) = 0$, $h(x_3) = 1$ and $h(z) = f(z)$ for any other vertex z, is an IID-function of G of weight less than $i_I(H)$.
If $\sum_{x \in N_G(x_1)-\{x_2\}} f(x) = 1$ and $\sum_{x \in N_G(x_3)-\{x_1,x_2\}} f(x) \geq 1$, then the function h defined by $h(x_1) = h(x_2) = h(x_3) = 0$, $h(u) = 2$ for some vertex $u \in V_1^f \cap N_G(x_1)$, $h(u') = \min\{2, f(u') + 1\}$ for some vertex $u' \in (V_1^f \cup V_2^f) \cap (N_G(x_3) - \{x_1, x_2\})$ and $h(z) = f(z)$ otherwise, is an IID-function of G of weight less than $i_I(H)$. If $\sum_{x \in N_G(x_1)-\{x_2\}} f(x) = 1$, $\sum_{x \in N_G(x_3)-\{x_1,x_2\}} f(x) = 0$ and $x_1x_3 \in E(G)$ then the function h defined by $h(x_1) = h(x_2) = 0, h(x_3) = 1$ and $h(z) = f(z)$ otherwise, is an IID-function of G of weight less than $i_I(H)$. If $\sum_{x \in N_H(x_1)} f(x) = 1$, $\sum_{x \in N_G(x_3)-\{x_1,x_2\}} f(x) = 0$ and $x_1x_3 \notin E(G)$, then the function h defined by $h(x_1) = h(x_2) = 0, h(x_3) = 1$, $h(u) = 2$ for some vertex $u \in V_1^f \cap N_G(x_1)$ and $h(z) = f(z)$ otherwise, is an IID-function of G of weight less than $i_I(H)$.
Based on the previous cases, we can assume now that $\sum_{x \in N_G(x_1)-\{x_2\}} f(x) = 0$.
If $\sum_{x \in N_G(x_3)-\{x_1,x_2\}} f(x) \geq 1$, then the function h defined by $h(x_2) = h(x_3) = 0$, $h(u') = \min\{2, f(u') + 1\}$ for some vertex $u' \in (V_1^f \cup V_2^f) \cap N_G(x_3) - \{x_1, x_2\}$ and $h(z) = f(z)$ otherwise, is an IID-function of G of weight less than $i_I(H)$. If $\sum_{x \in N_G(x_3)-\{x_1,x_2\}} f(x) = 0$ and $x_1x_2 \in E(G)$, then the function h defined by $h(x_1) = 2$, $h(x_2) = 0$ and $h(z) = h(z)$ for the remaining vertices, is an IID-function of G of weight less than $i_I(H)$. Finally, if $\sum_{x \in N_G(x_3)-\{x_1,x_2\}} f(x) = 0$ and $x_1x_2 \notin E(G)$, then the function h defined by $h(x_1) = h(x_3) = 1, h(x_2) = 0$ and $h(z) = h(z)$ for the remaining vertices, is an IID-function of G of weight less than $i_I(H)$. In any case considered above, we have shown that $i_I(H) > i_I(G)$.

Case 2. $\sum_{x \in N_G(x_2)-\{x_1\}} f(x) = 1$.

Assume that $y \in V_1^f \cap (N_G(x_2) - \{x_1\})$. If $\sum_{x\in N_G(x_1)-\{x_2\}} f(x) \geq 2$ and $\sum_{x\in N_G(x_3)-\{x_1,x_2\}} f(x) \geq 2$, then the function h defined by $h(y) = 2$, $f(x_1) = f(x_2) = 0$ and $h(z) = f(z)$ for any other vertex z, is an IID-function of G of weight less than $i_I(H)$. If $\sum_{x\in N_G(x_1)-\{x_2\}} f(x) \geq 2$ and $\sum_{x\in N_G(x_3)-\{x_1,x_2\}} f(x) = 1$, then the function h defined by $h(x_1) = h(x_2) = 0$, $h(y) = 2$, $h(w) = 2$ for some vertex $w \in (V_1^f - \{x_1\}) \cap N_G(x_3)$ and $h(z) = f(z)$ for any other vertex z, is an IID-function of G of weight less than $i_I(H)$. If $\sum_{x\in N_G(x_1)-\{x_2\}} f(x) \geq 2$ and $\sum_{x\in N_H(x_3)-\{x_1,x_2\}} f(x) = 0$, then the function h defined by $h(x_1) = h(x_2) = 0$, $h(x_3) = 1$ and $h(z) = f(z)$ for any other vertex z, is an IID-function of G of weight less than $i_I(H)$. From the above, we can assume now that $\sum_{x\in N_G(x_1)-\{x_2\}} f(x) \leq 1$. If $\sum_{x\in N_G(x_1)-\{x_2\}} f(x) = 1$ and $\sum_{x\in N_G(x_3)-\{x_1,x_2\}} f(x) \geq 2$, then the function h defined by $h(x_1) = h(x_2) = h(x_3) = 0$, $f(y) = 2$, $f(w) = 2$ for some vertex $w \in V_1^f \cap N_G(x_1)$ and $h(z) = f(z)$ otherwise, is an IID-function of G of weight less than $i_I(H)$. If $\sum_{x\in N_G(x_1)-\{x_2\}} f(x) = 1$ and $\sum_{x\in N_G(x_3)-\{x_1,x_2\}} f(x) = 1$, then the function h defined by $h(x_1) = h(x_2) = 0$, $h(y) = 2$, $h(u) = 2$ for some vertex $u \in V_1^f \cap N_G(x_1)$, $h(u') = 2$ for some vertex $u' \in V_1^f \cap N_G(x_3)$ and $h(z) = f(z)$ otherwise, is an IID-function of G of weight $\omega(f) = i_I(H)$ which assigns a 2 to some neighbor of x_i for each $i \in \{1,2,3\}$, resulting by assumption in $i_I(H) = \omega(h) > i_I(G)$. If $\sum_{x\in N_G(x_1)-\{x_2\}} f(x) = 1$, $\sum_{x\in N_G(x_3)-\{x_1,x_2\}} f(x) = 0$ and $x_1x_3 \in E(G)$, then the function h defined by $h(x_1) = h(x_2) = 0$, $h(x_3) = 1$ and $h(z) = f(z)$ otherwise, is an IID-function of G of weight less than $i_I(H)$. If $\sum_{x\in N_G(x_1)-\{x_2\}} f(x) = 1$, $\sum_{x\in N_G(x_3)-\{x_1,x_2\}} f(x) = 0$ and $x_1x_3 \notin E(G)$, then the function h defined by $h(x_1) = h(x_2) = 0$, $h(x_3) = 1$, $f(w) = 2$ for some vertex $w \in V_1^f \cap N_G(x_1)$ and $h(z) = f(z)$ otherwise, is an IID-function of G of weight less than $i_I(H)$. Hence we assume that $\sum_{x\in N_G(x_1)-\{x_2\}} f(x) = 0$.

If $\sum_{x\in N_G(x_1)-\{x_2\}} f(x) = 0$, $\sum_{x\in N_G(x_3)-\{x_1,x_2\}} f(x) \geq 1$ and $x_1x_3 \in E(G)$, then the function h defined by $h(x_2) = h(x_3) = 0$ and $h(z) = f(z)$ otherwise, is an IID-function of G of weight less than $i_I(H)$. If $\sum_{x\in N_G(x_1)-\{x_2\}} f(x) = 0$, $\sum_{x\in N_G(x_3)-\{x_1,x_2\}} f(x) \geq 1$ and $x_1x_3 \notin E(G)$, then the function h defined by $h(x_2) = h(x_3) = 0$, $h(w) = \min\{2, f(w)+1\}$ for some $w \in (V_1^f \cup V_2^f) \cap N_G(x_3) - \{x_2\}$ and $h(z) = f(z)$ otherwise, is an IID-function of G of weight less than $i_I(H)$.

If $\sum_{x\in N_H(x_1)} f(x) = 0$, $\sum_{x\in N_H(x_3)-\{x_1,x_2\}} f(x) = 0$ and $x_1x_3 \notin E(G)$, then the function h defined by $h(x_2) = 0$, $h(x_1) = h(x_3) = 1$, and $h(z) = f(z)$ otherwise, is an IID-function of G of weight less than $i_I(H)$. If $\sum_{x\in N_H(x_1)} f(x) = 0$, $\sum_{x\in N_H(x_3)-\{x_1,x_2\}} f(x) = 0$ and $x_1x_2 \in E(G)$, then the function h defined by $h(x_2) = 0$, $h(x_1) = 2$, and $h(z) = f(z)$ otherwise, is an IID-function of G of weight less than $i_I(H)$. □

Let H be a bipartite graph with bipartite sets $X = \{y_1, y_2, y_3\}$ and $Y = \{u_i, v_i, w_i \mid 1 \leq i \leq m\}$ ($m \geq 5$) and edge set $E(G) = \{y_1u_i, y_2u_i, y_1v_i, y_3v_i, y_2w_i, y_3w_i \mid 1 \leq i \leq m\}$, and let G be the graph obtained from H by adding a path $x_1x_2x_3$ and adding the disjoint edges x_iy_i for $i \in \{1,2,3\}$. Clearly, the function f defined by $f(y_1) = f(y_2) = f(y_3) = 2$ and $f(z) = 0$ otherwise, is the unique $i_I(G)$-function. Let $F = \{x_1x_2, x_1y_1, x_2y_2, x_3y_3\}$ and define the function g on $G - F$ by $g(x_1) = 1$,

$g(x_2) = 2$, $g(y_i) = 1$ for $i \in \{1, 2, 3\}$, is an IID-function of $G - F$ of weight $i_I(G)$. This example shows that the condition that G has no $i_I(G)$-function f assigning a 2 to some neighbor of x_i for each $i \in \{1, 2, 3\}$ simultaneously, is necessary.
Theorem 2 and its proof result in the following corollaries.

Corollary 2. Let G be a connected graph. If $x_1x_2x_3$ is a path of length 2 in G with $d_G(x_1) = 1$, then $b_{iI}(G) \le d_G(x_2) + d_G(x_3) - 2$.

Proof. Let f be a $i_I(G)$-function. If $f(x_2) \ge 1$, then we must have $f(x) = 0$ for each $x \in N(x_2)$, that is f does not assign 2 to no neighbor of x_2. On the other hand, if $f(x_2) = 0$, then f does not assign 2 to no neighbor of x_1. Hence G satisfies the condition specified in the statement of Theorem 2, and consequently, $b_{iI}(G) \le d_G(x_1) + d_G(x_2) + d_G(x_3) - 3 = d_G(x_2) + d_G(x_3) - 2$ and the proof is complete. □

Corollary 3. Let G be a connected graph. If $x_1x_2x_3$ is a path of length 2 in G with $d_G(x_2) = 2$, then $b_{iI}(G) \le 2\Delta(G) - 1$.

Restricted to the class of trees of order at least three, we will show that the independent Italian bondage number is at most two. We note that such an upper bound has been also proved for the Italian bondage number by Moradi et al. [10]. In the proof we give, several cases are considered and discussed. But before presenting this proof, we give some additional definitions and notations. A *path* joining two vertices x and y is called a (x, y)-path. The *diameter* of a connected graph G, denoted $\mathrm{diam}(G)$, is the length of the shortest path between the most distanced vertices. A *diametral path* of a graph G is a shortest path whose length is equal to $\mathrm{diam}(G)$. We are also considering *rooted trees* distinguished by one vertex r called the *root*. For a vertex $v \neq r$ in a rooted tree T, the *parent* of v is the neighbor of v on the unique (r, v)-path, while a *child* of v is any other neighbor of v. A *descendant* of v is a vertex $w \neq v$ such that the unique (r, w)-path contains v. The set of children of a vertex v is denoted by $C(v)$ while $D(v)$ denote the set of its descendants. The *maximal subtree* at v denoted by T_v is the subtree of T induced by v and all its descendants. The *depth* of v is the largest distance from v to a descendant of v.

Theorem 3. If T is a tree of order $n \ge 3$, then

$$b_{iI}(T) \le 2.$$

Furthermore, this bound is sharp for the double star $DS_{p,p}$ for $p \ge 2$.

Proof. Obviously $\mathrm{diam}(T) \ge 2$, since $n \ge 3$. If $\mathrm{diam}(T) = 2$, then T is a star and for any edge e of T we have $i_I(T - e) = 3 > i_I(T) = 2$ leading to $b_{iI}(T) = 1$. Assume now that $\operatorname{diam}(T) = 3$. Then T is a double star $DS_{p,q}$ for some integers $q \geq p \geq 1$. Let x, y be the support vertices of the double star, and let x', y' be the leaf neighbors of x and y, respectively. If $p = 1$, then $i_I(T - xy) = 4 > i_I(T) = 3$ and hence $b_{iI}(T) = 1$. Thus let $p \geq 2$. Then removing edges $x'x$ and $y'y$ provides a forest F with three components consisting of the two single vertices and a double star $DS_{p-1,q-1}$. In this case, $i_I(F) = 2 + (2 + (p - 1)) = 3 + p > i_I(T) = 2 + p$ yielding $b_{iI}(T) \leq 2$. In the sequel, we can assume that $\operatorname{diam}(T) \geq 4$. Let $x_1x_2 \ldots x_k$ ($k \geq 5$) be a diametral path in T chosen so that (i) $d_T(x_2)$ is as large as possible, and (ii) subject to (i) $d_T(x_3)$ is maximized. We root T at x_k.

If $d_T(x_3) = 2$, then let F be the forest obtained from T by removing edges x_3x_4 and x_3x_2. Clearly any $i_I(F)$-function f such that $f(x_2)$ is maximized, assigns 1 to x_3 and 2 to x_2, and thus the function h defined on $V(T)$ by $h(x_3) = 0$ and $h(x) = f(x)$ otherwise, is an IID-function of T of weight less than $\omega(f)$ leading to $b_{iI}(T) \leq 2$. Therefore we can assume that $d_T(x_3) \geq 3$ and by similarity, every child of x_4 with depth 2 has degree at least 3. Let $N_T(x_3) - \{x_2, x_4\} = \{y_1, y_2, \ldots, y_t\}$. We proceed with the following cases.

Case 1. x_3 has a child y with depth 1 and degree 3.

Let z_1 and z_2 be the leaf neighbors of y and let F be the forest obtained from T by removing the edges yz_1 and x_3y. Note that z_1 is isolated in F and the vertices z_2 and y induce a P_2 component in F. If f is an $i_I(F)$-function, then we have $f(z_1) = 1$ and either $f(z_2) = 2$ or $f(y) = 2$, say $f(z_2) = 2$ and thus $f(y) = 0$. In this case, the function g defined on $V(T)$ by $g(z_2) = 1$ and $g(x) = f(x)$ otherwise, is an IID-function of T of weight less than $\omega(f)$, leading to $b_{iI}(T) \leq 2$.

Thus we may assume that x_3 has no child with depth 1 and degree 3. In particular, $d_T(x_2) \neq 3$.

Case 2. $d_T(x_2) \geq 4$ and x_3 has a child of degree 2.

Without loss of generality, let y_1 be a child of x_3 with degree two and let y_1' be the leaf neighbor of y_1. Let F be the forest obtained from T by removing the edges x_2x_1 and x_3y_1. As in Case 1, if f is an $i_I(F)$-function such that $f(x_2)$ is as large as possible, then $f(x_1) = 1$ and either $f(y_1) = 2$ or $f(y_1') = 2$, say $f(y_1') = 2$ and thus $f(y_1) = 0$. Now, if $f(x_3) \geq 1$, then the function g defined on $V(T)$ by $g(y_1') = 1$ and $g(x) = f(x)$ otherwise, is an IID-function of T of weight less than $\omega(f)$. Hence let $f(x_3) = 0$. Since x_2 has at least two leaf neighbors in F, by the choice of f, we have $f(x_2) = 2$ and thus the function g defined on $V(T)$ by $g(x_1) = 0$ and $g(x) = f(x)$ otherwise, is an IID-function of T of weight less than $\omega(f)$. In either case, $b_{iI}(T) \leq 2$.

Case 3. $d_T(x_2) \geq 4$ and x_3 is a support vertex.

According to Cases 1 and 2 we may assume that each child of x_3 is a leaf or has degree at least four. Without loss of generality, let y_1 be a leaf neighbor of x_3, and let F be the forest obtained from T by removing edges x_1x_2 and x_3y_1. Let f be an $i_I(F)$-function. Clearly, $f(x_1) = f(y_1) = 1$. Now, if $f(x_2) = 2$, then $f(x_3) = 0$ and the function g defined on $V(T)$ by $g(x_1) = 0$ and $g(x) = f(x)$ for the remaining vertices, is an IID-function of T of weight less than $\omega(f)$. If $f(x_3) = 2$, then $f(x_2) = 0$ and the function g defined on $V(T)$ by $g(y_1) = 0$ and $g(x) = f(x)$ otherwise, is an

IID-function of T of weight less than $\omega(f)$. Hence we can assume that $f(x_2) \leq 1$ and $f(x_3) \leq 1$. Then all leaves adjacent to x_2 in F must be assigned a 1 under f and thus $f(x_2) = 0$. Recall that $d_T(x_2) \geq 4$. Now, if $f(x_3) = 0$, then the function g defined on $V(T)$ by $g(x) = 0$ for $x \in N_T(x_2)$, $g(x_2) = 2$ and $g(x) = f(x)$ for the remaining vertices, is an IID-function of T of weight less than $\omega(f)$. Thus we can assume that $f(x_3) = 1$. Then $f(x_4) = 0$ and y_1 is the unique leaf adjacent to x_3. Thus x_4 has a neighbor $w \neq x_3$ with positive weight. In this case, the function g defined on $V(T)$ by $g(w) = 2$, $g(x_3) = 0$, $g(x_2) = 2$, $g(x) = 0$ for $x \in N_T(x_2) - \{x_3\}$ and $g(x) = f(x)$ for the remaining vertices, is an IID-function of T of weight less than $\omega(f)$. All the situations examined lead to $b_{iI}(T) \leq 2$.
Taking into account the above three cases, we conclude that if $d_T(x_2) \geq 4$, then that each child of x_3 has degree at least four.

Case 4. $d_T(x_2) \geq 4$ and every child of x_3 has degree at least 4.
By the assumption, every y_i has at least three leaf neighbors, say y_i^1, y_i^2, y_i^3. Let F be the forest obtained from T by removing the edges x_2x_1 and $y_1y_1^1$ and let $f = (V_0, V_1, V_2)$ be an $i_I(F)$-function F such that $f(x_2) + f(y_1)$ is as large as possible. Since x_1 and y_1^1 are isolated in F, $f(x_1) = f(y_1^1) = 1$. Note that since x_2 has at least two leaf neighbors in F, then $f(x_2) \neq 1$, and likewise $f(y_1) \neq 1$. Now, if $f(x_2) = f(y_1) = 2$, then the function g defined on $V(T)$ by $g(x_1) = g(y_1^1) = 0$ and $g(x) = f(x)$ otherwise, is an IID-function of T of $\omega(f) - 2$. Also, if $f(x_2) = 2$ and $f(y_1) = 0$ (the case $f(x_2) = 0$ and $f(y_1) = 2$ is similar), then the function g defined on $V(T)$ by $g(x_1) = 0$ and $g(x) = f(x)$ otherwise, is an IID-function of T of weight less than $\omega(f) - 1$. Hence we can assume that $f(x_2) = f(y_1) = 0$. The choice of f implies that $f(x_3) \in \{1, 2\}$, and thus $f(y_i) = 0$ for each i, $N_T(x_2) - \{x_3\} \subseteq V_1$ and $N_T(y_i) - \{x_3\} \subseteq V_1$ for each $i \in \{1, \ldots, t\}$. If $(N_T(x_4) - \{x_3\}) \cap V_2 \neq \emptyset$, then the function g defined on $V(T)$ by $g(x_3) = 0$, $g(x_2) = g(y_i) = 2$ for all $i \in \{1, \ldots, t\}$, $g(x) = 0$ for $x \in L(v_2) \cup (\cup_{i=1}^t L(y_i))$ and $g(x) = f(x)$ otherwise, is an IID-function of T of weight less than $\omega(f)$. Therefore we may assume that $(N_T(x_4) - \{x_3\}) \cap V_2 = \emptyset$. If $(N_T(x_4) - \{x_3\}) \cap V_1 \neq \emptyset$, then pick a vertex $w \in (N_T(x_4) - \{x_3\}) \cap V_1$, and consider the function g defined on $V(T)$ by $g(w) = 2$, $g(x_3) = 0$, $g(x_2) = g(y_i) = 2$ for all $i \in \{1, \ldots, t\}$, $g(x) = 0$ for $x \in L(x_2) \cup (\cup_{i=1}^t L(y_i))$ and $g(x) = f(x)$ otherwise. One can see that g is an IID-function of T of weight less than $\omega(f)$. Hence we can assume that $(N(x_4) - \{x_3\}) \cap (V_1 \cup V_2) = \emptyset$. Again one can define an IID-function g on $V(T)$ of weight less than $\omega(f)$ by $g(x_4) = 1$, $g(x_3) = 0$, $g(x_2) = g(y_i) = 2$ for all $i \in \{1, \ldots, t\}$, $g(x) = 0$ for $x \in L(x_2) \cup (\cup_{i=1}^t L(y_i))$ and $g(x) = f(x)$ otherwise. All the situations that have been considered show that $i_I(T) < i_I(F)$ and thus $b_{iI}(T) \leq 2$.

Case 5. $d_T(x_2) = 2$.
It follows from the choice of the diametral path that each child of v_3 with depth one has degree 2. Thus the maximal subtree rooted at x_3 is a spider, that is a tree obtained from a star of order at least three by subdividing at least one of its edges. This remains valid for every maximal subtree rooted at any child of x_4 with depth 2 is a spider. Let us examine the following situations.

Subcase 5.1. Assume first that $d_T(x_3) = 3$ and x_3 has a leaf child.

Let w be the leaf child of x_3 and let F be the forest obtained from T by removing the edges x_3x_2, x_3x_4. Let f be an $i_I(F)$-function such that $f(x_2)+f(w)$ is maximized. Since x_3 and w as well as x_1 and x_2 induce a P_2 component of F, by the choice of f we have $f(x_2)=f(w)=2$. In this case, the function g defined on $V(T)$ by $g(w)=1$ and $g(x)=f(x)$ otherwise, is an IID-function of T of weight $i_I(F)-1$, implying that $b_{iI}(T)\leq 2$.

By Subcase 5.1, we may assume that $d_T(x_3)\geq 4$ or $d_T(x_3)=3$ and x_3 has two children with depth 1 and degree 2.

Subcase 5.2. $d_T(x_4)=2$.
Let F be the forest obtained from T by removing edges x_3x_2, x_4x_5. Clearly the component of F containing x_4 is a spider for which x_3 is one of its central vertices. Note that in this component x_3 is a support vertex and x_4 is its leaf neighbor. Let f be an $i_I(F)$-function such that $f(x_3)$ is maximized. Clearly $f(x_1)+f(x_2)=2$ and by the choice of f we must have $f(x_3)\geq 1$. Thus the function g defined on $V(T)$ by $g(x_2)=0$ and $g(x_1)=1$ and $g(x)=f(x)$ otherwise, is an IID-function of T of weight $i_I(F)-1$, implying that $b_{iI}(T)\leq 2$.

Subcase 5.3. $d_T(x_4)\geq 3$ and x_4 has a child w which is a support vertex of degree two.
Let w' be a leaf neighbor of w and consider the forest F obtained from T by removing edges wx_4, x_3x_2. Let f be an $i_I(F)$-function such that $f(x_1)+f(w')$ is maximized. Then $f(x_1)=f(w')=2$. Now if $f(x_4)\geq 1$, then reassigning a 1 to w' provides an IID-function T of weight smaller than $i_I(F)$. Likewise, if $f(x_3)\geq 1$, then reassigning a 1 to x_1 provides an IID-function of T of weight smaller than $i_I(F)$. Hence we can assume that $f(x_4)=f(x_3)=0$. Thus the function g defined on $V(T)$ by $g(x_3)=1$, $g(x)=0$ for $x\in N_T(x_3)$, $g(x)=1$ for $x\in D(x_3)-C(x_3)$ and $g(x)=f(x)$ otherwise, when x_3 is not a support vertex, and by $g(x_3)=2$, $g(x)=0$ for $x\in N_T(x_3)$, $g(x)=1$ for $x\in D(x_3)-C(x_3)$ and $g(x)=f(x)$ otherwise, when x_3 is a support vertex, is an IID-function of T of weight less than $\omega(f)$, and therefore $b_{iI}(T)\leq 2$.

Subcase 5.4. $d_T(x_4)\geq 3$ and x_4 is a support vertex.
Let w be a leaf neighbor of x_4 and consider the forest F obtained from T by removing the edges wx_4, x_3x_2. Let f be an $i_I(F)$-function such that $f(x_1)$ is maximized. Obviously, $f(w)=1$ and $f(x_1)=2$. Now if $f(x_4)=2$, then reassigning a 0 to w we get an IID-function T of weight $i_I(F)-1$. Moreover, if $f(x_3)\geq 1$, then reassigning a 1 to x_1 provides again an IID-function of T of weight $i_I(F)-1$. Hence we assume that $f(x_4)\leq 1$ and $f(x_3)=0$. Then $f(z)=1$ for each leaf neighbor z of x_3 and $f(a)+f(a')=2$ for each child a of x_3 with depth 1 and degree 2, where a' is the leaf adjacent to a. Now, if $f(x_4)=0$, then define the function g on $V(T)$ by $g(x_3)=2$, $g(x)=0$ or $x\in N(x_3)$, $g(x)=1$ for $x\in D(x_3)-C(x_3)$ and $g(x)=f(x)$ otherwise, when x_3 is a support vertex, and $g(x_3)=1$, $g(x)=0$ for $x\in N(x_3)$, $g(x)=1$ for $x\in D(x_3)-C(x_3)$ and $g(x)=f(x)$ otherwise, when x_3 is not a support vertex. Recall that $d_T(x_3)\geq 4$ or $d_T(x_3)=3$ and x_3 is not a support vertex. In this case, the function g defined above is an IID-function of T of weight less than $i_I(F)$. Finally let $f(x_4)=1$. It follows that w is the only leaf neighbor of x_4. Also according to

Subcase 5.3, we may assume that any child of x_4 with depth 1 has degree at least three. On the other hand, we may assume that the maximal subtree rooted at each child of x_4 with depth 2 is either a P_5 whose center vertex is adjacent to x_4 or a spider with maximum degree at least three. Thus f can be chosen such that each child of x_4 with depth 2 is Italian dominated by its children. Now, if x_5 has a neighbor in V_2, then define the function g on $V(T)$ by $g(x_4) = 0$, $g(x_3) = 2$, $g(x) = 0$ for $x \in N(v_3)$, $g(x) = 1$ for $x \in D(x_3) - C(x_3)$ and $g(x) = f(x)$ otherwise, when x_3 is a support vertex, and $g(x_3) = 1$, $g(x) = 0$ for $x \in N(v_3)$, $g(x) = 1$ for $x \in D(x_3) - C(x_3)$ and $g(x) = f(x)$ otherwise, if x_3 is not a support vertex. In either case, g is an IID-function of T of weight less than $i_I(F)$. If x_5 has a neighbor z in $V_1 - \{x_4\}$, then the function g on $V(T)$ by $g(z) = 2$, $g(x_4) = 0$, $g(x_3) = 2$, $g(x) = 0$ for $x \in N(v_3)$, $g(x) = 1$ for $x \in D(x_3) - C(x_3)$ and $g(x) = f(x)$ otherwise, when x_3 is a support vertex, and $g(x_3) = 1$, $g(x) = 0$ for $x \in N(v_3)$, $g(x) = 1$ for $x \in D(x_3) - C(x_3)$ and $g(x) = f(x)$ otherwise, when x_3 is not a support vertex, is an IID-function of T of weight less than $i_I(F)$. In either case, $b_{iI}(T) \leq 2$.

According to Subcases 5.1, 5.2, 5.3 and 5.4, we may assume that the maximal subtree rooted at each child of x_4 either is a star of order at least three or a path P_5 whose center vertex is adjacent to x_4 or a spider with maximum degree at least 3. Consider the tree F obtained from T by removing the edges x_3x_2 and let f be an $i_IF)$-function such that each child of x_4 has positive weight under f or is Italian dominated by its neighbor (such a property is possible seeing the subtrees rooted at any child of x_4). Clearly $f(x_1) + f(x_2) = 2$. If $f(x_3) \geq 1$, then the function g defined on $V(T)$ by $g(x_1) = 1$, $g(x_2) = 0$ and $g(x) = f(x)$ otherwise, is an IID-function of T of weight less than $\omega(f)$. Hence we assume that $f(x_3) = 0$. If $f(x_4) = 0$, then the function g on $V(T)$ by $g(x_3) = 2$, $g(x) = 0$ for $x \in N(x_3)$, $g(x) = 1$ for $x \in D(x_3) - C(x_3)$ and $g(x) = f(x)$ otherwise, when x_3 is a support vertex, and $g(x_3) = 1$, $g(x) = 0$ for $x \in N(x_3)$, $g(x) = 1$ for $x \in D(x_3) - C(x_3)$ and $g(x) = f(x)$ otherwise, when x_3 is not a support vertex, is an IID-function of T of weight less than $i_I(F)$. Hence we assume that $f(x_4) \geq 1$. Now, if x_5 has a neighbor in $V_2 - \{x_4\}$, then define the function g defined on $V(T)$ by $g(x_4) = 0$, $g(u) = \min\{2, 1 + |L(u)|\}$ for each $u \in C(x_4)$, $g(x) = 0$ for $x \in \cup_{u\in C(x_4)} N_T(u)$, $g(x) = 1$ for $x \in \cup_{u\in C(x_4)}(D(u) - C(u))$ and $g(x) = f(x)$ otherwise, is an IID-function of T of weight less than $i_I(F)$. If x_5 has a neighbor in $V_1 - \{x_4\}$, then define the function g defined on $V(T)$ by $g(x_4) = 0$, $g(u) = \min\{2, 1 + |L(u)|\}$ for each $u \in C(x_4)$, $g(x) = 0$ for $x \in \cup_{u\in C(x_4)} N_T(u)$, $g(x) = 1$ for $x \in \cup_{u\in C(x_4)}(D(u) - C(u))$ and $g(x) = f(x)$ otherwise, is an IID-function of T of weight less than $i_I(F)$. Finally, if x_5 has no neighbor in $V_1 \cup V_2 - \{x_4\}$, then we must have $f(x_4) = 2$ (because of x_5), and thus the function g defined on $V(T)$ by $g(x_5) = 1$, $g(x_4) = 0$, $g(u) = \min\{2, 1 + |L(u)|\}$ for each $u \in C(x_4)$, $g(x) = 0$ for $x \in \cup_{u\in C(x_4)} N_T(u)$, $g(x) = 1$ for $x \in \cup_{u\in C(x_4)}(D(u) - C(u))$ and $g(x) = f(x)$ otherwise, is an IID-function of T of weight less than $i_I(F)$. For each of the situations discussed above, we conclude that $b_{iI}(T) \leq 2$, and this completes the proof. □

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