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Research Article

Lower bound on the KG-Sombor index

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Abstract: In 2021, a novel degree-based topological index was introduced by Gutman, called the Sombor index. Recently Kulli and Gutman introduced a vertex-edge variant of the Sombor index, is caled KG-Sombor index. In this paper, we establish lower bound on the KG-Sombor index and determine the extremal trees achieve this bound.

Keywords: Sombor index, KG-Sombor index, lower bound

AMS Subject classification: 05C07, 05C05

1. Introduction

Let G be a simple connected graph of order n with vertex set $V(G)$ and edge set $E(G)$. For a vertex $x \in V(G)$, we denote by $N_G(x)$ the open neighborhood of x in G which is the set of vertices adjacent to x. The *degree* $d_G(x)$ of x in G is the cardinality of $N_G(x)$. We write $\Delta = \Delta(G)$ for the maximum degree of a graph G. The *distance* between the vertices $x, y \in V(G)$, denoted by $d_G(x, y)$, is defined as the length of any shortest path in G connecting x and y.

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In 2021, a novel degree-based topological index was introduced by Gutman [7], called the *Sombor index*. The Sombor index is defined as:

$$SO(G) = \sum_{uv \in E(G)} \sqrt{d_G^2(u) + d_G^2(v)}.$$

Recently Kulli and Gutman [10] was introduced the novel variant of Sombor index, called the *KG-Sombor index*. The KG-Sombor index defined as:

$$KG(G) = \sum_{uv \in E(G)} \sqrt{d_G^2(u) + (d_G(u) + d_G(v) - 2)^2} + \sum_{uv \in E(G)} \sqrt{d_G^2(v) + (d_G(u) + d_G(v) - 2)^2}.$$

For more information about Sombor index and its variants see [1–6, 8, 9, 11–23] and the references therein.

In this paper we establish a best possible lower bound for the KG-Sombor index of trees in terms of their order and maximum degree. Finally we determine the extremal trees achieve this bound.

We will use the following result.

Observation 1. Let G be a graph. Then for every edge $e \notin E(G)$,

$$KG(G + e) > KG(G).$$

2. A lower bound on the KG-Sombor index

A *tree* is a connected simple graph with no cycles. A leaf of a tree T is a vertex of degree 1 and a support vertex is a vertex adjacent to a leaf. A strong support vertex is a support vertex adjacent to at least two leaves. A *rooted tree* is a tree in which a distinguished vertex is selected as the *root*. A *spider* is a tree with at most one vertex of degree greater than 2 which is known as the *center* of the spider. (If there exists no vertex of degree more than two, then each vertex of degree 2 of the spider can be assumed as its center. In P_2, the only tree without vertices of degree two or more, any vertex can be the center.) A *leg* of a spider is a path from the center to a leaf. If this path is of length one, we say that the leg is *short*; otherwise, the leg is *long*. Hence, a star graph with t edges can be considered as a spider with t short legs, and a path of length at least two is a spider with 2 legs. P_2 is the only one-legged spider. Throughout this section, T denote a rooted tree with root v where v is a vertex of maximum degree. We denote the set of all n-vertex trees with maximum degree Δ by $\mathcal{T}(n, \Delta)$. We start with some lemmas.

Lemma 1. *Let $T \in \mathcal{T}(n, \Delta)$. If T has a strong support vertex of degree at least three different from v, then there is a tree $T' \in \mathcal{T}(n, \Delta)$ such that $KG(T') < KG(T)$.*

Proof. Let w be a strong support vertex of degree $d_T(w) \geq 3$ with maximum distance from v and let $d_T(w) = \alpha$. Assume that $N_T(w) = \{w_1, w_2, \ldots, w_\alpha\}$, where w_α lies on the unique path from w to v. Under our assumption, all neighbors of w except for w_α are of degree 1 or 2 in T. Since w is a strong support vertex, without lack of generality, let w_1 and w_2 be leaves. Let T' be the tree obtained from T by removing the edge ww_1 and adding the edge w_1w_2. Since $\alpha \geq 3$, we have

$$\begin{aligned}
KG(T) - KG(T') =& \sqrt{d_T^2(w_1) + (d_T(w) + d_T(w_1) - 2)^2} + \sqrt{d_T^2(w) + (d_T(w) + d_T(w_1) - 2)^2} \\
&+ \sqrt{d_T^2(w_2) + (d_T(w) + d_T(w_2) - 2)^2} + \sqrt{d_T^2(w) + (d_T(w) + d_T(w_2) - 2)^2} \\
&+ \sum_{i=3}^{\alpha} \sqrt{d_T^2(w_i) + (d_T(w) + d_T(w_i) - 2)^2} + \sum_{i=3}^{\alpha} \sqrt{d_T^2(w) + (d_T(w) + d_T(w_i) - 2)^2} \\
&- \sqrt{d_{T'}^2(w_1) + (d_{T'}(w_1) + d_{T'}(w_2) - 2)^2} - \sqrt{d_{T'}^2(w_2) + (d_{T'}(w_1) + d_{T'}(w_2) - 2)^2} \\
&- \sqrt{d_{T'}^2(w_2) + (d_{T'}(w) + d_{T'}(w_2) - 2)^2} - \sqrt{d_{T'}^2(w) + (d_{T'}(w) + d_{T'}(w_2) - 2)^2} \\
&- \sum_{i=3}^{\alpha} \sqrt{d_{T'}^2(w_i) + (d_{T'}(w) + d_{T'}(w_i) - 2)^2} \\
&\quad - \sum_{i=3}^{\alpha} \sqrt{d_{T'}^2(w) + (d_{T'}(w) + d_{T'}(w_i) - 2)^2} \\
&\quad \geq 2\sqrt{\alpha^2 + (\alpha - 1)^2} + 2\sqrt{1 + (\alpha - 1)^2} \\
&\quad - \sqrt{2} - \sqrt{5} - \sqrt{2(\alpha - 1)^2} - \sqrt{(\alpha - 1)^2 + 4} \\
&\quad > 0.
\end{aligned}$$

This completes the proof. □

Lemma 2. *Let $T \in \mathcal{T}(n, \Delta)$. If T has a support vertex of degree at least three different from v, then there is a tree $T' \in \mathcal{T}(n, \Delta)$ such that $KG(T') < KG(T)$.*

Proof. Let w be a support vertex of degree $d_T(w) \geq 3$ with maximum distance from v and let $d_T(w) = \alpha$. Assume that $N_T(w) = \{w_1, w_2, \ldots, w_\alpha\}$, where w_α lies on the unique path from w to v. Since w is a support vertex, without lack of generality, let $d_T(w_1) = 1$ and by Lemma 1, $d_T(w_i) = 2$ for $2 \leq i \leq \alpha - 1$. Let $wx_1x_2 \ldots x_l$ be a path in T such that $l \geq 2$ and $x_1 = w_2$. Assume that T' be the tree obtained from T by removing the edge ww_1 and adding the edge x_lw_1. Since $\alpha \geq 3$, we have

$$\begin{aligned}
KG(T) - KG(T') =& \sqrt{d_T^2(w_1) + (d_T(w) + d_T(w_1) - 2)^2} + \sqrt{d_T^2(w) + (d_T(w) + d_T(w_1) - 2)^2} \\
&+ \sqrt{d_T^2(x_l) + (d_T(x_l) + d_T(x_{l-1}) - 2)^2} + \sqrt{d_T^2(x_{l-1}) + (d_T(x_l) + d_T(x_{l-1}) - 2)^2} \\
&+ \sum_{i=2}^{\alpha} \sqrt{d_T^2(w_i) + (d_T(w) + d_T(w_i) - 2)^2} + \sum_{i=2}^{\alpha} \sqrt{d_T^2(w) + (d_T(w) + d_T(w_i) - 2)^2} \\
&- \sqrt{d_{T'}^2(w_1) + (d_{T'}(w_1) + d_{T'}(x_l) - 2)^2} - \sqrt{d_{T'}^2(x_l) + (d_{T'}(w_1) + d_{T'}(x_l) - 2)^2} \\
&- \sqrt{d_{T'}^2(x_l) + (d_{T'}(x_l) + d_{T'}(x_{l-1}) - 2)^2} \\
&- \sqrt{d_{T'}^2(x_{l-1}) + (d_{T'}(x_l) + d_{T'}(x_{l-1}) - 2)^2}
\end{aligned}$$

$$
\begin{aligned}
&- \sum_{i=2}^{\alpha} \sqrt{d_{T'}^2(w_i) + (d_{T'}(w) + d_{T'}(w_i) - 2)^2} \\
&- \sum_{i=2}^{\alpha} \sqrt{d_{T'}^2(w) + (d_{T'}(w) + d_{T'}(w_i) - 2)^2} \\
&\geq \sqrt{\alpha^2 + (\alpha-1)^2} + \sqrt{1 + (\alpha-1)^2} + \sqrt{2} + \sqrt{5} - \sqrt{2} - \sqrt{5} - 2\sqrt{8} \\
&\geq \sqrt{13} + \sqrt{5} - 2\sqrt{8} \approx 0.1847 > 0.
\end{aligned}
$$

This completes the proof. □

Lemma 3. *Let $T \in \mathcal{T}(n, \Delta)$. If T has a vertex of degree at least three different from v, then there is a tree $T' \in \mathcal{T}(n, \Delta)$ such that $KG(T') < KG(T)$.*

Proof. Let w be a vertex of degree $d_T(w) \geq 3$ with maximum distance from v and let $d_T(w) = \alpha$. Assume that $N_T(w) = \{w_1, w_2, \ldots, w_\alpha\}$, where w_α lies on the unique path from w to v.
By Lemmas 1 and 2, $d_T(w_i) = 2$ for $1 \leq i \leq \alpha - 1$. Let $wx_1x_2 \ldots x_t$ and $wy_1y_2 \ldots y_s$, $t, s \geq 2$, be two paths in T with $x_1 = w_1$ and $y_1 = w_2$. Let T' be the tree obtained from T by removing the edge ww_1 and adding the edge y_sw_1. Since $\alpha \geq 3$, then we have

$$
\begin{aligned}
KG(T) - KG(T') =& \sqrt{d_T^2(w_1) + (d_T(w) + d_T(w_1) - 2)^2} + \sqrt{d_T^2(w) + (d_T(w) + d_T(w_1) - 2)^2} \\
&+ \sqrt{d_T^2(y_s) + (d_T(y_s) + d_T(y_{s-1}) - 2)^2} + \sqrt{d_T^2(y_{s-1}) + (d_T(y_s) + d_T(y_{s-1}) - 2)^2} \\
&+ \sum_{i=2}^{\alpha} \sqrt{d_T^2(w_i) + (d_T(w) + d_T(w_i) - 2)^2} + \sum_{i=2}^{\alpha} \sqrt{d_T^2(w) + (d_T(w) + d_T(w_i) - 2)^2} \\
&- \sqrt{d_{T'}^2(w_1) + (d_{T'}(w_1) + d_{T'}(y_s) - 2)^2} - \sqrt{d_{T'}^2(y_s) + (d_{T'}(w_1) + d_{T'}(y_s) - 2)^2} \\
&- \sqrt{d_{T'}^2(y_s) + (d_{T'}(y_s) + d_{T'}(y_{s-1}) - 2)^2} \\
&- \sqrt{d_{T'}^2(y_{s-1}) + (d_{T'}(y_s) + d_{T'}(y_{s-1}) - 2)^2} \\
&- \sum_{i=2}^{\alpha} \sqrt{d_{T'}^2(w_i) + (d_{T'}(w) + d_{T'}(w_i) - 2)^2} \\
&- \sum_{i=2}^{\alpha} \sqrt{d_{T'}^2(w) + (d_{T'}(w) + d_{T'}(w_i) - 2)^2} \\
&\geq \sqrt{\alpha^2 + \alpha^2} + \sqrt{4 + \alpha^2} + \sqrt{2} + \sqrt{5} \\
&- \sqrt{2} - \sqrt{5} - 2\sqrt{8} \\
&\geq \sqrt{18} + \sqrt{13} - 2\sqrt{8} > 0.
\end{aligned}
$$

This completes the proof. □

Lemma 4. *Let T be a spider of order n with $k \geq 3$ legs. If T has a leg of length 1 and a leg of length at least 3, then there is a spider T' of order n with k legs such that $KG(T) > KG(T')$.*

Proof. Let x be the center of T and $N_T(x) = \{x_1, \ldots, x_k\}$. Root T at x. We may assume that $d(x_1) = 1$ and let $x_2y_1y_2\ldots y_t$, $t \geq 2$ be a longest leg of T. Let T' be the tree obtained from T be deleting the edge y_ty_{t-1} and adding the pendant edge x_1y_t. By definition we have

$$\begin{aligned} KG(T) - KG(T') = &\sqrt{\Delta^2 + (\Delta-1)^2} + \sqrt{1 + (\Delta-1)^2} + 2\sqrt{8} \\ &- \sqrt{2\Delta^2} - \sqrt{\Delta^2+4} - \sqrt{2} - \sqrt{5} > 0. \end{aligned}$$

This complete the proof. □

Now we prove the main theorems of this section.

Theorem 2. *For any tree $T \in \mathcal{T}(n, \Delta)$ of order $n \geq 3$,*

$$KG(T) \geq \Delta\sqrt{\Delta^2+4} + \sqrt{2}\Delta^2 + 2\sqrt{8}(n - 2\Delta - 1) + [\sqrt{2} + \sqrt{5}]\Delta,$$

when $\Delta \leq \frac{n-1}{2}$ and

$$\begin{aligned} KG(T) \geq &(2\Delta + 1 - n)[\sqrt{\Delta^2 + (\Delta-1)^2} + \sqrt{1 + (\Delta-1)^2}] \\ &+(n - \Delta - 1)[\sqrt{\Delta^2+4} + \sqrt{2}\Delta + \sqrt{5} + \sqrt{2}], \end{aligned}$$

when $\Delta > \frac{n-1}{2}$ and the equality holds if and only if T is a spider whose all legs have length at most two or all legs have length at least two.

Proof. Let $T^* \in \mathcal{T}(n, \Delta)$ such that $KG(T^*) \leq KG(T)$ for each $T \in \mathcal{T}(n, \Delta)$. Choose a vertex v of T^* with degree Δ as the root of T^*. If $\Delta = 2$, then T is a path of order n and $KG(T) = (n-3)\sqrt{8} + 2\sqrt{2} + 2\sqrt{5}$ as desired. Let $\Delta \geq 3$. By the choice of T^*, we deduce from Lemmas 1, 2 and 3 that T^* is a spider with center v. It follows from Lemma 4 and the choice of T^* that all legs of T^* either have length at most two or have length at least two. First let all legs of T^* have length at least two. Then clearly $\Delta \leq \frac{n-1}{2}$ and

$$KG(T^*) = \Delta\sqrt{\Delta^2+4} + \sqrt{2}\Delta^2 + 2\sqrt{8}(n - 2\Delta - 1) + [\sqrt{2} + \sqrt{5}]\Delta$$

as desired. Now let all legs of T^* have length at most two. Considering above case, we may assume that T^* has a leg of length 1. If T^* is a star, then the result is immediate. Assume T^* is not a star. Then the number of leaves adjacent to v is $2\Delta + 1 - n$ and hence

$$\begin{aligned} KG(T^*) = &(2\Delta + 1 - n)[\sqrt{\Delta^2 + (\Delta-1)^2} + \sqrt{1 + (\Delta-1)^2}] \\ &+(n - \Delta - 1)[\sqrt{\Delta^2+4} + \sqrt{2}\Delta + \sqrt{5} + \sqrt{2}]. \end{aligned}$$

This completes the proof. □

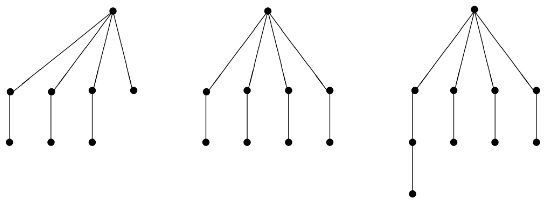

Figure 1. **Trees with** $n = 8, 9, 10$ **and** $\Delta = 4$

In Figure 1, three trees of orders $n = 8, 9, 10$ with maximum degree $\Delta = 4$ and with minimum KG-Sombor index are illustrated..
By Observation 1, we obtain the following corollary.

Corollary 1. *Let G be a graph of order n and maximum degree Δ. Then*

$$KG(G) \geq \Delta\sqrt{\Delta^2 + 4} + \sqrt{2}\Delta^2 + 2\sqrt{8}(n - 2\Delta - 1) + [\sqrt{2} + \sqrt{5}]\Delta,$$

when $\Delta \leq \frac{n-1}{2}$ *and*

$$\begin{aligned} KG(G) \geq & (2\Delta + 1 - n)[\sqrt{\Delta^2 + (\Delta - 1)^2} + \sqrt{1 + (\Delta - 1)^2}] \\ & +(n - \Delta - 1)[\sqrt{\Delta^2 + 4} + \sqrt{2}\Delta + \sqrt{5} + \sqrt{2}], \end{aligned}$$

when $\Delta > \frac{n-1}{2}$ *and the equality holds if and only if* G *is a spider whose all legs have length at most two or all legs have length at least two.*

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Data Availability. Data sharing is not applicable to this article as no datasets were generated or analyzed during the current study.

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