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On γ-free, γ-totally-free and γ-fixed sets in graphs

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Abstract: Let $G = (V, E)$ be a connected graph. A subset S of V is called a γ-free set if there exists a γ-set D of G such that $S \cap D = \emptyset$. If further the induced subgraph $H = G[V - S]$ is connected, then S is called a *cc*-γ-free set of G. We use this concept to identify connected induced subgraphs H of a given graph G such that $\gamma(H) \leq \gamma(G)$. We also introduce the concept of γ-totally-free and γ-fixed sets and present several basic results on the corresponding parameters.

Keywords: Domination, domination number, γ-set, γ-free set, γ-totally-free set, γ-fixed set

AMS Subject classification: 05C69

1. Introduction

By a graph $G = (V, E)$, we mean a finite, undirected and connected graph with neither loops nor multiple edges. For graph theoretic terminologies we refer to [1]. For domination related concepts we refer to [2].
A subset S of V is called a dominating set of G if every vertex in $V - S$ is adjacent to a vertex in S. The domination number γ of G is the minimum cardinality of a dominating set of G. A dominating set S of G with $|S| = \gamma$ is called a γ-set of G. A

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dominating set S is called a connected dominating set if the induced subgraph $G[S]$ is connected. The connected domination number γ_c of G is the minimum cardinality of a connected dominating set of G. A vertex v of G is called a support vertex if v is adjacent to a pendent vertex. If the number of pendent vertices adjacent to v is at least two, then v is a strong support vertex. The corona of two graphs G and H, denoted by $G \circ H$, is the graph obtained from one copy of G and $|V(G)|$ copies of H and i^{th} vertex of G is joined to every vertex in the i^{th} copy of H.
Let G be a connected graph and let H be a connected induced subgraph of G. Then $\gamma(H)$ may be equal to or less than or greater than $\gamma(G)$.

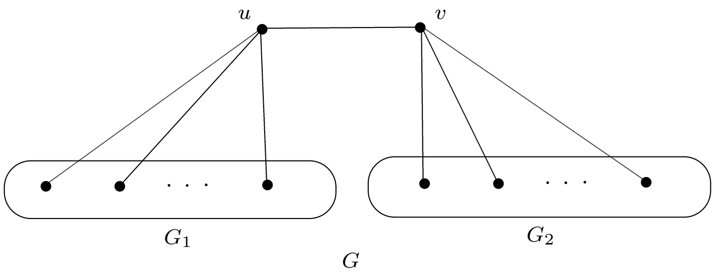

Figure 1. **A graph G with all three types of induced subgraphs**

Example 1. Consider the graph G given in Figure 1, where G_1 is any graph with $\gamma(G_1) \geq 3$. Clearly $\gamma(G) = 2$. Let $H_1 = G_1$, $H_2 = G_1 + u$, $H_3 = (v_1, u, v, w_1)$, where $v_1 \in V(G_1)$ and $w_1 \in V(G_2)$. Clearly H_1, H_2, H_3 are connected induced subgraphs of G. Also $\gamma(H_1) > \gamma(G)$, $\gamma(H_2) < \gamma(G)$ and $\gamma(H_3) = \gamma(G)$. Thus G contains all three types of induced subgraphs.

The following is a fundamental problem in domination:

Problem 1. Let H be a connected induced subgraph of a connected graph G. Under what conditions $\gamma(H) \leq \gamma(G)$?

In this paper we introduce the concepts of γ-free set, γ-totally-free set, γ-fixed set and cc-γ-free set in graphs. We use the concept of cc-γ-free set to identify connected induced subgraphs H of a given connected graph G such that $\gamma(H) \leq \gamma(G)$. The significance and use of this concept is given in the concluding section of the paper.

2. γ-free sets

Sampathkumar and Neeralagi introduced the concept of γ-free vertex, γ-totally free vertex and γ-fixed vertex in [5]. We extend this notion to subsets of V and define new parameters based on these concepts.

Definition 1. Let $G = (V, E)$ be a connected graph. A subset S of V is called a γ-free set if there exists a γ-set D of G such that $D \cap S = \emptyset$.

A subset of a γ-free set is γ-free and hence γ-freeness is a hereditary property. A γ-free set S is a maximal γ-free set if and only if $S \cup \{v\}$ is not a γ-free set for all $v \in V - S$.

Definition 2. The minimum cardinality of a maximal γ-free set of G is called the γ-free number of G and is denoted by $\gamma_{fr}(G)$.

Lemma 1. *Let G be any connected graph of order n. Then $\gamma_{fr}(G) = n - \gamma(G)$.*

Proof. Let D be a γ-set of G and let $S = V - D$. Since $D \cap S = \emptyset$, S is a γ-free set of G. Also for any $v \in D$, $S_1 = S \cup \{v\}$ is not a γ-free set of G since $|V - S_1| = |D - \{v\}| = \gamma(G) - 1$. Thus S is a maximal γ-free set and $|S| = n - \gamma(G)$. Hence $\gamma_{fr}(G) \leq n - \gamma(G)$. Now, let X be any maximal γ-free set of G. Let D be a γ-set of G such that $D \cap X = \emptyset$. Hence $D \subseteq V - X$. If there exists a vertex $v \in (V - X) - D$, then $X \cup \{v\}$ is a γ-free set which is a contradiction. Thus $\gamma_{fr}(G) \geq n - \gamma(G)$ and hence $\gamma_{fr}(G) = n - \gamma(G)$. □

Definition 3. A γ-free set S of G is called a *cc*-γ-free set if the induced subgraph $H = G[V - S]$ is connected.

Lemma 2. *Every connected graph G admits a cc-γ-free set.*

Proof. Let D be a γ-set of G. It follows from Lemma 1 that $S = V - D$ is a maximal γ-free set of G. Let $\mathcal{F} = \{H : H$ is a connected induced subgraph of G and $D \subseteq V(H)\}$. Since $G \in \mathcal{F}$, $\mathcal{F} \neq \emptyset$. Choose $H \in \mathcal{F}$, such that $|V(H)|$ is minimum. Since $D \subseteq V(H)$, we have $S_1 = V(G) - V(H) \subseteq S$. Since S is γ-free, S_1 is γ-free and $H = G[V - S_1]$ is connected. Hence S_1 is a *cc*-γ-free set of G. □

Definition 4. Let G be a connected graph. Then $\max\{|S| : S$ is a *cc*-γ-free set of $G\}$ is called the *cc*-γ-free number of G and is denoted by $cc\gamma_{fr}(G)$.

Example 2. For the path $P_6 = (v_1, v_2, v_3, v_4, v_5, v_6)$, $D = \{v_2, v_5\}$ is the unique γ-set and hence any γ-free set is a subset of $V - D = \{v_1, v_3, v_4, v_6\}$. Clearly $S_1 = \{v_1, v_6\}$ is a γ-free set of P_6 and $H = P_6[V - S_1] = P_4$. Thus $cc\gamma_{fr}(P_6) = 2$ and $\gamma_{fr}(P_6) = 4$.

Theorem 1. *Let G be a connected graph. Then $cc\gamma_{fr}(G) \leq \gamma_{fr}(G)$ and equality holds if and only if $\gamma(G) = \gamma_c(G)$, where $\gamma_c(G)$ is the connected domination number of G.*

Proof. Let S be a cc-γ-free set of G. Since S is a γ-free set, there exists a γ-set D of G such that $D \cap S = \emptyset$. Hence $cc\gamma_{fr}(G) = |S| \leq |V - D| = n - \gamma(G) = \gamma_{fr}(G)$. Thus $cc\gamma_{fr}(G) \leq \gamma_{fr}(G)$. Now, suppose $cc\gamma_{fr}(G) = \gamma_{fr}(G) = n - \gamma(G)$. Let S be a cc-γ-free set of G. Since $|S| = n - \gamma(G)$, it follows that $S = V - D$, where D is a γ-set of G. Also $G[V - S] = G[D]$ is connected and hence D is a connected dominating set of G. Hence $\gamma(G) = \gamma_c(G) = |D|$. The converse is obvious. □

Example 3. Let G be a connected graph of order n with $V(G) = \{v_1, v_2, \ldots, v_n\}$. Let H_i be any graph of order k_i, $1 \leq i \leq n$. Then the graph G^* obtained from G by joining v_i to all the vertices of H_i is called a generalized corona of G and is denoted by $G \circ (H_1, H_2, \ldots, H_n)$. Clearly $\gamma(G^*) = \gamma_c(G^*) = |V(G)| = n$. Hence $cc\gamma_{fr}(G^*) = \gamma_{fr}(G^*)$.

Also for the path P_n with $n \leq 4$, the cycle C_n with $n \leq 4$, the complete graph K_n and the complete k-partite graph $K_{n_1,n_2,\ldots,n_k}$ with $n_i \geq 2$ for all i, we have $\gamma = \gamma_c$. Hence for all these graphs, $cc\gamma_{fr}(G) = \gamma_{fr}(G) = n - \gamma(G)$. We now proceed to determine $cc\gamma_{fr}(G)$ for graphs with $\gamma \neq \gamma_c$.

Theorem 2. *Let T be any tree of order n. Then $cc\gamma_{fr}(T) = k$, where k is the number of pendant vertices of T.*

Proof. Let L and A denote respectively the set of all pendant vertices and the set of all support vertices of T. Clearly L is cc-γ-free set of T. Now let $v \in A$ and let $u \in L$ be the leaf adjacent to v. Since any γ-set D of T contains one of the vetices u, v, it follows that $L \cup \{v\}$ is not a γ-free set. Also if $w \in V - (L \cup A)$, then $T[V - (L \cup \{w\})]$ is not connected and hence $L \cup \{w\}$ is not a cc-γ-free set of T. Thus L is a maximal cc-γ-free set of T and hence $cc\gamma_{fr}(T) \geq |L| = k$. Now, let S be any maximal cc-γ-free set of T. If $u \in L$, $v \in A$ and $uv \in E(T)$, then S contains exactly one of the vertices u and v. Hence we may assume without loss of generality that $L \subseteq S$ and $S \cap A = \emptyset$. Since S is a cc-γ-free set; it follows that $S = L$. Hence $cc\gamma_{fr}(T) = k$. □

Corollary 1. *Let T be a tree of order n. Then $2 \leq cc\gamma_{fr}(T) \leq n-1$. Also $cc\gamma_{fr}(T) = 2$ if and only if T is the path P_n and $cc\gamma_{fr}(T) = n-1$ if and only if T is the star $K_{1,n-1}$.*

It follows from Theorem 1 that $cc\gamma_{fr}(K_n) = n - 1$. Thus $K_{1,n-1}$ and K_n are two graphs of order n with $cc\gamma_{fr}(G) = n-1$. The following result gives a characterization of all graphs of order n with $cc\gamma_{fr}(G) = n - 1$.

Theorem 3. *Let G be a connected graph of order n. Then $cc\gamma_{fr}(G) = n - 1$ if and only if $\gamma(G) = 1$.*

Proof. Suppose $cc\gamma_{fr}(G) = n - 1$. Let S be a cc-γ-free set of G with $|S| = n - 1$ and $V - S = \{v\}$. Since S is a γ-free set of G, it follows that $\{v\}$ is a dominating set of G and hence $\gamma(G) = 1$. Conversely suppose $\gamma(G) = 1$ and let $\{v\}$ be a dominating set of G. Then $V - \{v\}$ is a maximal cc-γ-free set of G. Hence $cc\gamma_{fr}(G) = n - 1$. □

3. On γ-totally-free sets

Definition 5. A subset S of V is called a γ-totally-free set if $D \cap S = \emptyset$ for all γ-sets D of G. The maximum cardinality of a γ-totally-free set of G is called as the γ-totally-free number of G and is denoted by $\gamma_{tf}(G)$.

Observation 4.

(i) Let A denote the union of all γ-sets of a graph G. Then G admits a γ-totally-free set if and only if $A \neq V$. In this case $V - A$ is a totally γ-free set and $\gamma_{tf}(G) = |V - A|$.

(ii) Since a γ-totally free set is a γ-free set, it follows that $\gamma_{tf}(G) \leq \gamma_{fr}(G) = n - \gamma(G)$. Furthermore, $\gamma_{tf}(G) = \gamma_{fr}(G)$ if and only if G has a unique γ-set.

(iii) If G is a vertex transitive graph, then every vertex of G lies in a γ-set of G and hence G does not admit a γ-totally free set. In particular the complete graph K_m and the cycle C_n do not admit a γ-totally-free set. Also $K_{r,s}$ where $r, s \geq 2$ does not admit a γ-totally-free set.

Theorem 5. *A path $P_n = (v_1, v_2, \cdots, v_n)$ admits a γ-totally-free set if and only if $n \not\equiv 1$(mod 3). Also*

$$\gamma_{tf}(P_n) = \begin{cases} \frac{2n}{3} & \text{if } n \equiv 0 (\text{mod } 3); \\ \lfloor \frac{n}{3} \rfloor & \text{if } n \equiv 2 (\text{mod } 3). \end{cases}$$

Proof. We consider the following cases.

Case 1. $n \equiv 1 \pmod 3$.
Let $n = 3k+1$. Then the γ-sets of P_n are given by, $D_1 = \{v_i : i \equiv 2 \pmod 3\} \cup \{v_n\}$, $D_2 = \{v_i : i \equiv 1 \pmod 3\} \cup \{v_{n-2}\}$, $D_3 = \{v_i : i \equiv \ \pmod 3\} \cup \{v_1\}$ and $D_4 = \{v_i : i \equiv 1 \pmod 3\} \cup \{v_3\}$. Clearly $\cup_{i=1}^{4} D_i = V$ and hence P_n does not admit γ-totally-free set.

Case 2. $n \equiv 0 \pmod 3$.
In this case $D = \{v_i : i \equiv 2 \pmod 3\}$ is the unique γ-set of P_n and $|D| = \frac{n}{3}$. Hence $\gamma_{tf}(P_n) = n - \frac{n}{3} = \frac{2n}{3}$.

Case 3. $n \equiv 2 \pmod 3$.
Let $n = 3k + 2$. Any γ-set containing v_n is of the form $\{v_n\} \cup D$, where D is a γ-set of $P_{n-2} = \{v_1, v_2, \ldots, v_{3k}\}$. This path has unique γ-set $\{v_i : i \equiv 2 \pmod 3\}$. Hence only γ-set containing v_n is $D_1 = \{v_n\} \cup \{v_i : i \equiv 2 \pmod 3\}$.
Similarly only γ-set containing v_1 is of the form $D_2 = \{v_1\} \cup \{v_i : i \equiv 1 \pmod 3\}$.
Any γ-set containing v_{n-1} is of the form $\{v_{n-1}\} \cup D$, where D is a γ-set of $P_n - \{v_{3k}, v_{3k+1}, v_{3k+2}\}$ or $P_n - \{v_{3k+1}, v_{3k+2}\}$. The only γ-sets containing v_{n-1} are given by, $D_3 = \{v_{n-1}\} \cup \{v_i : i \equiv 1 \pmod 3\}$ and $D_4 = \{v_{n-1}\} \cup \{v_i : i \equiv 2 \pmod 3\}$. Similarly the only γ-sets containing v_2 are given by, $D_5 = \{v_2\} \cup \{v_i : i \equiv 1 \pmod 3\}$

and $D_6 = \{v_2\} \cup \{v_i : i \equiv 2 \pmod 3\}$. Hence $\cup_{i=1}^{6} D_i = \{v_i : i \not\equiv 0 \pmod 3\}$. Therefore $\{v_i : i \equiv 0 \pmod 3\}$ is a γ-totally-free set of P_n of maximum cardinality. Hence $\gamma_{tf}(P_n) = \lfloor \frac{n}{3} \rfloor$. □

Definition 6. A γ-totally-free set S of G is called a cc-γ-totally-free set if $G[V-S]$ is connected.

Example 4. Let H be any connected graph with $\gamma(H) \geq 2$ and let $G = H + K_k$, $k \geq 2$. Then $V(H)$ is a cc-γ-totally-free set of G and $\gamma(H) > \gamma(G)$.

4. On γ-fixed sets

Definition 7. Let $G = (V, E)$ be a connected graph. A subset S of V is called a γ-fixed set if $D \cap S \neq \emptyset$ for all γ-sets D in G.

A superset of a γ-fixed set is γ-fixed and hence γ-fixedness is a super-hereditary property. A γ-fixed set S is a minimal γ-fixed set if and only if $S - \{v\}$ is not a γ-fixed set for all $v \in S$.

Definition 8. The minimum cardinality of a minimal γ-fixed set of G is called the γ-fixed number of G and is denoted by $\gamma_{fi}(G)$. The maximum cardinality of a minimal γ-fixed set of G is called the Γ-fixed number of G and is denoted by $\Gamma_{fi}(G)$.

Theorem 6. *A γ-fixed set S of G is a minimal γ-fixed set if and only if for every $v \in S$, there exists a γ-set D such that $D \cap S = \{v\}$.*

Proof. Suppose S is a minimal γ-fixed set of G. Hence for all $v \in S$, $S - \{v\}$ is not a γ-fixed set. Therefore there exists a γ-set D of G such that $D \cap (S - \{v\}) = \emptyset$. Hence $D \cap S = \{v\}$. The Converse is obvious. □

Example 5. If G is a graph with $\gamma(G) = 1$, then $S = \{v : deg(v) = n-1\}$ is the only minimal γ-fixed set in G. Hence $\gamma_{fi}(G) = \Gamma_{fi}(G) = |S|$. In particular $\gamma_{fi}(K_n) = \Gamma_{fi}(K_n) = n$.

Example 6. Let $G = K_{r,s}$ be a complete bipartite graph with $r \leq s$. Let V_1 and V_2 be the partite sets of G with $|V_1| = r$ and $|V_2| = s$. Then V_1 and V_2 are minimal γ-fixed sets in G. Hence $\gamma_{fi}(G) = r$ and $\Gamma_{fi}(G) = s$.

Observation 7. If G has a unique γ-set D, then $S = \{v\}$ is a minimal γ-fixed set for any $v \in D$. Hence $\gamma_{fi}(G) = \Gamma_{fi}(G) = 1$.

Lemma 3. *Let G be a graph of order $n \geq 3$ with $\delta = 1$. Then*

$$\gamma_{fi}(G) = \begin{cases} 1 & \text{if } G \text{ has a strong support vertex;} \\ 2 & \text{otherwise.} \end{cases}$$

Proof. If G has a strong vertex v, let $S = \{v\}$. Otherwise let $S = \{v, u\}$, where v is a support vertex and u is the pendent neighbor of v. Clearly S is a minimal γ-fixed set of G and hence the result follows. □

Corollary 2. $\gamma_{fi}(P_n) = \begin{cases} 1 & \text{if } n \leq 3; \\ 2 & \text{otherwise.} \end{cases}$

Corollary 3. *Let G be a graph of order $n \geq 2$. Then $\gamma_{fi}(G \circ K_1) = 2$.*

Theorem 8. *For any graph G, $\gamma_{fi}(G) \leq \delta + 1$. Equality holds if and only if $N[v]$ is a minimal γ-fixed set of G for every vertex v with $deg(v) = \delta$.*

Proof. Let $deg(v) = \delta$. Then $D \cap N[v] \neq \emptyset$ for all γ-sets D of G. Hence $N[v]$ is a γ-fixed set. Let $S \subseteq V$ be a minimal γ-fixed set of G. Then $\gamma_{fi}(G) \leq |S| \leq \delta + 1$ and equality holds if and only if $N[v]$ is a minimal γ-fixed set of G. □

Definition 9. A subset S of V is said to be a *cc*-γ-fixed set of G if S is γ-fixed and $G[V - S]$ is connected.

Question 1. Which graphs admit *cc*-γ-fixed set?

5. Conclusion and scope

Let G be a connected graph and let S be a *cc*-γ-free set of G. Then the induced subgraph $H = G[V - S]$ is connected and there exists a γ-set D of G such that $D \subseteq V(H)$. Hence it follows that $\gamma(H) \leq \gamma(G)$. Thus *cc*-γ-free set serves as an useful tool in identifying connected induced subgraphs H of G with $\gamma(H) \leq \gamma(G)$. The queens graph Q_n has vertex set V of order n^2(in $1 - 1$ correspondence with the n^2 cells of an $n \times n$ chessboard) where two vertices u and v are adjacent if and only if a queen at u can reach v in a single move. Hence $\gamma(Q_n)$ is the minimum number of queens to be placed on an $n \times n$ chessboard such that every cell is either occupied by a queen or attached by a queen. One of the most interesting open problems on $\gamma(Q_n)$ is the following:

Problem 2. Is $\gamma(Q_n) \leq \gamma(Q_{n+1})$?

Though the problem appears to be obviously true, no proof has yet been formed and for a solution to this problem a \$ 100 price is offered by S.T. Hedetniemi ([3], pp 157). This problem is a special case of Problem 1.2.
We observe that if it can be proved that $V(Q_{n+1}) - V(Q_n)$ is a *cc*-γ-free set of Q_{n+1}, then we get an affirmative answer to the above problem. Also similar problems for other domination related parameters such as connected domination number, total domination number [4], independent domination number, super domination number and 2-domination number can be investigated.

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