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Research Article

Some Properties and Identities of Hyperbolic Generalized k-Horadam Quaternions and Octonions

Kalika Prasad[1,†], Munesh Kumari[1,*], Can Kızılateş[2]

[1]Department of Mathematics, Central University of Jharkhand, 835205, India
[†]klkaprsd@gmail.com
[*]muneshnasir94@gmail.com

[2]Department of Mathematics, Zonguldak Bülent Ecevit University, 67100, Turkey
[‡]cankizilates@gmail.com

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Abstract: The aim of this paper is to introduce the hyperbolic generalized k-Horadam quaternions and octonions and investigate their algebraic properties. We present some properties and identities of these quaternions and octonions for generalized k-Horadam numbers. Moreover, we give some determinants related to the hyperbolic generalized k-Horadam quaternions and octonions. Finally, we evaluate its determinants through the Chebyshev polynomials of the second kind and give an illustrative example as well.

Keywords: Horadam number, Hyperbolic quaternions, Hyperbolic octonions, Binet Formula, Catalan Identity, Generating Function, Chebyshev polynomials, Determinant

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1. Introduction

Special number sequences are of great interests in research due to their wide uses in other branches like geometry, combinatorics, approximation theory, statistics, cryptography, physics, etc. Especially many numbers defined with the help of second-order recurrence relations have been widely investigated by mathematicians in recent years. One of such interesting sequence is Horadam sequence [13] which generalizes many second order recurrences. In recent years, several works have been done on Horadam

* *Corresponding Author*

numbers. For example, Frontczak [8] studied the Horadam identities via binomial coefficients and gave many other known identities. G. Y. Şentürk et al. [28] obtained the fundamental and algebraic properties of Horadam numbers and gave special matrix representations of them. Horzum and Kocer [15] studied the polynomials associated with Horadam numbers. T.D. Şentürk et. al. [29] presented study on the Horadam hybrid numbers. C. Kızılateş [18] studied the Horadam hybrinomials. Yazlık, et al. [32] investigated the k-Horadam sequence in more generalized way. Prasad, et al. [27] extended the k-Horadam numbers to third order and studied their properties. It is worthful to start with the definition of the Horadam numbers in generalized form.

Definition 1. ([32]) Let $\phi(k)$ and $\psi(k)$ be scalar valued polynomials and $k \in \mathbb{R}^+$. Then the k-Horadam sequence $\{H_{k,n}\}$ in generalized form is defined by

$$H_{k,n+2} = \phi(k)H_{k,n+1} + \psi(k)H_{k,n}, \quad n \geq 0, \quad \text{where} \quad H_{k,0} = a, \; H_{k,1} = b. \tag{1}$$

Note that $\alpha = \left(\phi(k) + \sqrt{\phi^2(k) + 4\psi(k)}\,\right)/2$ and $\beta = \left(\phi(k) - \sqrt{\phi^2(k) + 4\psi(k)}\,\right)/2$ are two roots of the characteristic equation $x^2 - \phi(k)x - \psi(k) = 0$ corresponding to Eqn. (1). Thus, they satisfy the following properties:

$$\begin{gathered} \alpha + \beta = \phi(k), \quad \alpha - \beta = \sqrt{\phi^2(k) + 4\psi(k)}, \quad \alpha\beta = -\psi(k), \\ \alpha^2 + \beta^2 = \phi^2(k) + 2\psi(k) \quad \text{and} \quad \alpha^3 + \beta^3 = \phi^3(k) + 3\phi(k)\psi(k). \end{gathered} \tag{2}$$

The sequence defined by (1) is usually denoted by $H_{k,n}(a, b; \phi, \psi)$ and for simplicity we use $H_{k,n}$ if it does not cause ambiguity. We have listed some of the well-known k-Horadam numbers in the following table for particular values of a, b and ϕ, ψ.

Name of Sequences	Coefficient (ϕ, ψ)	Initial values (a, b)
Horadam sequence	(p, q)	a, b
k-Fibonacci / k-Lucas sequence	$(k, 1)$	$a = 0, b = 1$ / $a = 2, b = 1$
Fibonacci / Lucas sequence	$(1, 1)$	$a = 0,\; b = 1$ / $a = 2,\; b = 1$
k-Jacobsthal / k-Jacobsthal-Lucas sequence	$(k, 2)$	$a = 0, b = 1$ / $a = 2, b = k$
k-Pell / k-Pell-Lucas sequence	$(2, k)$	$a = 0,\; b = 1$ / $a = b = 2$
k-Mersenne / k-Mersenne-Lucas sequence	$(3k, -2)$	$a = 0,\; b = 1$ / $a = 2, b = 3k$
k-balancing / k-Lucas-balancing sequence	$(6k, -1)$	$a = 0, b = 1$ / $a = 1, b = 3$

Table 1. **List of some integer sequences $H_{k,n}$.**

In [27, 32], the authors have studied the generalized k-Horadam sequence $\{H_{k,n}\}$ and presented some interesting properties of them. The closed form formula for sequence (1) is given by

$$H_{k,n} = \frac{P\alpha^n - Q\beta^n}{\alpha - \beta} \quad \text{and} \quad H_{k,-n} = (-1)^n \frac{P\beta^n - Q\alpha^n}{(\psi(k))^n(\alpha - \beta)}, \quad n \in \mathbb{N}, \tag{3}$$

where $P = b - a\beta$ and $Q = b - a\alpha$.

Before proceeding, it is worth noting some information on the progress of quaternion and octonion. These hyper-complex numbers have wide application in various areas like string theory, quantum physics, computer sciences, differential geometry, etc. The concept of quaternions ($\mathbb{H}$) was introduced in 1843 by W.R Hamilton [12], which is an associative and non-commutative 4-dimensional algebra over $\mathbb{R}$. Later in 1963, Horadam [14] introduced the Fibonacci quaternions (Q_n) and Lucas quaternions (R_n) using the Fibonacci numbers F_n and Lucas numbers L_n , respectively, which are defined as

$$Q_n = \sum_{r=0}^{3} F_{n+r} e_s \qquad \text{and} \qquad R_n = \sum_{r=0}^{3} L_{n+r} e_s.$$

Motivated by Hamilton's work, in 1843 J. Graves introduced the set of octonions ($\mathbb{O}$) which is a non-associative and non-commutative 8-dimensional algebra over $\mathbb{R}$. Many works on octonions with a number sequence have been done. For example, the Fibonacci and Lucas octonions were studied by Keçilioğlu et al. [16] as

$$Q_n = \sum_{r=0}^{7} F_{n+r} e_s \qquad \text{and} \qquad T_n = \sum_{r=0}^{7} L_{n+r} e_s.$$

For some recent works on variants of Fibonacci like quaternions, octonions and their applications, see [1, 2, 6, 11, 17, 20–22, 25, 31].
Nowadays, research works are going on in more generalizations such as hyperbolic quaternions, octonions, etc. involving a special Horadam number sequence. Some recent works on the hyperbolic quaternions and octonions with known number sequences are due to A. Godase for hyperbolic k-Fibonacci and k-Fibonacci Lucas octonions [9] (and quaternions [10]), Özkan et al. [24] for hyperbolic k-Jacobsthal and k-Jacobsthal Lucas quaternions, etc.
In the light of the above papers, here, we define and study the hyperbolic generalized k-Horadam quaternions and octonions, respectively. Then we obtain some properties of these newly established hyper-complex numbers. Finally, we present matrix representations of these hyper-complex numbers and some nice determinant computations.

2. Hyperbolic Generalized k-Horadam Quaternions

In this section, we introduce the hyperbolic generalized k-Horadam quaternions and study their some algebraic properties. First we establish the Binet formula and then give some combinatorial identities of this sequence.
A hyperbolic quaternion $\mathcal{Q}$ is an expression of the form

$$\mathcal{Q} = \mathcal{Q}_1 + \mathcal{Q}_2 e_1 + \mathcal{Q}_3 e_2 + \mathcal{Q}_4 e_3 = \langle\, \mathcal{Q}_1, \mathcal{Q}_2, \mathcal{Q}_3, \mathcal{Q}_4 \rangle\,,$$

where $\mathcal{Q}_1, \mathcal{Q}_2, \mathcal{Q}_3, \mathcal{Q}_4$ are real components and e_0, e_1, e_2, e_3 are hyperbolic quaternion units. The bases of hyperbolic quaternions is $B_4 = \{e_0 = 1, e_1, e_2, e_3\}$, where e_0 is identity satisfy the non-commutative multiplication rule, defined as:

$$e_0 = 1, e_1^2 = e_2^2 = e_3^2 = 1, e_1 e_2 e_3 = 1,$$
$$e_1 e_2 = e_3 = -e_2 e_1,\ e_2 e_3 = e_1 = -e_3 e_2,\ e_3 e_1 = e_2 = -e_1 e_3.$$

Here, Q_1 is the real part of $\mathcal{Q}$ and $\sum_{r=1}^{3} Q_{r+1} e_r$ is vector part of $\mathcal{Q}$ and it is denoted by $R(\mathcal{Q})$ and $V(\mathcal{Q})$, respectively. Thus, $\mathcal{Q} = R(\mathcal{Q}) + V(\mathcal{Q})$.

Definition 2. The hyperbolic generalized k-Horadam quaternions $\{\mathcal{QW}_{k,n}\}_{n\geq 0}$ are defined as

$$\begin{aligned}\mathcal{QW}_{k,n} &= H_{k,n} + H_{k,n+1}e_1 + H_{k,n+2}e_2 + H_{k,n+3}e_3 \\ &= \langle H_{k,n}, H_{k,n+1}, H_{k,n+2}, H_{k,n+3} \rangle .\end{aligned}$$

Here, $R(\mathcal{QW}_{k,n}) = H_{k,n}$ and $V(\mathcal{QW}_{k,n}) = \langle H_{k,n+1}, H_{k,n+2}, H_{k,n+3} \rangle$.
The hyperbolic generalized k-Horadam quaternions $\mathcal{QW}_{k,n}$ can be also extended to negative indices n and they are given as

$$\mathcal{QW}_{k,-n} = H_{k,-n} + H_{k,-n+1}e_1 + H_{k,-n+2}e_2 + H_{k,-n+3}e_3.$$

Definition 3. The conjugate of hyperbolic generalized k-Horadam quaternions $\overline{\mathcal{QW}}_{k,n}$ is defined as

$$\begin{aligned}\overline{\mathcal{QW}}_{k,n} &= H_{k,n} - H_{k,n+1}e_1 - H_{k,n+2}e_2 - H_{k,n+3}e_3 \\ &= \langle H_{k,n}, -H_{k,n+1}, -H_{k,n+2}, -H_{k,n+3} \rangle .\end{aligned}$$

Here, $R(\overline{\mathcal{QW}}_{k,n}) = H_{k,n}$ and $V(\overline{\mathcal{QW}}_{k,n}) = \langle -H_{k,n+1}, -H_{k,n+2}, -H_{k,n+3} \rangle$.

Theorem 1. *The hyperbolic generalized k-Horadam quaternions and their conjugates are related as*

$$\mathcal{QW}_{k,n} + \overline{\mathcal{QW}}_{k,n} = 2H_{k,n}.$$

The norm of $\mathcal{QW}_{k,n}$ is given as

$$N(\mathcal{QW}_{k,n}) = \mathcal{QW}_{k,n}\overline{\mathcal{QW}}_{k,n} = \sqrt{H_{k,n}^2 - H_{k,n+1}^2 - H_{k,n+2}^2 - H_{k,n+3}^2}.$$

In this section, we define the following notations:

$$\tilde{\alpha} = 1 + \alpha e_1 + \alpha^2 e_2 + \alpha^3 e_3 \qquad \text{and} \qquad \tilde{\beta} = 1 + \beta e_1 + \beta^2 e_2 + \beta^3 e_3.$$

Now we give the following useful lemma for proofs of theorems.

Lemma 1. *For $\tilde{\alpha}$ and $\tilde{\beta}$, the following identities are provided.*

1. $\tilde{\alpha}\tilde{\beta} = 2(1 + \beta e_1 + \beta^2 e_2) + (\alpha^3 + \beta^3 + \alpha - \beta)e_3$.
2. $\tilde{\beta}\tilde{\alpha} = 2(1 + \alpha e_1 + \alpha^2 e_2) + (\alpha^3 + \beta^3 - \alpha + \beta)e_3$.
3. $\tilde{\alpha} + \tilde{\beta} = 2 + \phi(k)e_1 + (\phi^2(k) + 2\psi(k))e_2 + (\phi^3(k) + 3\phi(k)\psi(k))e_3$.
4. $\tilde{\alpha}\tilde{\beta} - \tilde{\beta}\tilde{\alpha} = 2\phi(k)(-e_1 - \phi(k)e_2 + e_3)$.
5. $\tilde{\alpha}\tilde{\beta} + \tilde{\beta}\tilde{\alpha} = 2(\tilde{\alpha} + \tilde{\beta})$.

Proof. The identities can be can easily proved by definition of $\tilde{\alpha}$ and $\tilde{\beta}$ using relation (2). □

Theorem 2. *The Binet formula for hyperbolic generalized k-Horadam quaternions $\mathcal{QW}_n$ is given by*

$$\mathcal{QW}_{k,n} = \frac{P\alpha^n\tilde{\alpha} - Q\beta^n\tilde{\beta}}{\alpha - \beta}.$$

Proof. From (3), we have

$$\begin{aligned}
\mathcal{QW}_{k,n} &= \sum_{r=0}^{3} \frac{P\alpha^{n+r} - Q\beta^{n+r}}{\alpha - \beta} e_r \\
&= \frac{1}{\alpha - \beta}\Big(\sum_{r=0}^{3} P\alpha^{n+r}e_r - \sum_{r=0}^{3} Q\beta^{n+r}e_r\Big) \\
&= \frac{1}{\alpha - \beta}\Big(P\alpha^n \sum_{r=0}^{3} \alpha^r e_r - Q\beta^n \sum_{r=0}^{3} \beta^r e_r\Big) \\
&= \frac{1}{\alpha - \beta}\Big(P\alpha^n\tilde{\alpha} - Q\beta^n\tilde{\beta}\Big).
\end{aligned}$$

Thus, this completes the proof. □

Theorem 3. *For hyperbolic generalized k-Horadam quaternions $\mathcal{QW}_{k,n}$, the generating function is given by*

$$G(t) = \frac{\mathcal{QW}_{k,0} + (\mathcal{QW}_{k,1} - \phi(k)\mathcal{QW}_{k,0})t}{1 - \phi(k)t - \psi(k)t^2}.$$

Proof. Let $G(t) = \sum_{n=0}^{\infty} \mathcal{QW}_{k,n}t^n$ be an ordinary generating function for $\mathcal{QW}_{k,n}$. Namely,

$$G(t) = \mathcal{QW}_{k,0} + \mathcal{QW}_{k,1}t + \mathcal{QW}_{k,2}t^2 + \mathcal{QW}_{k,3}t^3 + \dots.$$

Then using Binet formula for hyperbolic generalized k-Horadam quaternions, we obtain

$$\begin{aligned}
G(t) &= \sum_{n=0}^{\infty} \mathcal{QW}_{k,n}t^n \\
&= \sum_{n=0}^{\infty} \frac{P\alpha^n\tilde{\alpha} - Q\beta^n\tilde{\beta}}{\alpha - \beta} t^n
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{\alpha-\beta}\Big(P\tilde{\alpha}\sum_{n=0}^{\infty}\alpha^n t^n - Q\tilde{\beta}\sum_{n=0}^{\infty}\beta^n t^n\Big)\\
&= \frac{1}{\alpha-\beta}\Big(\frac{P\tilde{\alpha}}{1-\alpha t} - \frac{Q\tilde{\beta}}{1-\beta t}\Big)\\
&= \frac{1}{\alpha-\beta}\Big(\frac{P\tilde{\alpha}-Q\tilde{\beta}-t(P\tilde{\alpha}\beta-Q\tilde{\beta}\alpha)}{(1-\alpha t)(1-\beta t)}\Big)\\
&= \frac{1}{\alpha-\beta}\Big(\frac{(P\tilde{\alpha}-Q\tilde{\beta})+t((P\tilde{\alpha}\alpha-Q\tilde{\beta}\beta)-(\alpha+\beta)(P\tilde{\alpha}-Q\tilde{\beta})}{1-(\alpha+\beta)t+\alpha\beta t^2}\Big).
\end{aligned}
$$

Thus using Theorem 2, we have

$$G(t) = \frac{\mathcal{QW}_{k,0} + (\mathcal{QW}_{k,1} - \phi(k)\mathcal{QW}_{k,0})t}{1-\phi(k)t-\psi(k)t^2}.$$

□

Theorem 4. *For hyperbolic generalized k-Horadam quaternions, the exponential generating function is given by*

$$\sum_{n=0}^{\infty}\mathcal{QW}_{k,n}\frac{t^n}{n!} = \frac{P\tilde{\alpha}e^{\alpha t} - Q\tilde{\beta}e^{\beta t}}{\sqrt{\phi^2(k)+4\psi(k)}}.$$

Proof. Using Theorem 2, the result can be easily proved. □

Theorem 5. *For $n, s \in \mathbb{Z}$, we have*

$$\mathcal{QW}_{k,n-s}\mathcal{QW}_{k,n+s} - \mathcal{QW}_{k,n}^2 = \frac{PQ(-\psi(k))^{n-s}(\alpha^s-\beta^s)(\tilde{\alpha}\tilde{\beta}\beta^s - \tilde{\beta}\tilde{\alpha}\alpha^s)}{\phi^2(k)+4\psi(k)}. \tag{4}$$

Proof. Using Theorem 2 in the LHS, we have

$$\mathcal{QW}_{k,n-s}\mathcal{QW}_{k,n+s} - \mathcal{QW}_{k,n}^2 = \Big(\frac{P\alpha^{n-s}\tilde{\alpha} - Q\beta^{n-s}\tilde{\beta}}{\alpha-\beta}\Big)\Big(\frac{P\alpha^{n+s}\tilde{\alpha} - Q\beta^{n+s}\tilde{\beta}}{\alpha-\beta}\Big) - \Big(\frac{P\alpha^{n}\tilde{\alpha} - Q\beta^{n}\tilde{\beta}}{\alpha-\beta}\Big)^2.$$

By using (2) and after some calculations, we find that

$$
\begin{aligned}
\mathcal{QW}_{k,n-s}\mathcal{QW}_{k,n+s} - \mathcal{QW}_{k,n}^2 &= \frac{PQ}{(\alpha-\beta)^2}\Big(-\alpha^{n-s}\beta^{n+s}\tilde{\alpha}\tilde{\beta} - \beta^{n-s}\alpha^{n+s}\tilde{\beta}\tilde{\alpha} + \alpha^n\beta^n\tilde{\alpha}\tilde{\beta} + \beta^n\alpha^n\tilde{\beta}\tilde{\alpha}\Big)\\
&= \frac{PQ(\alpha\beta)^n}{(\alpha-\beta)^2}\left[\tilde{\alpha}\tilde{\beta}\left(1-\frac{\beta^s}{\alpha^s}\right) + \tilde{\beta}\tilde{\alpha}\left(1-\frac{\alpha^s}{\beta^s}\right)\right]\\
&= \frac{PQ(\alpha\beta)^n}{(\alpha-\beta)^2}(\alpha^s-\beta^s)\Big(\frac{\tilde{\alpha}\tilde{\beta}}{\alpha^s} - \frac{\tilde{\beta}\tilde{\alpha}}{\beta^s}\Big)\\
&= \frac{PQ(\alpha\beta)^n(\alpha^s-\beta^s)}{(\alpha-\beta)^2}\Big(\frac{\tilde{\alpha}\tilde{\beta}\beta^s - \tilde{\beta}\tilde{\alpha}\alpha^s}{\alpha^s\beta^s}\Big)\\
&= \frac{PQ(-\psi(k))^{n-s}(\alpha^s-\beta^s)(\tilde{\alpha}\tilde{\beta}\beta^s - \tilde{\beta}\tilde{\alpha}\alpha^s)}{\phi^2(k)+4\psi(k)}.
\end{aligned}
$$

□

Setting $s = 1$ in Theorem 5 gives the following identity for hyperbolic generalized k-Horadam quaternion which is known as the Cassini's identity.

Corollary 1.

$$\mathcal{QW}_{k,n-1}\mathcal{QW}_{k,n+1} - \mathcal{QW}_{k,n}^2 = \frac{PQ(-\psi(k))^{n-1}(\beta\tilde{\alpha}\tilde{\beta} - \alpha\tilde{\beta}\tilde{\alpha})}{\sqrt{\phi^2(k) + 4\psi(k)}}. \tag{5}$$

Theorem 6. *For $n, r \in \mathbb{Z}$, we have*

$$\mathcal{QW}_{k,r}\mathcal{QW}_{k,n+1} - \mathcal{QW}_{k,r+1}\mathcal{QW}_{k,n} = \frac{PQ(-\psi(k))^n}{\sqrt{\phi^2(k) + 4\psi(k)}}\Big(\tilde{\alpha}\tilde{\beta}\alpha^{r-n} - \tilde{\beta}\tilde{\alpha}\beta^{r-n}\Big).$$

Proof. By virtue of Theorem 2, we get

$$\begin{aligned}\mathcal{QW}_{k,r}\mathcal{QW}_{k,n+1} - \mathcal{QW}_{k,r+1}\mathcal{QW}_{k,n} &= \Big(\frac{P\alpha^r\tilde{\alpha} - Q\beta^r\tilde{\beta}}{\alpha - \beta}\Big)\Big(\frac{P\alpha^{n+1}\tilde{\alpha} - Q\beta^{n+1}\tilde{\beta}}{\alpha - \beta}\Big) \\ &\quad -\Big(\frac{P\alpha^{r+1}\tilde{\alpha} - Q\beta^{r+1}\tilde{\beta}}{\alpha - \beta}\Big)\Big(\frac{P\alpha^n\tilde{\alpha} - Q\beta^n\tilde{\beta}}{\alpha - \beta}\Big).\end{aligned}$$

On simplification and using (2), it gives

$$\begin{aligned}\mathcal{QW}_{k,r}\mathcal{QW}_{k,n+1} - \mathcal{QW}_{k,r+1}\mathcal{QW}_{k,n} &= \frac{PQ}{(\alpha - \beta)^2}\Big(\tilde{\alpha}\tilde{\beta}\alpha^r\beta^n(\alpha - \beta) + \tilde{\beta}\tilde{\alpha}\beta^r\alpha^n(\beta - \alpha)\Big) \\ &= \frac{PQ(\alpha - \beta)}{(\alpha - \beta)^2}(\tilde{\alpha}\tilde{\beta}\alpha^r\beta^n - \tilde{\beta}\tilde{\alpha}\beta^r\alpha^n) \\ &= \frac{PQ(-\psi(k))^n}{\sqrt{\phi^2(k) + 4\psi(k)}}\Big(\tilde{\alpha}\tilde{\beta}\alpha^{r-n} - \tilde{\beta}\tilde{\alpha}\beta^{r-n}\Big).\end{aligned}$$

□

Theorem 7 (Finite sum formula). *For $\phi(k) + \psi(k) \neq 1$, we have*

$$\sum_{r=0}^{n}\mathcal{QW}_{k,r} = \frac{\mathcal{QW}_{k,1} + (1 - \phi(k))\mathcal{QW}_{k,0} - \psi(k)\mathcal{QW}_{k,n} - \mathcal{QW}_{k,n+1}}{1 - \phi(k) - \psi(k)}.$$

Proof. From Theorem 2, we write L.H.S. as

$$\begin{aligned}\sum_{r=0}^{n}\Big(\frac{P\alpha^r\tilde{\alpha} - Q\beta^r\tilde{\beta}}{\alpha - \beta}\Big) &= \frac{P\tilde{\alpha}}{\alpha - \beta}\sum_{r=0}^{n}\alpha^r - \frac{Q\tilde{\beta}}{\alpha - \beta}\sum_{r=0}^{n}\beta^r \\ &= \frac{P\tilde{\alpha}}{\alpha - \beta}\frac{\alpha^{n+1} - 1}{\alpha - 1} - \frac{Q\tilde{\beta}}{\alpha - \beta}\frac{1 - \beta^{n+1}}{1 - \beta} \\ &= \frac{1}{\alpha - \beta}\Big(\frac{P\tilde{\alpha}\alpha^{n+1} - P\tilde{\alpha}}{\alpha - 1} - \frac{Q\tilde{\beta} - Q\tilde{\beta}\beta^{n+1}}{1 - \beta}\Big) \\ &= \frac{1}{\alpha - \beta}\Big(\frac{P\tilde{\alpha}\alpha^{n+1}\beta - P\tilde{\alpha}\beta - P\tilde{\alpha}\alpha^{n+1} + P\tilde{\alpha}}{(\alpha - 1)(\beta - 1)} \\ &\quad - \frac{Q\tilde{\beta}\beta^{n+1}\alpha - Q\tilde{\beta}\alpha - Q\tilde{\beta}\beta^{n+1} + Q\tilde{\beta}}{(\alpha - 1)(\beta - 1)}\Big).\end{aligned}$$

Since $(\alpha-1)(\beta-1)=1-\phi(k)-\psi(k)$, thus using Theorem 2, we get

$$\sum_{r=0}^{n}\mathcal{QW}_{k,r}=\frac{\mathcal{QW}_{k,1}+(1-\phi(k))\mathcal{QW}_{k,0}-\psi(k)\mathcal{QW}_{k,n}-\mathcal{QW}_{k,n+1}}{1-\phi(k)-\psi(k)}.$$

□

3. Hyperbolic Generalized k-Horadam Octonions

Now, we introduce the hyperbolic generalized k-Horadam octonions and study their properties. We establish the Binet formula and then we present some well-known identities of them.
Cariow, et al. [3, 4] defined the concept of hyperbolic octonion $\mathcal{O}$, which is expressed as

$$\begin{aligned}\mathcal{O} &= h_0+h_1i_1+h_2i_2+h_3i_3+h_4i_4+h_5i_5+h_6i_6+h_7i_7\\ &= <h_0,h_1,h_2,h_3,h_4,h_5,h_6,h_7>\end{aligned}$$

where $h_0,h_1,h_2,h_3,h_4,h_5,h_6,h_7$ are the real components and i_1,i_2,i_3 are quaternion imaginary units, $i_4({i_4}^2=1)$ is a counter imaginary unit. For hyperbolic octonions, bases are defined as follows:

$$i_1i_4=i_5,\ i_2i_4=i_6,\ i_3i_4=i_7,\ {i_4}^2={i_5}^2={i_6}^2={i_7}^2=1.$$

The multiplication rules for the bases of hyperbolic octonions $\mathcal{O}$ as mentioned in [3, 4] are given in the Table 2.

.	i_1	i_2	i_3	i_4	i_5	i_6	i_7
i_1	-1	i_3	$-i_2$	i_5	i_4	$-i_7$	i_6
i_2	$-i_3$	-1	i_1	i_6	i_7	i_4	$-i_5$
i_3	i_2	$-i_1$	-1	i_7	$-i_6$	i_5	i_4
i_4	$-i_5$	$-i_6$	$-i_7$	1	i_1	i_2	i_3
i_5	$-i_4$	$-i_7$	i_6	$-i_1$	1	i_3	$-i_2$
i_6	i_7	$-i_4$	$-i_5$	$-i_2$	$-i_3$	1	i_1
i_7	$-i_6$	i_5	$-i_4$	$-i_3$	i_2	$-i_1$	1

Table 2. **Multiplication rules for hyperbolic octonions units.**

Definition 4. Let $\overline{\mathcal{O}}$ be the conjugate of hyperbolic octonion $\mathcal{O}$. Then the norm is defined as

$$N(\mathcal{O})=\sqrt{\mathcal{O}\overline{\mathcal{O}}}.$$

Definition 5. The hyperbolic generalized k-Horadam octonion $\{\mathcal{OW}_{k,n}\}_{n\geq 0}$ is defined by

$$\begin{aligned}\mathcal{OW}_{k,n} &= H_{k,n} + H_{k,n+1}i_1 + \cdots + H_{k,n+7}i_7\\ &= \langle H_{k,n}, H_{k,n+1}, \ldots, H_{k,n+7}\rangle,\end{aligned}$$

where $H_{k,n}$is the nth generalized k-Horadam numbers.

Definition 6. The conjugate of hyperbolic generalized k-Horadam octonion i.e. $\overline{\mathcal{OW}}_{k,n}$ is defined by

$$\begin{aligned}\overline{\mathcal{OW}}_{k,n} &= H_{k,n} - H_{k,n+1}i_1 - \cdots - H_{k,n+7}i_7\\ &= \langle H_{k,n}, -H_{k,n+1}, \ldots, -H_{k,n+7}\rangle.\end{aligned}$$

Theorem 8. *For $n \geq 0$, the following recurrences are verified.*

1. $\mathcal{OW}_{k,n+2} = \phi(k)\mathcal{OW}_{k,n+1} + \psi(k)\mathcal{OW}_{k,n}$,
2. $\overline{\mathcal{OW}}_{k,n+2} = \phi(k)\overline{\mathcal{OW}}_{k,n+1} + \psi(k)\overline{\mathcal{OW}}_{k,n}$.

Theorem 9. *Let $\psi(k) \neq 0$, then in negative indices n, the hyperbolic generalized k-Horadam octonions are given by*

1. $\mathcal{OW}_{k,-n} = \frac{1}{\psi(k)}(\mathcal{OW}_{k,-n+2} - \phi(k)\mathcal{OW}_{k,-n+1})$,
2. $\overline{\mathcal{OW}}_{k,-n} = \frac{1}{\psi(k)}(\overline{\mathcal{OW}}_{k,-n+2} - \phi(k)\overline{\mathcal{OW}}_{k,-n+1})$.

Let $R(\mathcal{OW}_{k,n})$ and $V(\mathcal{OW}_{k,n})$ represent the real and vector parts, respectively, of $\mathcal{OW}_{k,n}$ and they are defined as follows:

$$\begin{aligned}R(\mathcal{OW}_{k,n}) &= H_{k,n},\\ V(\mathcal{OW}_{k,n}) &= \langle H_{k,n+1}, H_{k,n+2}, H_{k,n+3}, \ldots, H_{k,n+7}\rangle.\end{aligned}$$

Thus,

$$\mathcal{OW}_{k,n} = R(\mathcal{OW}_{k,n}) + V(\mathcal{OW}_{k,n}).$$

Similarly, $\overline{\mathcal{OW}}_{k,n} = R(\mathcal{OW}_{k,n}) - V(\mathcal{OW}_{k,n})$.

Theorem 10. *The following equations are provided.*

1. $\mathcal{OW}_{k,n} + \overline{\mathcal{OW}}_{k,n} = 2R(\mathcal{OW}_{k,n}) = 2H_{k,n}$,
2. $\mathcal{OW}_{k,n} - \overline{\mathcal{OW}}_{k,n} = 2V(\mathcal{OW}_{k,n})$.

Now we give Binet like formula and establish some results for the hyperbolic generalized k-Horadam octonions. For rest of the paper, the following symbols has been defined

$$\overline{\alpha} = \sum_{r=0}^{7} \alpha^r i_r, \quad \overline{\alpha}^* = 1 - \sum_{r=1}^{7} \alpha^r i_r, \quad \overline{\beta} = \sum_{r=0}^{7} \beta^r i_r, \quad \text{and} \quad \overline{\beta}^* = 1 - \sum_{r=1}^{7} \beta^r i_r.$$

Theorem 11. *The Binet's formulas for hyperbolic octonions $\mathcal{OW}_n$ and $\overline{\mathcal{OW}}_{k,n}$ are given by*

$$\mathcal{OW}_{k,n} = \frac{P\alpha^n\overline{\alpha} - Q\beta^n\overline{\beta}}{\alpha-\beta} \quad \textit{and} \quad \overline{\mathcal{OW}}_{k,n} = \frac{P\alpha^n\overline{\alpha}^* - Q\beta^n\overline{\beta}^*}{\alpha-\beta}.$$

Proof. From Definition 5 and relation (3), we have

$$\begin{aligned}
\mathcal{OW}_{k,n} &= \sum_{r=0}^{7} H_{k,n+r} i_r = \sum_{r=0}^{7} \frac{P\alpha^{n+r} - Q\beta^{n+r}}{\alpha-\beta} i_r \\
&= \frac{1}{\alpha-\beta}\Big(\sum_{r=0}^{7} P\alpha^{n+r} i_r - \sum_{r=0}^{7} Q\beta^{n+r} i_r\Big) \\
&= \frac{1}{\alpha-\beta}\Big(P\alpha^n \sum_{r=0}^{7} \alpha^r i_r - Q\beta^n \sum_{r=0}^{7} \beta^r i_r\Big) \\
&= \frac{1}{\alpha-\beta}\Big(P\alpha^n\overline{\alpha} - Q\beta^n\overline{\beta}\Big).
\end{aligned}$$

The proofs of second part is similar to first part by using the Definition 6. □

Lemma 2. *The following identities are verified:*

1. $P\overline{\alpha} - Q\overline{\beta} = (\alpha-\beta)\mathcal{OW}_{k,0}$,
2. $P\overline{\alpha}^* - Q\overline{\beta}^* = (\alpha-\beta)\overline{\mathcal{OW}}_{k,0}$,
3. $\overline{\alpha} + \overline{\alpha}^* = \overline{\beta} + \overline{\beta}^* = 2$.

Theorem 12 (Finite sum formula). *For $\phi(k) + \psi(k) \neq 1$, we have*

$$\sum_{r=0}^{n} \mathcal{OW}_{k,r} = \frac{\mathcal{OW}_{k,1} + (1-\phi(k))\mathcal{OW}_{k,0} - \psi(k)\mathcal{OW}_{k,n} - \mathcal{OW}_{k,n+1}}{1-\phi(k)-\psi(k)}.$$

Proof. From Theorem 11, we have

$$\begin{aligned}
\sum_{r=0}^{n} \mathcal{OW}_{k,r} &= \sum_{r=0}^{n} \left(\frac{P\alpha^r\overline{\alpha} - Q\beta^r\overline{\beta}}{\alpha-\beta}\right) \\
&= \frac{P\overline{\alpha}}{\alpha-\beta}\sum_{r=0}^{n} \alpha^r - \frac{Q\overline{\beta}}{\alpha-\beta}\sum_{r=0}^{n} \beta^r
\end{aligned} \tag{6}$$

$$
\begin{aligned}
&= \frac{P\overline{\alpha}(1-\beta-\alpha^{n+1}+\beta\alpha^{n+1}) - Q\overline{\beta}(1-\alpha-\beta^{n+1}+\alpha\beta^{n+1})}{(\alpha-\beta)(1-(\alpha+\beta)+\alpha\beta)} \\
&= \frac{(P\overline{\alpha}-Q\overline{\beta}) - \alpha\beta(P\overline{\alpha}\alpha^{-1}-Q\overline{\beta}\beta^{-1}) - (P\overline{\alpha}\alpha^{n+1}-Q\overline{\beta}\beta^{n+1}) + \alpha\beta(P\overline{\alpha}\alpha^{n}-Q\overline{\beta}\beta^{n})}{(\alpha-\beta)(1-\phi(k)-\psi(k))} \\
&= \frac{\mathcal{OW}_{k,0} + \psi(k)\mathcal{OW}_{k,-1} - \mathcal{OW}_{k,n+1} - \psi(k)\mathcal{QW}_{k,n}}{1-\phi(k)-\psi(k)}.
\end{aligned}
$$

Thus, by virtue of Theorems 8 and 9, we have

$$
\sum_{r=0}^{n} \mathcal{OW}_{k,r} = \frac{\mathcal{OW}_{k,1} + (1-\phi(k))\mathcal{OW}_{k,0} - \psi(k)\mathcal{OW}_{k,n} - \mathcal{OW}_{k,n+1}}{1-\phi(k)-\psi(k)}.
$$

□

Theorem 13. *For hyperbolic generalized k-Horadam octonions, the generating function is given by*

$$
\sum_{n=0}^{\infty} \mathcal{OW}_{k,n} t^n = \frac{\mathcal{OW}_{k,0} + (\mathcal{OW}_{k,1} - \phi(k)\mathcal{OW}_{k,0})t}{1-\phi(k)t-\psi(k)t^2}.
$$

Proof. Consider the ordinary generating function $f(t) = \sum_{n=0}^{\infty} \mathcal{OW}_{k,n} t^n$ i.e.

$$
f(t) = \mathcal{OW}_{k,0} + \mathcal{OW}_{k,1}t + \mathcal{OW}_{k,2}t^2 + \mathcal{OW}_{k,3}t^3 + \dots .
$$

In order to obtain the generating function, we need to calculate $\sum_{n=0}^{\infty} \mathcal{OW}_{k,n+2} t^{n+2}$ and $\sum_{n=0}^{\infty} \mathcal{OW}_{k,n+1} t^{n+2}$ that are given as

$$
\sum_{n=0}^{\infty} \mathcal{OW}_{k,n+2} t^{n+2} = f(t) - \mathcal{OW}_{k,0} - \mathcal{OW}_{k,1}t \quad \text{and} \quad \sum_{n=0}^{\infty} \mathcal{OW}_{k,n+1} t^{n+2} = t(f(t) - \mathcal{OW}_{k,0}).
$$

On performing necessary mathematical operations with Theorem 8(1), we get the following.

$$
\begin{aligned}
f(t) - \mathcal{OW}_{k,0} - \mathcal{OW}_{k,1}t - \phi(k)t(f(t) - \mathcal{OW}_{k,0}) - \psi(k)t^2 f(t) = 0 \\
f(t)(1-\phi(k)t-\psi(k)t^2) = \mathcal{OW}_{k,0} + \mathcal{OW}_{k,1}t - \phi(k)t\mathcal{OW}_{k,0}.
\end{aligned}
$$

Thus, we have

$$
f(t) = \frac{\mathcal{OW}_{k,0} + (\mathcal{OW}_{k,1} - \phi(k)\mathcal{OW}_{k,0})t}{1-\phi(k)t-\psi(k)t^2},
$$

as required. □

Theorem 14. *For hyperbolic generalized k-Horadam octonions, the exponential generating function is given by*

$$
\sum_{n=0}^{\infty} \mathcal{OW}_{k,n} \frac{t^n}{n!} = \frac{P\overline{\alpha}e^{\alpha t} - Q\overline{\beta}e^{\beta t}}{\alpha-\beta}.
$$

Proof. It can be easily proved using Theorem 11. $\square$

Theorem 15. *For hyperbolic generalized k-Horadam octonions, the norm is given by*

$$N(\mathcal{OW}_{k,n})=\sqrt{\frac{(1-\alpha^8)P^2\alpha^{2n}}{(\alpha-\beta)^2}\hat{\alpha}+\frac{(1-\beta^8)Q^2\beta^{2n}}{(\alpha-\beta)^2}\hat{\beta}+\frac{2PQ(\psi(k)^4-1)(-\psi(k))^n}{(\alpha-\beta)^2}\hat{g}},$$

where $\hat{\alpha}=\sum_{r=0}^{3}\alpha^{2r}$, $\hat{\beta}=\sum_{r=0}^{3}\beta^{2r}$ *and* $\hat{g}=\sum_{r=0}^{3}(-\psi(k))^r$.

Proof. From Definition 4 and Theorem 11, we have

$$\begin{aligned}
N^2(\mathcal{OW}_{k,n}) &= H_{k,n}^2+H_{k,n+1}^2+H_{k,n+2}^2+H_{k,n+3}^2-H_{k,n+4}^2-H_{k,n+5}^2-H_{k,n+6}^2-H_{k,n+7}^2\\
&=\sum_{r=0}^{3}(H_{k,n+r}^2-H_{k,n+r+4}^2)\\
&=\sum_{r=0}^{3}\left[\left(\frac{P\alpha^{n+r}-Q\beta^{n+r}}{\alpha-\beta}\right)^2-\left(\frac{P\alpha^{n+r+4}-Q\beta^{n+r+4}}{\alpha-\beta}\right)^2\right]\\
&=\frac{1}{(\alpha-\beta)^2}\sum_{r=0}^{3}\Big(P^2\alpha^{2(n+r)}+Q^2\beta^{2(n+r)}-P^2\alpha^{2(n+r+4)}-Q^2\beta^{2(n+r+4)}\\
&\quad+2PQ(\alpha\beta)^{n+r}((\alpha\beta)^4-1)\Big)\\
&=\frac{1}{(\alpha-\beta)^2}\sum_{r=0}^{3}\Big(P^2\alpha^{2(n+r)}(1-\alpha^8)+Q^2\beta^{2(n+r)}(1-\beta^8)+2PQ(-\psi(k))^{n+r}(\psi(k)^4-1)\Big)\\
&=\frac{(1-\alpha^8)P^2\alpha^{2n}}{(\alpha-\beta)^2}\sum_{r=0}^{3}\alpha^{2r}+\frac{(1-\beta^8)Q^2\beta^{2n}}{(\alpha-\beta)^2}\sum_{r=0}^{3}\beta^{2r}\\
&\quad+\frac{2PQ(\psi(k)^4-1)(-\psi(k))^n}{(\alpha-\beta)^2}\sum_{r=0}^{3}(-\psi(k))^r\\
&=\frac{(1-\alpha^8)P^2\alpha^{2n}}{(\alpha-\beta)^2}\hat{\alpha}+\frac{(1-\beta^8)Q^2\beta^{2n}}{(\alpha-\beta)^2}\hat{\beta}+\frac{2PQ(\psi(k)^4-1)(-\psi(k))^n}{(\alpha-\beta)^2}\hat{g},
\end{aligned}$$

where $\hat{\alpha}=\sum_{r=0}^{3}\alpha^{2r}$, $\hat{\beta}=\sum_{r=0}^{3}\beta^{2r}$ and $\hat{g}=\sum_{r=0}^{3}(-\psi(k))^r$. $\square$

Theorem 16 (Catalan's Identity). *For any integers n and s, we write*

$$\mathcal{OW}_{k,n-s}\mathcal{OW}_{k,n+s}-\mathcal{OW}_{k,n}^2=\frac{PQ(\alpha\beta)^{n-s}(\alpha^s-\beta^s)(\overline{\alpha}\overline{\beta}\beta^s-\overline{\beta}\overline{\alpha}\alpha^s)}{(\alpha-\beta)^2}. \tag{7}$$

Proof. It can be easily proved using Binet's formula (Theorem 11) of hyperbolic octonions $\mathcal{OW}_n$. $\square$

Substituting $s=1$ in the Theorem 16 proves the Cassini's identity of the hyperbolic generalized k-Horadam octonions, provided in the next corollary.

Corollary 2. *For integer n, we have*

$$\mathcal{OW}_{k,n-1}\mathcal{OW}_{k,n+1} - \mathcal{OW}_{k,n}^2 = \frac{PQ(\alpha\beta)^{n-1}(\overline{\alpha}\overline{\beta}\beta - \overline{\beta}\overline{\alpha}\alpha)}{(\alpha-\beta)}.$$

Theorem 17. *For $n, r \in \mathbb{Z}$, the d'Ocagne identity is given as*

$$\mathcal{OW}_{k,r}\mathcal{OW}_{k,n+1} - \mathcal{OW}_{k,r+1}\mathcal{OW}_{k,n} = \frac{PQ(-\psi(k))^n}{(\alpha-\beta)}\Big(\overline{\alpha}\overline{\beta}\alpha^{r-n} - \overline{\beta}\overline{\alpha}\beta^{r-n}\Big).$$

Proof. Using Binet formula as given in Theorem 11, we have

$$\begin{aligned}
\mathcal{OW}_{k,r}\mathcal{OW}_{k,n+1} - \mathcal{OW}_{k,r+1}\mathcal{OW}_{k,n} &= \Big(\frac{P\alpha^r\overline{\alpha} - Q\beta^r\overline{\beta}}{\alpha-\beta}\Big)\Big(\frac{P\alpha^{n+1}\overline{\alpha} - Q\beta^{n+1}\overline{\beta}}{\alpha-\beta}\Big) \\
&\quad -\Big(\frac{P\alpha^{r+1}\overline{\alpha} - Q\beta^{r+1}\overline{\beta}}{\alpha-\beta}\Big)\Big(\frac{P\alpha^{n}\overline{\alpha} - Q\beta^{n}\overline{\beta}}{\alpha-\beta}\Big) \\
&= \frac{PQ}{(\alpha-\beta)^2}\Big(\overline{\alpha}\overline{\beta}\alpha^r\beta^n(\alpha-\beta) + \overline{\beta}\overline{\alpha}\beta^r\alpha^n(\beta-\alpha)\Big) \\
&= \frac{PQ(\alpha-\beta)}{(\alpha-\beta)^2}(\overline{\alpha}\overline{\beta}\alpha^r\beta^n - \overline{\beta}\overline{\alpha}\beta^r\alpha^n) \\
&= \frac{PQ(-\psi(k))^n}{(\alpha-\beta)}\Big(\overline{\alpha}\overline{\beta}\alpha^{r-n} - \overline{\beta}\overline{\alpha}\beta^{r-n}\Big).
\end{aligned}$$

□

4. Some Applications of Hyperbolic Generalized k-Horadam Quaternions and Octonions to Matrices

Now, we obtain the matrix representation of hyperbolic generalized k-Horadam quaternions and octonions. After that, we give closed form formula for the hyperbolic generalized k-Horadam quaternions $\mathcal{QW}_{k,n}$, and hyperbolic generalized k-Horadam octonions $\mathcal{OW}_{k,n}$, in terms of tridiagonal determinants. Based on papers [5, 7, 19, 23, 26, 30, 33], we give the following results with a similar approach.

Theorem 18. *Let $n \in \mathbb{N}$, then the following equalities hold:*

$$\begin{pmatrix} \mathcal{QW}_{k,n+3} & \mathcal{QW}_{k,n+2} \\ \mathcal{QW}_{k,n+2} & \mathcal{QW}_{k,n+1} \end{pmatrix} = \begin{pmatrix} \mathcal{QW}_{k,3} & \mathcal{QW}_{k,2} \\ \mathcal{QW}_{k,2} & \mathcal{QW}_{k,1} \end{pmatrix}\begin{pmatrix} \phi(k) & 1 \\ \psi(k) & 0 \end{pmatrix}^n, \tag{8}$$

$$\begin{pmatrix} \mathcal{OW}_{k,n+3} & \mathcal{OW}_{k,n+2} \\ \mathcal{OW}_{k,n+2} & \mathcal{OW}_{k,n+1} \end{pmatrix} = \begin{pmatrix} \mathcal{OW}_{k,3} & \mathcal{OW}_{k,2} \\ \mathcal{OW}_{k,2} & \mathcal{OW}_{k,1} \end{pmatrix}\begin{pmatrix} \phi(k) & 1 \\ \psi(k) & 0 \end{pmatrix}^n. \tag{9}$$

Proof. We prove it by induction on n. Clearly, the equality (8) is true for $n = 1$. Now assume that the result is true for $n > 1$. So, to verify the result is true for $n+1$, we have

$$\begin{aligned}
\begin{pmatrix} \mathcal{QW}_{k,3} & \mathcal{QW}_{k,2} \\ \mathcal{QW}_{k,2} & \mathcal{QW}_{k,1} \end{pmatrix} \begin{pmatrix} \phi(k) & 1 \\ \psi(k) & 0 \end{pmatrix}^{n+1} &= \begin{pmatrix} \mathcal{QW}_{k,3} & \mathcal{QW}_{k,2} \\ \mathcal{QW}_{k,2} & \mathcal{QW}_{k,1} \end{pmatrix} \begin{pmatrix} \phi(k) & 1 \\ \psi(k) & 0 \end{pmatrix}^{n} \begin{pmatrix} \phi(k) & 1 \\ \psi(k) & 0 \end{pmatrix} \\
&= \begin{pmatrix} \mathcal{QW}_{k,n+3} & \mathcal{QW}_{k,n+2} \\ \mathcal{QW}_{k,n+2} & \mathcal{QW}_{k,n+1} \end{pmatrix} \begin{pmatrix} \phi(k) & 1 \\ \psi(k) & 0 \end{pmatrix} \\
&= \begin{pmatrix} \mathcal{QW}_{k,n+4} & \mathcal{QW}_{k,n+3} \\ \mathcal{QW}_{k,n+3} & \mathcal{QW}_{k,n+2} \end{pmatrix}.
\end{aligned}$$

So the proof is completed. The assertion (9) can be proved similarly. $\square$

Now we give different versions of Cassini identity for hyperbolic generalized k-Horadam quaternions and hyperbolic generalized k-Horadam octonions.

Corollary 3. *For $n \in \mathbb{N}$, we have*

$$\mathcal{QW}_{k,n+1}\mathcal{QW}_{k,n-1} - (\mathcal{QW}_{k,n})^2 = (-\psi(k))^{n-2}\left(\mathcal{QW}_{k,3}\mathcal{QW}_{k,1} - (\mathcal{QW}_{k,2})^2\right),$$

$$\mathcal{QW}_{k,n-1}\mathcal{QW}_{k,n+1} - (\mathcal{QW}_{k,n})^2 = (-\psi(k))^{n-2}\left(\mathcal{QW}_{k,1}\mathcal{QW}_{k,3} - (\mathcal{QW}_{k,2})^2\right),$$

$$\mathcal{OW}_{k,n+1}\mathcal{OW}_{k,n-1} - (\mathcal{OW}_{k,n})^2 = (-\psi(k))^{n-2}\left(\mathcal{OW}_{k,3}\mathcal{OW}_{k,1} - (\mathcal{OW}_{k,2})^2\right),$$

$$\mathcal{OW}_{k,n-1}\mathcal{OW}_{k,n+1} - (\mathcal{OW}_{k,n})^2 = (-\psi(k))^{n-2}\left(\mathcal{OW}_{k,1}\mathcal{OW}_{k,3} - (\mathcal{OW}_{k,2})^2\right).$$

Proof. The first claim of the Theorem can be achieved by taking the determinant on the both sides of Eqn. (8). Other assertions of the theorem are obtained similarly. $\square$

The nth term of hyperbolic k-Horadam quaternions can be obtained via the computation of the determinant of the tridiagonal matrix $M\mathcal{QW}_{k,n-1}$.
The hyperbolic k-Horadam quaternions are expressed via the determinant of the following matrix:

$$M\mathcal{QW}_{k,n} = \begin{pmatrix} \mathcal{QW}_{k,2} & \mathcal{QW}_{k,1} & & & & \\ -\psi(k) & \phi(k) & 1 & & & \\ & -\psi(k) & \phi(k) & 1 & & \\ & & \ddots & \ddots & \ddots & \\ & & & -\psi(k) & \phi(k) & 1 \\ & & & & -\psi(k) & \phi(k) \end{pmatrix}, \tag{10}$$

satisfy

$$|M\mathcal{QW}_{k,n}| = \mathcal{QW}_{k,n+1}.$$

Moreover, the hyperbolic k-Horadam octonions are expressed via the determinant of the following matrix:

$$M\mathcal{OW}_{k,n}=\begin{pmatrix} \mathcal{OW}_{k,2} & \mathcal{OW}_{k,1} & & & & \\ -\psi(k) & \phi(k) & 1 & & & \\ & -\psi(k) & \phi(k) & 1 & & \\ & & \ddots & \ddots & \ddots & \\ & & & -\psi(k) & \phi(k) & 1 \\ & & & & -\psi(k) & \phi(k) \end{pmatrix}, \tag{11}$$

satisfy

$$|M\mathcal{OW}_{k,n}|=\mathcal{OW}_{k,n+1}.$$

Note that one of the special cases of the generalized k-Horadam polynomials is Chebyshev polynomial of the second kind which satisfy the following relation:

$$U_{n+1}(x)=2xU_n(x)-U_{n-1}(x),\ n\geq 0 \quad \text{with} \quad U_{-1}(x)=0, U_1(x)=1.$$

Now we examine the matrix $M\mathcal{QW}_{k,n}$. We can present the hyperbolic generalized k-Horadam quaternions, by means of the Chebyshev polynomials of the second kind, given as

$$\mathcal{QW}_{k,n+1}=\left(-i\sqrt{\psi(k)}\right)^{n-1}\left(U_{n-1}\left(\frac{\phi(k)}{2\sqrt{\psi(k)}}\right)\mathcal{QW}_{k,2}+i\sqrt{\psi(k)}U_{n-2}\left(\frac{\phi(k)}{2\sqrt{\psi(k)}}\right)\mathcal{QW}_{k,1}\right), \tag{12}$$

where

$$\begin{aligned}\mathcal{QW}_{k,1}=be_1+(a\psi(k)+b\phi(k))\,e_2+\left(a\phi(k)\psi(k)+b\left[\phi^2(k)+\psi(k)\right]\right)e_3\\ +\left(a\left[\phi^2(k)\psi(k)+\psi(k)\right]+b\left[\phi^3(k)+2\phi(k)\psi(k)\right]\right)e_4,\end{aligned}$$

and

$$\begin{aligned}\mathcal{QW}_{k,2}=(a\psi(k)+b\phi(k))\,e_1+\left(a\phi(k)\psi(k)+b\left[\phi^2(k)+\psi(k)\right]\right)e_2\\ +\left(a\left[\phi^2(k)\psi(k)+\psi(k)\right]+b\left[\phi^3(k)+2\phi(k)\psi(k)\right]\right)e_3\\ +\left(a\phi(k)\psi(k)\left[\phi^2(k)+2\psi(k)\right]+b\left[\phi^4(k)+3\phi^2(k)\psi(k)+\psi^2(k)\right]\right)e_4.\end{aligned}$$

As an illustrative example, substituting $n=10$, $a=0$, $b=\phi(k)=\psi(k)=1$, in Eqn. (12), we have the hyperbolic Fibonacci quaternion $\mathcal{QF}_{k,11}$. Namely,

$$\mathcal{QF}_{k,11}=(-i)\left(U_9\left(\frac{1}{2}\right)\mathcal{QF}_{k,2}+iU_8\left(\frac{1}{2}\right)\mathcal{QF}_{k,1}\right),$$

where

$$\mathcal{QF}_{k,1}=e_1+e_2++2e_3+3e_4 \quad \text{and} \quad \mathcal{QF}_{k,2}=e_1+2e_2+3e_3+5e_4.$$

5. Conclusion

In summary, we have examined the hyperbolic generalized k-Horadam quaternions and hyperbolic k-Horadam octonions. We have obtained some new properties and identities of these types of hypercomplex numbers. By the use of tridiagonal matrix, we have achieved formula for the nth element of hyperbolic generalized k-Horadam quaternions and hyperbolic k-Horadam octonions. Moreover, the determinants of the tridiagonal matrix has been obtained through the Chebyshev polynomials of the second kind. For particular cases of a, b, $\phi(k)$, and $\psi(k)$, all results are applicable to hyperbolic quaternions and octonions whose components are special polynomials and numbers defined by a second-order recurrence relation.

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