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Short Note

A note on the small quasi-kernels conjecture in digraphs

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Abstract: A subset K of vertices of digraph $D = (V(D), A(D))$ is a kernel if the following two conditions are fulfilled: (i) no two vertices of K are connected by an arc in any direction and (ii) every vertex not in K has an ingoing arc from some vertex in K. A quasi-kernel of D is a subset Q of vertices satisfying condition (i) and furthermore every vertex can be reached in at most two steps from Q. A vertex is source-free if it has at least one ingoing arc. In 1976, P.L. Erdös and L.A. Székely conjectured that every source-free digraph D has a quasi-kernel of size at most $|V(D)|/2$. Recently, this conjecture has been shown to be true by Allan van Hulst for digraphs having kernels. In this note, we provide a short and simple proof of van Hulst's result. We additionally characterize all source-free digraphs D having kernels with smallest quasi-kernels of size $|V(D)|/2$.

Keywords: Digraphs, kernel, quasi-kernel

AMS Subject classification: 05C20, 05C69

1. Introduction

We make use of the standard terminology and notation on digraphs as given in [1]. However, we still provide most of the necessary definitions for the convenience of the reader. Let D be a digraph with vertex set $V(D)$ and arc set $A(D)$. In our definitions of a digraph, we do not allow neither multiple arcs joining two vertices in the same direction nor loops. If xy is an arc of a digraph D, then we say x *dominates* y or y is *dominated* by x. We will say moreover that y is an *out-neighbor* of x, and x is an *in-neighbor* of y. The *outset* of a vertex x, denoted by $N_D^+(x)$, consists of all out-neighbors of x while the *inset* of x, denoted by $N_D^-(x)$, consists of all in-neighbors

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of x. Any vertex of D without in-neighbors is called a *source*. A *source-free digraph* is a digraph without a source vertex. For a subset S of $V(D)$, the outset of S is $N_D^+(S) = \cup_{u \in S} N_D^+(u)$ and the inset of S is $N_D^-(S) = \cup_{u \in S} N_D^-(u)$. A digraph D is *connected* if its underlying graph (graph obtained from D by deleting directions) is a connected graph. We denote by $\overrightarrow{C_2}$ the digraph with two vertices and two arcs, one in each direction. For $n \geq 3$, $\overrightarrow{C_n} = (x_1, x_2, \ldots, x_n, x_1)$ denotes the directed cycle with arcs $x_n x_1$ and $x_i x_{i+1}$, for $i = 1, \ldots, n-1$.
A subset $S \subseteq V(D)$ is *independent* if no two vertices of S are connected by an arc in any direction. The set S is a *dominating set* if every vertex of $V(D) - S$ is dominated by at least one vertex of S, that is every vertex not in S has an in-neighbor in S. A subset $K \subseteq V(D)$ is a *kernel* of D if and only if K is both independent and dominating in D. A *quasi-kernel* Q is an independent set such that every vertex can be reached in at most two steps from Q, that is for every vertex not in Q, either $\left|N_D^-(x) \cap Q\right| \geq 1$ or $\left|N_D^-(N_D^-(x)) \cap Q\right| \geq 1$. By this definition, it is clear that any kernel is also a quasi-kernel.
In 1976, Erdös and Székely proposed the following conjecture that can be found in [2, 3].

Conjecture 1. Every source-free digraph has a quasi-kernel of size at most $|V(D)|/2$.

The conjecture that remains open has been proven recently by van Hulst in [3] for source-free digraphs having kernels.
In this note, we give a short and simple proof of van Hulst's result. Moreover, a characterization of all source-free digraphs having kernels with smallest quasi-kernels of size $|V(D)|/2$ is provided.

2. Main Result

In this section we will prove our main result. It is worth noting that the digraphs we consider do not contain isolated vertices, i.e. any vertex is incident with at least one arc.

Theorem 2. *Let D be a source-free digraph having a kernel. Then D has a minimum quasi-kernel of size at most $|V(D)|/2$, with equality if and only if each component of D belongs to $\{\overrightarrow{C_2}, \overrightarrow{C_4}\}$.*

Proof. Let K be a kernel of D. Since K dominates $V(D)-K$, let K^* be a smallest set of K such that every vertex of $V(D)-K$ is dominated by a vertex of K^*. Certainly, $|K^*| \leq |V(D) - K|$. Also $N_D^+(K^*) = V(D) - K$ because K is independent. Moreover, since D is source-free and K is independent, every vertex in $K - K^*$ has an in-neighbor in $V(D) - K$, and thus it can be reached in at most two steps from K^*. Hence K^* is a quasi-kernel of D. Therefore

$$|V(D)| = |K| + |V(D) - K| \geq |K^*| + |K^*| = 2|K^*|. \quad (1)$$

and thus $|K^*| \leq |V(D)|/2$, which confirms the first part of the result, and accordingly gives a shorter proof of the result given in [3].

In the sequel, we prove that D has a minimum quasi-kernel of size $|V(D)|/2$ if and only if each component of D is either $\overrightarrow{C_2}$ or $\overrightarrow{C_4}$. Clearly, if each component of D belongs to $\{\overrightarrow{C_2}, \overrightarrow{C_4}\}$, then a quasi-kernel of D is also a kernel of size $|V(D)|/2$.

Conversely, assume that D has a minimum quasi-kernel of size $|V(D)|/2$. Without loss of generality, we assume that D is connected since the argument we use is valid for any component of D. Let K be a kernel of D and let $K^* \subseteq K$ be a quasi-kernel of D as defined above. Clearly, $|K^*| \geq |V(D)|/2$. Furthermore, we have

$$|V(D)| = (|K| + |V(D) - K|) \geq 2|K^*| \geq |V(D)|,$$

and thus $|K^*| = |K|$ and $|K| = |V(D) - K|$. We conclude that every kernel is a quasi-kernel of size $|V(D)|/2$. For the rest of the proof, we need to show the following facts.

Claim 1. Every vertex of K dominates exactly one vertex of $V(D) - K$.

Proof of Claim 1. Let x be any vertex of K. If x has no out-neighbor in $V(D) - K$, then since D is a connected source-free digraph, x has at least one in-neighbor belonging to $V(D) - K$. In this case, $K - \{x\}$ is a quasi-kernel of D smaller than K, a contradiction. Hence every vertex of K dominates at least one vertex of $V(D) - K$. Now assume that some vertex in K dominates two or more vertices of $V(D) - K$. Since each vertex of K has at least one out-neighbor in $V(D) - K$, there must exist a vertex $y \in K$ such that every vertex of $N_D^+(y)$ has an in-neighbor in K other than y. But then $K - \{y\}$ would be a quasi-kernel of D smaller than K, a contradiction, which completes the proof of Claim 1.

Claim 2. $V(D) - K$ is an independent set.

Proof of Claim 2. Suppose to the contrary that $V(D) - K$ is not an independent set. Let x' and y' be two adjacent vertices from $V(D) - K$. Without loss of generality, assume that $x'y' \in A(D)$. Let x and y be two vertices of K that dominate x' and y', respectively. By Claim 1, $x \neq y$ and neither xy' or yx' belong to $A(D)$. If $x'y \in A(D)$, then certainly $K - \{y\}$ is a quasi-kernel of D smaller than K, a contradiction. Hence $x'y \notin A(D)$. For the next, since G is source-free, let y'' be an in-neighbor of y. Suppose that $y'x \in A(D)$. If $y'' \neq y'$, then $K - \{y\}$ is a quasi-kernel of D smaller than K, a contradiction. Hence we can assume that $y'' = y'$, that is y' is the unique out and in-neighbor of y. In this case, let $Y = N^+(y') \cap K$. Note that $|Y| \geq 2$ since $x, y \in Y$. Then $Q = \{y'\} \cup K - Y$ is an independent set and all vertices in $N^+(Y)$ can be reached in at most two steps from Q. Hence Q is a quasi-kernel of D smaller than K, a contradiction. We conclude that $y'x \notin A(D)$.

Now, as before, if $y'' \neq y'$, then certainly $K - \{y\}$ is a quasi-kernel of D which leads to a contradiction. Hence we can assume that $y'' = y'$. Let $X_1 = N^+(x') \cap K$ and $X_1' = N^+(X_1)$. Note that x may belong to X_1 and in this case $x' \in X_1'$. By Claim 1, $|X_1| = |X_1'|$. Now, if $x' \in N_D^-(x)$ or $N_D^-(x) - X_1' \neq \emptyset$, then $\{x'\} \cup K - (\{x, y\} \cup X_1)$ is a quasi-kernel of D which leads to a contradiction. Hence we assume that $N_D^-(x) \subseteq X_1' - \{x'\}$. Note that according to the discussion done above regarding x, y, x' and y', we deduce that no arc exists in any direction between x' and the vertices of $N_D^-(x)$. Let $x'' \in N_D^-(x)$, and $X_2 = N^+(x'') \cap K$. Note that $x \in X_2$ and $|X_1 \cup X_2| \geq 2$. Then one can check that $\{x', x''\} \cup K - (X_1 \cup X_2 \cup \{y\})$ is a quasi-kernel of D smaller than K, a contradiction. Therefore $V(D) - K$ is an independent set, and the proof of the claim is complete.

Claim 3. Every vertex of $V(D) - K$ has exactly one out-neighbor in K.
Proof of Claim 3. If $V(D) - K$ has a vertex with no out-neighbor in K, then since D is a connected source-free digraph, certainly there must exist in $V(D) - K$ a vertex having at least two out-neighbors in K. Hence, without loss of generality, we assume that some vertex $x' \in V(D) - K$ has at least two out-neighbors in K. Let $X_1 = N_D^+(x')$ and $X_1' = N^+(X_1)$.Thus $|X_1| \geq 2$ and $|X_1| = |X_1'|$ (by Claim 1). Let x be the unique in-neighbor of x' in K. Note that x may belong to X_1 and in this case $x' \in X_1'$. If $x' \in N_D^-(x)$ or $N_D^-(x) - X_1' \neq \emptyset$, then $\{x'\} \cup K - (X_1 \cup \{x\})$ is a quasi-kernel of D smaller than K, a contradiction. Hence we may assume that $N_D^-(x) \subseteq X_1' - \{x'\}$. Let $x'' \in N_D^-(x)$ and $X_2 = N^+(x'') \cap K$. Note that $x \in X_2$ and $|X_1 \cup X_2| \geq 3$. Then $\{x', x''\} \cup K - (X_1 \cup X_2)$ is a quasi-kernel of D smaller than K, a contradiction. Consequently, each vertex of $V(D) - K$ has exactly one out-neighbor in K.

Now, according to Claims 1,2 and 3, we conclude that D is an even directed cycle $\overrightarrow{C_n}$ of order n. If $n \in \{2, 4\}$, then clearly $D \in \{\overrightarrow{C_2}, \overrightarrow{C_4}\}$. Hence assume that $n \geq 6$ and is even. Let $V(\overrightarrow{C_n}) = \{x_1, x_2, \ldots, x_n\}$ with $x_n x_1$ and $x_i x_{i+1} \in A(\overrightarrow{C_n})$ for every $i \in \{1, \ldots, n-1\}$. If $n \equiv 0 (\mathrm{mod}\, 3)$, then let $Q_1 = \{x_{3i+1} \mid 0 \leq i \leq \frac{n}{3} - 1\}$ and if $n \equiv 1, 2 (\mathrm{mod}\, 3)$, then let $Q_2 = \{x_{n-1}, x_{3i+1} : 0 \leq i \leq \lfloor \frac{n}{3} \rfloor - 1\}$. For each case, Q_1 and Q_2 are quasi-kernels of $\overrightarrow{C_n}$ of size $\lceil n/3 \rceil$, a contradiction, and the proof of Theorem 2 is complete. □

In view of Theorem 2, we suggest a slight modification to the conjecture of Erdös and Székely as follows.

Conjecture 3. Every connected source-free digraph D of order at least 5 has a quasi-kernel of size less than $|V(D)|/2$.

Conflict of interest. The authors declare that they have no conflict of interest.

Data Availability. Data sharing is not applicable to this article as no datasets were generated or analyzed during the current study.

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