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*Research Article*

# On the complement of the intersection graph of subgroups of a group

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**Abstract:** The complement of the intersection graph of subgroups of a group  $G$ , denoted by  $\mathcal{I}^c(G)$ , is the graph whose vertex set is the set of all nontrivial proper subgroups of  $G$  and its two distinct vertices  $H$  and  $K$  are adjacent if and only if  $H \cap K = 1$ , where  $1$  denotes the trivial subgroup of  $G$ . In this paper, we classify all finite groups whose complement of the intersection graph of subgroups is one of totally disconnected, bipartite, complete bipartite, tree, star graph or  $C_3$ -free. Also we characterize all the finite groups whose complement of the intersection graph of subgroups is planar.

**Keywords:** Complement of intersection graph of subgroups, bipartite graph, planar graph

**AMS Subject classification:** 05C25, 05C10

## 1. Introduction

Bosak [3] initiated the study of the intersection graph of subsemigroups of a semigroup. Subsequently, Csákány and Pollák [5] defined the intersection graph of subgroups of a group. Over the past several years, many significant results on this graph have been established by several researchers; see, for instance [1, 2, 7, 9, 10, 12–17, 19]. Let  $G$  be a group. The intersection graph of subgroups of a group  $G$ , denoted by  $\mathcal{I}(G)$ , is

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the graph whose vertex set is the set of all nontrivial proper subgroups of  $G$  and its distinct vertices  $H$  and  $K$  are adjacent if and only if  $H \cap K \neq 1$ , where  $1$  denotes the trivial subgroup of  $G$ .

The complement of the intersection graph of subgroups of a group  $G$ , denoted by  $\mathcal{I}^c(G)$ , is the graph whose vertex set is the set of all nontrivial proper subgroups of  $G$  and its distinct vertices  $H$  and  $K$  are adjacent if and only if  $H \cap K = 1$ . This graph was considered firstly by Visveswaran and Vadhel in [18]. Therein, they have studied the connectedness, diameter, girth, clique number, chromatic number and completeness of this graph.

We use the standard terminology of graphs following [8]. Let  $G$  be a simple graph with vertex set  $V(G)$  and edge set  $E(G)$ .  $G$  is said to be bipartite if  $V(G)$  can be partitioned into two subsets  $V_1$  and  $V_2$  such that every edge joins a vertex of  $V_1$  to a vertex of  $V_2$ . A complete bipartite graph is a bipartite graph in which every vertex in one partition is adjacent with all the vertices in the other partition and is denoted by  $K_{m_1, m_2}$ , where  $m_i = |V_i|$ ,  $i = 1, 2$ . In particular,  $K_{1, m}$  is a star. The complete graph and the cycle graph on  $n$  vertices are denoted by  $K_n$  and  $C_n$ , respectively. A graph whose edge set is empty is called totally disconnected. A connected graph without a cycle is called a tree. A graph is said to be planar if there is a plane embedding of this graph. The complement of a graph  $G$  is denoted by  $\overline{G}$ . For given two graphs  $G_1$  and  $G_2$ , their join and union are denoted by  $G_1 + G_2$  and  $G_1 \cup G_2$  respectively. The generalized quaternion group of order  $2^\alpha$  ( $\alpha \geq 3$ ) is given by  $Q_{2^\alpha} = \langle a, b \mid a^{2^{\alpha-2}} = b^4 = e, a^{2^{\alpha-3}} = b^2, bab^{-1} = a^{-1} \rangle$ . The multiplicative order of a nonzero element  $x \in \mathbb{Z}_n$  is denoted by  $ord_n(x)$ .

This paper is organized as follows. In Section 2, we classify all finite groups whose complement of the intersection graph of subgroups is one of bipartite, complete bipartite, tree, star graph, totally disconnected and  $C_3$ -free. In Section 3, we characterize all finite groups whose complement of intersection graph of subgroups is planar.

## 2. Groups with specified complement of intersection graph of subgroups

**Theorem 1.** *Let  $G$  be a finite group. Then  $\mathcal{I}^c(G)$  is totally disconnected if and only if  $G$  is isomorphic to either  $\mathbb{Z}_{p^\alpha}$  ( $\alpha \geq 1$ ) or  $Q_{2^\alpha}$  ( $\alpha \geq 3$ ), where  $p$  is a prime number.*

*Proof.* Suppose  $|G| = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_k^{\alpha_k}$ , where  $p_i$ s are distinct primes and  $\alpha_i \geq 1$  for all  $i$ . If  $k \geq 2$ , then  $G$  has at least two subgroups of prime order and so they are adjacent in  $\mathcal{I}^c(G)$ . Now we assume that  $k = 1$ . Suppose  $G$  has at least two subgroups of prime order, then they are adjacent in  $\mathcal{I}^c(G)$  and so  $G$  must be isomorphic to either  $\mathbb{Z}_{p^\alpha}$  ( $\alpha \geq 1$ ) or  $Q_{2^\alpha}$  ( $\alpha \geq 3$ ). In either case,  $G$  has a unique subgroup of prime order and so all their subgroups intersect non-trivially. It follows that  $\mathcal{I}^c(G)$  is totally disconnected.  $\square$

**Theorem 2.** *Let  $G$  be a finite group. Then the following are equivalent.*

- (1)  $G$  is isomorphic to one of  $\mathbb{Z}_{p^\alpha}$  ( $\alpha \geq 1$ ),  $Q_{2^\alpha}$  ( $\alpha \geq 3$ ) or  $|G| = p^\alpha q^\beta$  ( $\alpha, \beta \geq 1$ ) with  $G$  has a unique subgroup of each of distinct prime orders  $p, q$ ;
- (2)  $\mathcal{I}^c(G)$  is bipartite;
- (3)  $\mathcal{I}^c(G)$  is  $C_3$ -free.

*Proof.* Suppose  $|G| = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_k^{\alpha_k}$ , where  $p_i$ s are pairwise distinct primes and  $\alpha_i \geq 1$ . If  $k \geq 3$ , then  $G$  has at least three subgroups of distinct prime orders and so they form  $C_3$  as a subgraph of  $\mathcal{I}^c(G)$ . Now we assume that  $k = 1$ . If  $G$  has a unique subgroup of order  $p_1$ , then  $G \cong \mathbb{Z}_{p_1^{\alpha_1}}$  ( $\alpha_1 \geq 1$ ) or  $Q_{2^{\alpha_1}}$  ( $\alpha_1 \geq 3$ ) and so by Theorem 1,  $\mathcal{I}^c(G)$  is bipartite. Otherwise,  $G$  has three subgroups of order  $p_1$ , so they form  $C_3$  as a subgraph of  $\mathcal{I}^c(G)$ . Next we assume that  $k = 2$ . Then  $\mathcal{I}^c(G)$  is bipartite if and only if  $G$  has a unique subgroup of each of orders  $p_1$  and  $p_2$ . For otherwise, the subgroups of prime orders forms  $C_3$  as a proper subgraph of  $\mathcal{I}^c(G)$ . In this case,  $\mathcal{I}^c(G)$  is bipartite with bipartition  $X$  and  $Y$ , where  $X$  is the set of all proper subgroups of  $G$  which contains the subgroup of order  $p_1$  and  $Y$  is the set of all proper subgroups of  $G$  which contains the subgroup of order  $p_2$ . So the proof follows.  $\square$

**Corollary 1.** *Let  $G$  be a finite group. Then the following are equivalent.*

- (1)  $G \cong \mathbb{Z}_{pq}$ , where  $p$  and  $q$  are distinct prime numbers;
- (2)  $\mathcal{I}^c(G)$  is a tree;
- (3)  $\mathcal{I}^c(G)$  is complete bipartite;
- (4)  $\mathcal{I}^c(G)$  is a star graph.

*Proof.* We use Theorem 2 to prove this result, since the three type of graphs mentioned in this result are bipartite. If  $G \cong \mathbb{Z}_{p^\alpha}$  ( $\alpha \geq 1$ ),  $Q_{2^\alpha}$  ( $\alpha \geq 3$ ), then by Theorem 1,  $\mathcal{I}^c(G)$  is neither a tree nor complete bipartite. Now we assume that  $|G| = p^\alpha q^\beta$  ( $\alpha, \beta \geq 1$ ) with  $G$  has a unique subgroup of each of distinct prime orders  $p$  and  $q$ . Suppose  $G$  has a subgroup of order  $pq$ , then this subgroup is an isolated vertex in  $\mathcal{I}^c(G)$  and so  $\mathcal{I}^c(G)$  is disconnected. Consequently,  $\mathcal{I}^c(G)$  is neither a tree nor complete bipartite. Finally, suppose  $G \cong \mathbb{Z}_{pq}$ , then  $\mathcal{I}^c(\mathbb{Z}_{pq}) \cong K_2$ , which is a tree and a star graph.  $\square$

### 3. Planarity of $\mathcal{I}^c(G)$

In this section, we characterize all finite groups whose complement of intersection graph of subgroups is planar. The well-known Kuratowski's theorem [8, Theorem 11.13] states that a graph is planar if and only if it does not contain a subdivision of  $K_5$  or  $K_{3,3}$ .

The main result of this section is the following.

**Theorem 3.** *Let  $G$  be a finite group. Then  $\mathcal{I}^c(G)$  is planar if and only if  $G$  is isomorphic to one the following groups.*

- (1)  $\mathbb{Z}_{p^\alpha}$  ( $\alpha \geq 1$ ),  $\mathbb{Z}_{p^\alpha q^\beta}$  ( $\alpha + \beta \leq 5$ ),  $\mathbb{Z}_{p^\alpha q^\beta r^\gamma}$  ( $\alpha + \beta + \gamma \leq 5$ ),  $\mathbb{Z}_{pqr s}$ ,  $\mathbb{Z}_2^\alpha \times \mathbb{Z}_2$  ( $\alpha \geq 2$ ),  $\mathbb{Z}_2 \times \mathbb{Z}_2$ ,  $\mathbb{Z}_3 \times \mathbb{Z}_3$ ,  $Q_{2^\alpha}$  ( $\alpha \geq 3$ ),  $S_3$ ,  $Q_{2^\alpha} \times \mathbb{Z}_p$  ( $\alpha \geq 3$ ),  $Q_{2^\alpha} \times \mathbb{Z}_{p^2}$  ( $\alpha \geq 3$ ), where  $p, q, r, s$  are distinct primes;
- (2)  $\langle a, b \mid a^q = b^{p^2} = 1, bab^{-1} = a^i, \text{ord}_q(i) = p^2 \rangle \cong \mathbb{Z}_q \rtimes \mathbb{Z}_{p^2}$ , where  $p, q$  are distinct primes with  $p < q$  and  $p^2 \mid (q - 1)$ .
- (3)  $|G| = p^\alpha q$  or  $p^\alpha q^2$  ( $\alpha \geq 3$ ) with  $G$  has a unique Sylow  $q$ -subgroup; Sylow  $p$ -subgroup is not unique and each of them is isomorphic to  $\mathbb{Z}_{p^\alpha}$  or  $Q_{2^\alpha}$  and they intersect with each other non-trivially, where  $p, q$  are distinct primes.

To prove this main result, we start with the following.

**Proposition 1.** *If  $G$  is a finite group whose order has at least five distinct prime factors, then  $\mathcal{I}^c(G)$  contains  $K_5$ .*

*Proof.* By [18, Proposition 3.1],  $\mathcal{I}^c(G)$  contains  $K_n$ ,  $n \geq 5$ , so the proof follows.  $\square$

**Proposition 2.** *Let  $G$  be a group of order  $p_1^{\alpha_1} p_2^{\alpha_2} p_3^{\alpha_3} p_4^{\alpha_4}$ , where  $p_i$ s are pairwise distinct prime numbers and  $\alpha_i \geq 1$  for  $i = 1, 2, 3, 4$ . Then  $\mathcal{I}^c(G)$  is planar if and only if  $G \cong \mathbb{Z}_{p_1 p_2 p_3 p_4}$ .*

*Proof.* Suppose  $G$  is nilpotent. Then  $G$  is the direct product of Sylow  $p_i$ -subgroups for  $i = 1, 2, 3, 4$ . If  $\alpha_i \geq 2$  for some  $i$ ; with out loss of generality, we assume that  $\alpha_1 \geq 2$ . Then  $\mathcal{I}^c(G)$  contains  $K_{3,3}$  as a subgraph with bipartition  $(X, Y)$ , where  $X$  is a set of three subgroups of  $G$  whose orders are  $p_1^{\alpha_1}$ ,  $p_1$ ,  $p_2$ , respectively and  $Y$  is a set of three subgroups of  $G$  whose orders are  $p_3$ ,  $p_4$ ,  $p_3 p_4$ , respectively. If  $\alpha_i = 1$  for every  $i = 1, 2, 3, 4$ , then  $G \cong \mathbb{Z}_{p_1 p_2 p_3 p_4}$  and so  $\mathcal{I}^c(G)$  is planar as shown in Figure 1, where  $H_i$ ,  $i = 1, 2, \dots, 14$  are the subgroups of  $G$  of order  $p_1$ ,  $p_2$ ,  $p_3$ ,  $p_4$ ,  $p_1 p_2$ ,  $p_1 p_3$ ,  $p_1 p_4$ ,  $p_2 p_3$ ,  $p_2 p_4$ ,  $p_3 p_4$ ,  $p_1 p_2 p_3$ ,  $p_1 p_2 p_4$ ,  $p_1 p_3 p_4$ ,  $p_2 p_3 p_4$  respectively.

Next, suppose  $G$  is non-nilpotent. With out loss of generality, we may assume that Sylow  $p_1$ -subgroup of  $G$  is not unique. Then  $G$  has at least three Sylow  $p_1$ -subgroups. In this case,  $\mathcal{I}^c(G)$  contains  $K_{3,3}$  as a subgraph with bipartition  $(X, Y)$ , where  $X$  is a set of three Sylow  $p_1$ -subgroups of  $G$  and  $Y$  is a set of three subgroups of  $G$  whose orders are  $p_2^{\alpha_2}$ ,  $p_3^{\alpha_3}$ ,  $p_4^{\alpha_4}$ , respectively.  $\square$

**Proposition 3.** *Let  $G$  be a group of order  $p_1^{\alpha_1} p_2^{\alpha_2} p_3^{\alpha_3}$ , where  $p_i$ s are pairwise distinct prime numbers and  $\alpha_i \geq 1$  for  $i = 1, 2, 3$ . Then  $\mathcal{I}^c(G)$  is planar if and only if  $G \cong \mathbb{Z}_{p_1^{\alpha_1} p_2^{\alpha_2} p_3^{\alpha_3}}$  with  $\alpha_1 + \alpha_2 + \alpha_3 \leq 4$ .*

*Proof.* Suppose  $G$  is nilpotent. Then  $G$  is the direct product of Sylow  $p_i$ -subgroups for  $i = 1, 2, 3$ . Then we have three cases to consider.

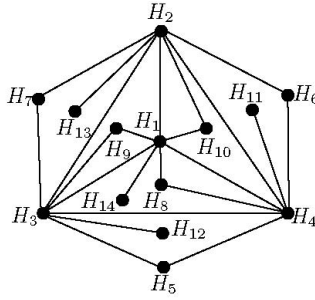


Figure 1.  $\mathcal{S}^c(\mathbb{Z}_{p_1 p_2 p_3 p_4})$

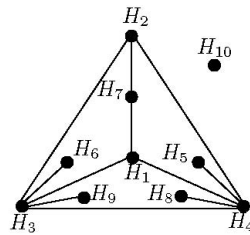


Figure 2.  $\mathcal{S}^c(\mathbb{Z}_{p_1^2 p_2 p_3})$

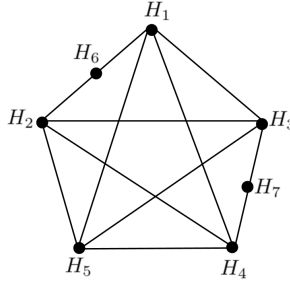
**Case 1.** If  $\alpha_1 + \alpha_2 + \alpha_3 \leq 4$ , then  $G$  is abelian. If  $G$  is cyclic, then  $\mathcal{S}^c(G)$  is planar as shown in Figure 2, where  $H_i, i = 1, 2, \dots, 10$  are subgroups of  $G$  of order  $p_1, p_1^2, p_2, p_3, p_1 p_2, p_1 p_3, p_2 p_3, p_1^2 p_3, p_1^2 p_3, p_1 p_2 p_3$  respectively. If  $G$  is non-cyclic, then  $G$  has at least five subgroups of prime orders and so they form  $K_5$  as a subgraph of  $\mathcal{S}^c(G)$ .

**Case 2.** If  $\alpha_1 \geq 3, \alpha_2 = \alpha_3 = 1$ , then  $\mathcal{S}^c(G)$  contains  $K_{3,3}$  as a proper subgraph with bipartition  $(X, Y)$ , where  $X$  is a set of three subgroups of  $G$  whose orders are  $p_1, p_1^2, p_1^3$ , respectively and  $Y$  is a set of three subgroups of  $G$  whose orders are  $p_2, p_3, p_2 p_3$ , respectively.

**Case 3.** If  $\alpha_1 \geq 2, \alpha_2 \geq 2, \alpha_3 = 1$ , then  $G$  has subgroups  $H_i, i = 1, 2, \dots, 7$  of order  $p_1, p_1^2, p_2, p_2^2, p_3, p_2 p_3, p_1 p_3$ , respectively. It follows that  $\mathcal{S}^c(G)$  contains a subdivision of  $K_5$  as shown in Figure 3.

Next, suppose  $G$  is non-nilpotent. With out loss of generality, we may assume that Sylow  $p_1$ -subgroup of  $G$  is not unique. Then  $G$  has at least three Sylow  $p_1$ - subgroups. It follows that  $\mathcal{S}^c(G)$  contains  $K_{3,3}$  as a subgraph with bipartition  $(X, Y)$ , where  $X$  is a set of three Sylow  $p_1$ -subgroups of  $G$  and  $Y$  is a set of three subgroups of  $G$  whose orders are  $p_2, p_3, p_2^i p_3^j$ , respectively.  $\square$

**Proposition 4.** Let  $G$  be an abelian group of order either  $p_1^{\alpha_1}$  or  $p_1^{\alpha_1} p_2^{\alpha_2}$ , where  $p_1, p_2$  are distinct prime numbers and  $\alpha_1, \alpha_2 \geq 1$ . Then  $\mathcal{S}^c(G)$  is planar if and only if  $G$  is isomorphic



**Figure 3.** A subdivision of  $K_5$  in  $\mathcal{S}^c(G)$

to either  $\mathbb{Z}_{p_1^{\alpha_1}}$ ,  $\mathbb{Z}_{p_1^{\alpha_1} p_2^{\alpha_2}}$  ( $\alpha_1 + \alpha_2 \leq 5$ ),  $\mathbb{Z}_{2^{\alpha_1}} \times \mathbb{Z}_2$ ,  $\alpha_1 \geq 1$  or  $\mathbb{Z}_3 \times \mathbb{Z}_3$ .

*Proof.* Proof is divided in to two cases.

**Case 1.** Suppose  $G$  is cyclic. If  $|G| = p_1^{\alpha_1}$ , then by Theorem 1,  $\mathcal{S}^c(G)$  is planar. If  $|G| = p_1^{\alpha_1} p_2^{\alpha_2}$ , then the subgroups of  $G$  are  $H_i, K_j, N_{ij}$ , where  $|H_i| = p_1^i$ ,  $|K_j| = p_2^j$ ,  $|N_{ij}| = p_1^i p_2^j$  for  $i = 1, 2, \dots, \alpha_1$ ,  $j = 1, 2, \dots, \alpha_2$ . It can be seen that  $\mathcal{S}^c(G) \cong K_{\alpha_1, \alpha_2} \cup K_{\alpha_1 \alpha_2 - 1}$ . Therefore,  $\mathcal{S}^c(G)$  is planar only when  $\alpha_1 + \alpha_2 \leq 5$ .

**Case 2.** Suppose  $G$  is non-cyclic.

**Subcase 2a.** If  $G \cong \mathbb{Z}_{p_1} \times \mathbb{Z}_{p_1}$ , then  $\mathcal{S}^c(G) \cong K_{p_1+1}$  and so it is planar only when  $p = 2, 3$ .

**Subcase 2b.** If  $G \cong \mathbb{Z}_{p_1^2} \times \mathbb{Z}_{p_1}$ , then from the subgroup lattice of  $G$ , it can be seen that

$$\mathcal{S}^c(G) \cong K_1 \cup (K_{p_1} + \overline{K}_{p_1+1}) \quad (1)$$

and so it is planar only when  $p = 2$ .

**Subcase 2c.** If  $G \cong \mathbb{Z}_{p_1 p_2} \times \mathbb{Z}_{p_1}$ , then  $G$  has exactly  $p_1 + 2$  subgroups of prime order and they have trivial intersection with each other. The remaining subgroups of  $G$  have non-trivial intersection with each other. It follows that  $\mathcal{S}^c(G)$  is planar only when  $p_1 = 2$ .

**Subcase 2d.** If  $G \cong \mathbb{Z}_{p_1^{\alpha_1}} \times \mathbb{Z}_{p_1}$ , then from the subgroup lattice of  $G$ , we have

$$\mathcal{S}^c(G) \cong K_{\alpha_1-2} \cup (K_{p_1} + \overline{K}_{(\alpha_1-2)p_1+1}) \quad (2)$$

It follows that  $\mathcal{S}^c(G)$  is planar only when  $p = 2$ .

**Subcase 2e.** If  $G \cong \mathbb{Z}_{p_1^2} \times \mathbb{Z}_{p_1^2} := \langle a, b \mid a^{p_1^2} = b^{p_1^2} = 1, ab = ba \rangle$ . Then  $K_{3,3}$  is a subgraph of  $\mathcal{S}^c(G)$  with bipartition  $\{\langle a \rangle, \langle a, b^2 \rangle, \langle a^2 \rangle\}$  and  $\{\langle a, b \rangle, \langle a^2, b \rangle, \langle a^3, b \rangle\}$ .

**Subcase 2f.** If  $G \cong \mathbb{Z}_{p_1^2 p_2} \times \mathbb{Z}_{p_1}$  or  $\mathbb{Z}_{p_1} \times \mathbb{Z}_{p_1} \times \mathbb{Z}_{p_1}$ , then  $G$  has at least five subgroups of prime order and so  $\mathcal{S}^c(G)$  contains  $K_5$ .

**Subcase 2g.** If  $G \cong \mathbb{Z}_{p_1^{\alpha_1}} \times \mathbb{Z}_{p_2^{\alpha_2}} \times \dots \times \mathbb{Z}_{p_k^{\alpha_k}}$ ,  $k \geq 2$ , then  $G$  has one of  $\mathbb{Z}_{p_1^2} \times \mathbb{Z}_{p_1^2}$ ,  $\mathbb{Z}_{p_1^2 p_2} \times \mathbb{Z}_{p_1}$  or  $\mathbb{Z}_{p_1} \times \mathbb{Z}_{p_1} \times \mathbb{Z}_{p_1}$  as a subgroup. Then by Subcases 2e and 2f,  $\mathcal{S}^c(G)$  contains either  $K_{3,3}$  or  $K_5$ .  $\square$

**Proposition 5.** *Let  $G$  be a non-abelian group of order  $p^\alpha$ , where  $p$  is a prime number and  $\alpha \geq 3$ . Then  $\mathcal{S}^c(G)$  is planar if and only if  $G \cong Q_{2^\alpha}$  or  $M_{2^\alpha}$ .*

*Proof.* Suppose  $\alpha = 3$ . Up to isomorphism, there are four groups of order  $p^3$ , including the group  $Q_8$ . Except  $Q_8$ , the remaining three groups have at least five subgroups of prime order and so they form  $K_5$  as a subgraph of  $\mathcal{S}^c(G)$ .

Suppose  $p > 2$ . Then  $G$  has a non-cyclic subgroup  $H$  of order  $p^{\alpha-1}$ . So by the above argument and Proposition 4,  $\mathcal{S}^c(H)$  is non-planar.

Suppose  $p = 2$  and  $\alpha \geq 4$ . If  $G \not\cong Q_{2^\alpha}$ ,  $Q_{2^{\alpha-1}} \times \mathbb{Z}_2$  and  $M_{2^\alpha}$ , then either  $G$  contains a non-cyclic subgroup, say  $H$  of order  $2^{\alpha-1}$  or it contains at least five subgroups of order 2. So by above argument and Proposition 4,  $\mathcal{S}^c(H)$  is non-planar or  $\mathcal{S}^c(G)$  contains  $K_5$ . Next we investigate the remaining possibilities.

If  $G \cong Q_{2^\alpha}$ , then by Proposition 1,  $\mathcal{S}^c(Q_{2^\alpha})$  is planar.

If  $G \cong Q_{2^{\alpha-1}} \times \mathbb{Z}_2$ , then we split the set of all non-trivial proper subgroups of  $G$  into five mutually disjoint subsets: The first subset consist of subgroups  $\langle a^{2^{\alpha-3}}c \rangle$  and  $\langle c \rangle$ , where  $c$  denotes the generator of  $\mathbb{Z}_2$ . The second subset consists of the subgroups  $\langle ac \rangle$ ,  $\langle a^2c \rangle$ ,  $\dots$ ,  $\langle a^{2^{\alpha-4}}c \rangle$ ,  $\langle bc \rangle$ ,  $\langle abc \rangle$ ,  $\langle a^2bc \rangle$ ,  $\dots$ ,  $\langle a^{2^{\alpha-3}-3}c \rangle$ . The third subset consists of the subgroup  $\langle a^{2^{\alpha-3}} \rangle$ . The fourth subset consists of all the subgroups of  $Q_{2^{\alpha-1}}$  except  $\{e\}$ . The fifth subset consists of the remaining subgroups of  $G$ . It can be seen that any two subgroups in the union of these subsets, except the first subset intersect non-trivially. Each subgroup in the first subset intersect trivially with the subgroups in the second, third and fourth subsets; at the same time it intersect non-trivially with the subgroups in the fifth subset. Also the two subgroups in the first subset intersect trivially. From the above description, it is easy to see that the structure of  $\mathcal{S}^c(G)$  as shown in Figure 4 and so it is planar.

If  $G \cong M_{2^\alpha}$  then its subgroup lattice is isomorphic to the subgroup lattice of  $\mathbb{Z}_{2^\alpha} \times \mathbb{Z}_2$ . By Theorem 4,  $\mathcal{S}^c(G)$  is planar and

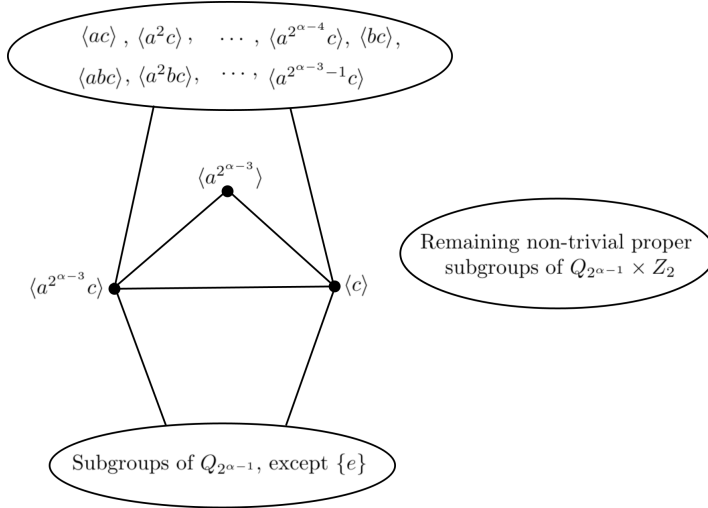
$$\mathcal{S}^c(M_{2^\alpha}) \cong K_{\alpha-2} \cup (K_2 + \overline{K}_{2^{\alpha-4}}). \quad (3)$$

This completes the proof. □

**Proposition 6.** *Let  $G$  be the non-abelian group of order  $pq$ , where  $p$  and  $q$  are distinct primes with  $p < q$  and  $p$  divides  $q - 1$ . Then  $\mathcal{S}^c(G)$  is planar if and only if  $G \cong S_3$ .*

*Proof.* Since  $G$  has  $q + 1$  subgroups of prime order and these are the only proper subgroups of  $G$ , it follows that  $\mathcal{S}^c(G) \cong K_{q+1}$ . Therefore,  $\mathcal{S}^c(G)$  is planar if and only if  $q = 3$ . □

**Proposition 7.** *Let  $G$  be a non-abelian group of order  $p^2q$ , where  $p, q$  are distinct primes. Then  $\mathcal{S}^c(G)$  is planar if and only if  $G \cong \langle a, b \mid a^q = b^{p^2} = 1, bab^{-1} = a^i, \text{ord}_q(i) = p^2 \rangle \cong \mathbb{Z}_q \rtimes \mathbb{Z}_{p^2}$ , where  $p, q$  are distinct primes with  $p < q$  and  $p^2 \mid (q - 1)$ .*



**Figure 4.** The structure of  $\mathcal{I}^c(Q_{2^{\alpha-1}} \times \mathbb{Z}_2)$

*Proof.* According to [4], there are eight groups of order  $p^2q$ . It can be seen that all these groups, except  $\mathbb{Z}_q \rtimes \mathbb{Z}_{p^2}$  have at least five subgroups of prime order and so they form  $K_5$  as a subgraph of  $\mathcal{I}^c(G)$ . If  $G \cong \mathbb{Z}_q \rtimes \mathbb{Z}_{p^2} = \langle a, b \mid a^q = b^{p^2} = 1, bab^{-1} = a^i, \text{ord}_q(i) = p^2 \rangle$ , where  $p, q$  are distinct primes with  $p < q$  and  $p^2 \mid (q-1)$ . Then  $\langle a \rangle$ ,  $\langle b^p \rangle$ ,  $\langle ab^p \rangle$  and  $\langle a^i b \rangle$ , where  $i = 1, 2, \dots, q$  are the only nontrivial proper subgroups of  $G$ . Here  $\langle b^p \rangle$  is contained in these subgroups, except  $\langle a \rangle$ . Also  $\langle a \rangle$  is a subgroup of  $\langle ab^p \rangle$ . It follows that

$$\mathcal{I}^c(\mathbb{Z}_q \rtimes \mathbb{Z}_{p^2}) \cong K_1 \cup K_{1,q+1}. \quad (4)$$

Therefore,  $\mathcal{I}^c(\mathbb{Z}_q \rtimes \mathbb{Z}_{p^2})$  is planar.  $\square$

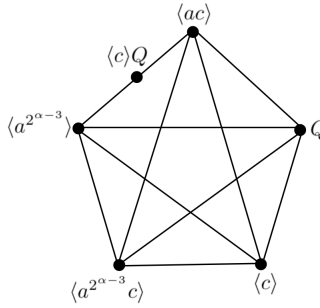
**Proposition 8.** Let  $G$  be a non-abelian group of order  $p^\alpha q$ , where  $p, q$  are distinct primes and  $\alpha \geq 3$ . Then  $\mathcal{I}^c(G)$  is planar if and only if  $G \cong Q_{2^\alpha} \times \mathbb{Z}_q$  or  $G$  has a unique Sylow  $q$ -subgroup; Sylow  $p$ -subgroup is not unique and each of them is isomorphic to  $\mathbb{Z}_{p^\alpha}$  or  $Q_{2^\alpha}$  and they intersect with each other non-trivially.

*Proof.* Let  $P$  and  $Q$  be a Sylow  $p$ -subgroup and a Sylow  $q$ -subgroup of  $G$ , respectively. Suppose  $\mathcal{I}^c(P)$  is non-planar, then  $\mathcal{I}^c(G)$  is so. Therefore, it is enough to consider the cases when  $\mathcal{I}^c(P)$  is planar. By Propositions 4 and 5,  $P \cong \mathbb{Z}_{p^\alpha}$ ,  $\mathbb{Z}_{2^\alpha} \rtimes \mathbb{Z}_2$ ,  $M_{p^\alpha}$ ,  $Q_{2^{\alpha-1}} \times \mathbb{Z}_2$  or  $Q_{2^\alpha}$ .

**Case 1.** Suppose Sylow  $q$ -subgroup of  $G$  is not unique. Then  $\mathcal{I}^c(G)$  has  $K_{3,3}$  as a subgraph with bipartition  $(X, Y)$ , where  $X$  is a set of three subgroups of  $G$  whose orders are  $p, p^2, p^3$ , respectively and  $Y$  is a set of three subgroups of  $G$  whose orders are  $q, q, q$ , respectively.

**Case 2.** Suppose Sylow  $q$ -subgroup of  $G$  is unique.





**Figure 5.** A subdivision of  $K_5$  in  $\mathcal{S}^c(G)$

If  $P \cong Q_{2^{\alpha-1}} \times \mathbb{Z}_2$ , then  $\langle c \rangle, \langle ac \rangle, \langle a^{2^{\alpha-3}}c \rangle, \langle a^{2^{\alpha-3}} \rangle$  are subgroups of  $P$  as mentioned in the proof of Proposition 5. These four subgroups together with  $Q, \langle c \rangle Q$  forms a subdivision of  $K_5$  in  $\mathcal{S}^c(G)$ , which is shown in Figure 5. So  $\mathcal{S}^c(G)$  is non-planar.

Next we investigate the remaining possibilities.

**Subcase 2a.** Suppose Sylow  $p$ -subgroup is not unique and these subgroups intersect with each other trivially. Then  $G$  has subgroups  $H_i, i = 1, 2, \dots, 6$  of order  $p^\alpha, p^\alpha, p^\alpha, q, p, p^2$ , respectively such that  $H_6$  is a subgroup of  $H_1$ ; but not a subgroup of  $H_2$  and  $H_3$ . These subgroups form a subdivision of  $K_5$  in  $\mathcal{S}^c(G)$ , which is isomorphic to the graph shown in Figure 5.

**Subcase 2b.** Suppose Sylow  $p$ -subgroup is not unique and these subgroups intersect with each other non-trivially. If  $P \cong \mathbb{Z}_{2^\alpha} \times \mathbb{Z}_2$  or  $M_{2^\alpha}$ , then by (2),  $\mathcal{S}^c(G)$  has  $K_{2,3}$  as a subgraph. Since  $\mathbb{Z}_q$  is adjacent with all the vertices of  $\mathcal{S}^c(P)$ , it follows that  $\mathcal{S}^c(G)$  has  $K_{3,3}$  as a subgraph. If  $P \cong \mathbb{Z}_{p^\alpha}$  or  $Q_{2^\alpha}$ , then  $G$  has exactly two subgroups, one having order 2 and the other having order  $q$ ; subgroups of  $\mathbb{Z}_{p^\alpha}$  are adjacent with  $\mathbb{Z}_q$  and the remaining subgroups of  $G$  intersect non-trivially. It follows that  $\mathcal{S}^c(G)$  is planar.

**Subcase 2c.** Suppose Sylow  $p$ -subgroup is unique. Then  $G$  is the direct product of its Sylow  $p$ -subgroup and Sylow  $q$ -subgroup. If  $P \cong \mathbb{Z}_{p^\alpha}, \mathbb{Z}_{2^\alpha} \times \mathbb{Z}_2$ , then  $G$  is abelian, which is not possible. So it forces that  $P \cong M_{2^4}$  or  $Q_{2^\alpha}$ . If  $G \cong M_{2^4} \times \mathbb{Z}_q$ , then by (2) and by the above argument,  $\mathcal{S}^c(G)$  has  $K_{3,3}$  as a subgraph. If  $G \cong Q_{2^\alpha} \times \mathbb{Z}_q$ , then  $G$  has exactly two subgroups, one of which has order 2 other has order  $q$ ; and so the remaining subgroups of  $G$  intersect non-trivially. It follows that  $\mathcal{S}^c(G)$  is planar.  $\square$

**Proposition 9.** Let  $G$  be a non-abelian group of order  $p^2q^2$ , where  $p, q$  are distinct prime numbers. Then  $\mathcal{S}^c(G)$  is non-planar.

*Proof.* According to [11], there are four subgroups of order  $p^2q^2$  when  $(p, q) \neq (3, 2)$  and nine groups of order 36 when  $(p, q) = (3, 2)$ . Using the subgroups information of these groups given in [6, pages 40-43], it can be directly seen that the complement of

intersection graph of subgroups of these groups have  $K_5$  as a subgraph, except the following two groups, which have to be considered separately.

The first group is  $(\mathbb{Z}_p \times \mathbb{Z}_p) \rtimes \mathbb{Z}_{q^2} := \langle a, b, c \mid a^p = b^p = c^{q^2} = 1, ab = ba, cac^{-1} = a^i b^j, cbc^{-1} = a^k b^l \rangle$ , where  $\begin{pmatrix} i & j \\ k & l \end{pmatrix}$  has order  $q^2$  in  $GL_2(p)$  and the second group is  $(\mathbb{Z}_p \times \mathbb{Z}_p) \rtimes (\mathbb{Z}_q \times \mathbb{Z}_q)$  where  $(p, q) \neq (3, 2)$ . Notice that in each of these groups  $p \geq 5$  and it has  $\mathbb{Z}_p \times \mathbb{Z}_p$  as its subgroup, so by Proposition 4,  $\mathcal{S}^c(\mathbb{Z}_p \times \mathbb{Z}_p)$  is non-planar. Consequently, the complement of the intersection graph of subgroups of these two groups are non-planar.  $\square$

**Proposition 10.** *Let  $G$  be a non-abelian group of order  $p^\alpha q^2$ , where  $p, q$  are distinct prime numbers,  $\alpha \geq 3$ . Then  $\mathcal{S}^c(G)$  is planar if and only if  $G \cong Q_{2^\alpha} \times \mathbb{Z}_{q^2}$  or  $G$  has a unique Sylow  $q$ -subgroup; Sylow  $p$ -subgroup is not unique and each of them is isomorphic to  $\mathbb{Z}_{p^\alpha}$  or  $Q_{2^\alpha}$  and they intersect with each other non-trivially.*

*Proof.* Let  $P$  and  $Q$  be a Sylow  $p$ -subgroup and a Sylow  $q$ -subgroup of  $G$ , respectively. Suppose either  $\mathcal{S}^c(P)$  or  $\mathcal{S}^c(Q)$  is non-planar, then  $\mathcal{S}^c(G)$  is so. Therefore, it is enough to consider the cases when both  $\mathcal{S}^c(P)$  and  $\mathcal{S}^c(Q)$  are planar. By Propositions 4 and 5,  $P \cong \mathbb{Z}_{p^\alpha}, \mathbb{Z}_{2^\alpha} \times \mathbb{Z}_2, M_{2^\alpha}, Q_{2^{\alpha-1}} \times \mathbb{Z}_2$  or  $Q_{2^\alpha}$  and  $Q \cong \mathbb{Z}_{q^2}, \mathbb{Z}_2 \times \mathbb{Z}_2$ , or  $\mathbb{Z}_3 \times \mathbb{Z}_3$ .

**Case 1.** Suppose  $Q$  is not unique. Then  $\mathcal{S}^c(G)$  contains  $K_{3,3}$  as a subgraph with bipartition  $(X, Y)$ , where  $X$  is a set of three subgroups of  $G$  whose orders are  $p^3, p^2, p$ , respectively and  $Y$  is a set of three subgroups of  $G$  whose orders are  $q, q, q^2$ , respectively.

**Case 2.** Suppose  $Q$  is unique.

If  $P \cong Q_{2^{\alpha-1}} \times \mathbb{Z}_2$ , then  $\langle c \rangle, \langle ac \rangle, \langle a^{2^{\alpha-3}} c \rangle, \langle a^{2^{\alpha-3}} \rangle$  are subgroups of  $P$  as mentioned in the proof of Proposition 5. These four subgroups together with  $Q, \langle c \rangle Q$  forms a subdivision of  $K_5$  in  $\mathcal{S}^c(G)$ , which is shown in Figure 5. So  $\mathcal{S}^c(G)$  is non-planar.

Next we investigate the remaining possibilities.

**Subcase 2a.** Suppose  $P$  is unique. Then  $G \cong P \times Q$  and so by the above argument,  $G$  is isomorphic to one of  $Q_{2^\alpha} \times \mathbb{Z}_{q^2}, Q_{2^\alpha} \times \mathbb{Z}_3 \times \mathbb{Z}_3, M_{2^\alpha} \times \mathbb{Z}_{q^2}$  or  $M_{2^\alpha} \times \mathbb{Z}_3 \times \mathbb{Z}_3$ .

If  $G \cong Q_{2^\alpha} \times \mathbb{Z}_3 \times \mathbb{Z}_3$ , then  $G$  has five subgroups of prime order and so they form  $K_5$  as a subgraph of  $\mathcal{S}^c(G)$ . If  $G \cong Q_{2^\alpha} \times \mathbb{Z}_{q^2}$ , then  $G$  contains unique subgroups  $H_1$  and  $H_2$  of order 2 and 3, respectively. Here  $H_1$  and  $H_2$  are adjacent with all the subgroups of  $\mathbb{Z}_{p^2}$  and  $Q_{2^\alpha}$ , respectively. The remaining proper subgroups of  $G$  contains  $H_1$  and  $H_2$ . It follows that  $\mathcal{S}^c(G)$  is planar. If  $G \cong M_{2^\alpha} \times \mathbb{Z}_{q^2}$ , then by (2),  $\mathcal{S}^c(M_{2^\alpha})$  has  $K_{2,3}$  as a subgraph. Notice that  $\mathbb{Z}_{p^2}$  has a trivial intersection with the subgroups corresponding to the vertices of  $K_{2,3}$  and so  $\mathcal{S}^c(G)$  has  $K_{3,3}$  as a subgraph. If  $G \cong M_{2^\alpha} \times \mathbb{Z}_3 \times \mathbb{Z}_3$ , then by a similar argument as above, it can be seen that  $\mathcal{S}^c(G)$  has  $K_{3,3}$  as a subgraph.

**Subcase 2b.** Suppose  $P$  is not unique and Sylow  $p$ -subgroups of  $G$  mutually intersect non-trivially.

Suppose  $Q \cong \mathbb{Z}_{q^2}$ . If  $P \cong \mathbb{Z}_{p^\alpha}$  or  $Q_{2^\alpha}$ , then  $G$  contains unique subgroups of each of orders  $p$  and  $q$  and these subgroups are contained in the remaining proper subgroups

of  $G$ . Therefore,  $\mathcal{I}^c(G)$  is planar. If  $P \cong \mathbb{Z}_{2^\alpha} \times \mathbb{Z}_2$  or  $M_{2^\alpha}$ , then by (2),  $\mathcal{I}^c(M_{2^\alpha})$  has  $K_{2,3}$  as a subgraph. Here  $\mathbb{Z}_{p^2}$  is adjacent with all the vertices of  $K_{2,3}$  and so  $\mathcal{I}^c(G)$  contains  $K_{3,3}$  as a subgraph.

Suppose  $Q \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ . Then  $\mathcal{I}^c(G)$  has  $K_{3,3}$  as a proper subgraph with bipartition  $(X, Y)$ , where  $X$  is a set of three subgroups of  $G$  each having order 2 and  $Y$  is a set of three subgroups of  $G$  whose orders are  $p, p^2, p^3$ , respectively. Suppose  $Q \cong \mathbb{Z}_3 \times \mathbb{Z}_3$ , then  $G$  has four subgroups each having order 3 and has a subgroup of order  $p$ . These five subgroups form  $K_5$  as a subgraph of  $\mathcal{I}^c(G)$ .

**Subcase 2c.** Suppose  $P$  is not unique and Sylow  $p$ -subgroups of  $G$  mutually intersect trivially.

If  $Q \cong \mathbb{Z}_{q^2}$ , then  $G$  contains distinct subgroups of order  $p, p, p^2, p^3, q, q^2$ . They form a subdivision of  $K_5$  in  $\mathcal{I}^c(G)$ , which is isomorphic to the graph shown in Figure 5.

For the remaining cases, we can apply the same argument as in Subcase 2b and obtain that  $\mathcal{I}^c(G)$  contains either  $K_5$  or  $K_{3,3}$ .  $\square$

**Proposition 11.** *Let  $G$  be a non-abelian group of order  $p^\alpha q^\beta$ , where  $p, q$  are distinct prime numbers,  $\alpha, \beta \geq 3$ ,  $\alpha + \beta \geq 6$ . Then  $\mathcal{I}^c(G)$  is non-planar.*

*Proof.* Here  $\mathcal{I}^c(G)$  has  $K_{3,3}$  as a subgraph with bipartition  $(X, Y)$ , where  $X$  is a set of three subgroups of  $G$  whose orders are  $p^\alpha, p, p^2$ , respectively and  $Y$  is a set of three subgroups of  $G$  whose orders are  $q^\alpha, q, q^2$ , respectively.  $\square$

*Proof of Theorem 3:* Combining all the results that have been established thus far in this section, we arrive at the desired outcome.  $\square$

Notice that  $\mathbb{Z}_q \rtimes_t \mathbb{Z}_{p^\alpha} := \langle a, b \mid a^q = b^{p^\alpha} = 1, bab^{-1} = a^i, \text{ord}_q(i) = p^t \rangle$ , where  $p^t \mid (q - 1)$  and  $\mathbb{Z}_{q^2} \rtimes_t \mathbb{Z}_{p^\alpha} := \langle a, b \mid a^{q^2} = b^{p^\alpha} = 1, bab^{-1} = a^i, \text{ord}_{q^2}(i) = p^t \rangle$ , where  $p^t \mid (q^2 - 1)$  shows the existence of groups of order  $p^\alpha q$  and  $p^\alpha q^2$ , respectively satisfying the condition (2) of Theorem 3. However, classifying these groups seems to be a challenging issue that requires more research.

## References

- [1] H. Ahmadi and B. Taeri, *Planarity of the intersection graph of subgroups of a finite group*, J. Algebra Appl. **15** (2016), no. 3, Article ID: 1650040 <https://doi.org/10.1142/S0219498816500407>.
- [2] S. Akbari, F. Heydari, and M. Maghasedi, *The intersection graph of a group*, J. Algebra Appl. **14** (2015), no. 5, Article ID: 1550065 <https://doi.org/10.1142/S0219498815500656>.
- [3] J. Bosák, *The graphs of semigroups*, in: Theory of Graphs and Application, Academic Press, New York (1964), 119–125.

- 
- [4] W. Burnside, *Theory of Groups of Finite Order*, Dover Publications, Cambridge, 1955.
- [5] B. Csákány and G. Pollák, *The graph of subgroups of a finite group*, Czechoslovak Math. J. **19** (1969), no. 94, 241–247  
<http://dx.doi.org/10.21136/CMJ.1969.100891>.
- [6] P. Devi, *Studies on graphs associated with subgroups of groups*, Ph.D. thesis, Gandhigram, India, 2016.
- [7] P. Devi and R. Rajkumar, *Intersection graph of abelian subgroups of a group*, Indian J. Discrete Math. **2** (2016), no. 2, 77–86.
- [8] F. Harary, *Graph Theory*, Addison-Wesley, Philippines, 1969.
- [9] S.H. Jafari and N. Jafari Rad, *Intersection graphs of normal subgroups of groups*, Quasigroups and related systems **18** (2010), no. 2, 137–142.
- [10] S. Kayacan, *Connectivity of intersection graphs of finite groups*, Comm. Algebra **46** (2018), no. 4, 1492–1505  
<https://doi.org/10.1080/00927872.2017.1347662>.
- [11] H.L. Lin, *On groups of order  $p^2q$ ,  $p^2q^2$* , Tamkang J. Math. **5** (1974), no. 2, 167–190.
- [12] X. Ma, *On the diameter of the intersection graph of a finite simple group*, Czechoslovak Math. J. **66** (2016), no. 2, 365–370  
<https://doi.org/10.1007/s10587-016-0261-2>.
- [13] R. Rajkumar and P. Devi, *Toroidality and projective-planarity of intersection graphs of subgroups of finite groups*, arXiv preprint arXiv:1505.08094 (2015).
- [14] ———, *Intersection graph of subgroups of some non-abelian groups*, Malaya J. Mat. **4** (2016), no. 02, 238–242.
- [15] ———, *Intersection graphs of cyclic subgroups of groups*, Electron. Notes Discrete Math. **53** (2016), 15–24  
<https://doi.org/10.1016/j.endm.2016.05.003>.
- [16] H. Shahsavari and B. Khosravi, *Characterization of some families of simple groups by their intersection graphs*, Comm. Algebra **48** (2020), no. 3, 1266–1280  
<https://doi.org/10.1080/00927872.2019.1682151>.
- [17] R. Shen, *Intersection graphs of subgroups of finite groups*, Czechoslovak Math. J. **60** (2010), no. 4, 945–950  
<https://doi.org/10.1007/s10587-010-0085-4>.
- [18] S. Visweswaran and P. Vadhel, *Some results on the complement of the intersection graph of subgroups of a finite group*, J. Algebraic Sys. **7** (2020), no. 2, 105–130  
<https://doi.org/10.22044/jas.2018.5917.1296>.
- [19] B. Zelinka, *Intersection graphs of finite abelian groups*, Czechoslovak Math. J. **25** (1975), no. 2, 171–174  
<https://doi.org/10.21136/CMJ.1975.101307>.