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Graphs with unique minimum edge-vertex dominating sets

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Abstract: An edge e of a simple graph $G = (V_G, E_G)$ is said to ev-dominate a vertex $v \in V_G$ if e is incident with v or e is incident with a vertex adjacent to v. A subset $D \subseteq E_G$ is an edge-vertex dominating set (or an *evd-set* for short) of G if every vertex of G is ev-dominated by an edge of D. The edge-vertex domination number of G is the minimum cardinality of an *evd-set* of G. In this paper, we initiate the study of the graphs with unique minimum *evd-sets* that we will call UEVD-graphs. We first present some basic properties of UEVD-graphs, and then we characterize UEVD-trees by equivalent conditions as well as by a constructive method.

Keywords: Edge-vertex dominating set, edge-vertex domination number, trees

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1. Introduction

Let G be a simple, connected and undirected graph with vertex set V_G and edge set E_G. The set $N_G(v) = \{x \in V_G : x \text{ is adjacent to } v \text{ in } G\}$ is the *open neighborhood* of a vertex $v \in V_G$ and the *closed neighborhood* of v is the set $N_G[v] = N_G(v) \cup \{v\}$. An edge $e \in E_G$ *edge-vertex dominates* (or simply *ev-dominates*) a vertex $v \in V_G$ if e is incident with v or e is incident with a vertex adjacent to v. In [10], Peters introduced edge-vertex dominating sets, abbreviated *evd-sets*, of a graph G as a subset

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$D \subseteq E_G$ such that every vertex of G is ev-dominated by an edge of D. The *edge-vertex domination number* of G, denoted as $\gamma_{ev}(G)$, is the minimum cardinality of an *evd-set* of G. A $\gamma_{ev}(G)$-set is an *evd-set* of G with minimum cardinality $\gamma_{ev}(G)$. For further details on edge-vertex domination, the reader is referred to [8, 9, 11].
Several studies on the graphs having a unique set for some domination parameters are available in the literature. But the first work on such graphs with respect to the domination number was done by Gunther et al. [4] who additionally gave a characterization of the trees having unique minimum dominating sets. For further details, we refer the reader for example to [1–3, 5–7, 12].
Our main purpose in this paper is to study the graphs G with unique $\gamma_{ev}(G)$-sets which we call UEVD-graphs. In section 2, some basic properties of UEVD-graphs are discussed while in Section 3, we establish equivalent conditions for the characterization of UEVD-trees. Moreover, a constructive characterization of UEVD-trees will be provided in the last section.
Before presenting our results, we need to introduce some further but standard notation and definitions. Given a simple and connected graph $G = (V_G, E_G)$. The *degree* of a vertex $v \in V_G$ is $d_G(v) = |N_G(v)|$. A vertex of degree one is a *leaf* and its neighbor is a *support vertex*. A support vertex is a *weak support vertex* if it is adjacent to exactly one leaf, otherwise it is called a *strong support vertex*. A *pendant edge* in G is an edge incident with a leaf. A *star* of order $n \geq 2$, denoted by $K_{1,n-1}$, is a tree with at least $n-1$ leaves. A *double star* $S_{p,q}$ is a tree with exactly two vertices that are not leaves. The *distance* between two vertices u and v in a connected graph G is the number of edges in a shortest path between u and v. The *diameter* of a connected graph G, denoted diam(G), is the maximum distance between two vertices.

2. Properties of the UEVD-graphs

In this section, we prove certain properties of the UEVD-graphs. We begin by defining a private-vertex of an edge.

Definition 1. Let D be an evd-set of a graph G. A vertex $v \in V_G$ is a private-vertex of an edge $e \in D$ with respect to D if v is ev-dominated by the edge e and no other edge in $D \setminus \{e\}$, ev-dominates v.

In accordance with Definition 1, let $P(e, D)$ denote the set of private vertices of an edge e with respect to the set D. The following result gives a necessary and sufficient condition for *evd-sets* to be minimal in a graph G.

Proposition 1. *Let D be an evd-set of a connected graph G. Then, D is minimal if and only if for every $e \in D$, we have $P(e, D) \neq \emptyset$.*

Proof. Let D be a minimal *evd-set* of G. Suppose that $P(e, D) = \emptyset$ for some $e \in D$. Since the vertices ev-dominated by e are already ev-dominated by $D \setminus \{e\}$, the set

$D \setminus \{e\}$ thus remains an *evd-set of* G, contradicting the minimality of D. Hence $P(e,D) \neq \emptyset$.
Conversely, assume that for every $e \in D$, we have $P(e,D) \neq \emptyset$. Suppose that D is not minimal. Then, $D \setminus \{e^*\}$ is an *evd-set* of G for some $e^* \in D$. It follows that $P(e^*,D) = \emptyset$, contradicting our assumption. □

According to Definition 1, for any edge $e = xy$ in a $\gamma_{ev}(G)$-set D, let $\alpha_D^e(x) = P(e,D) \cap (N(x) - \{y\})$ and $\alpha_D^e(y) = P(e,D) \cap (N(y) - \{x\})$. Observe that $x \notin \alpha_D^e(y)$ and $y \notin \alpha_D^e(x)$ even when $x, y \in P(e,D)$.

Proposition 2. *Let G be a connected graph of order at least three with a unique $\gamma_{ev}(G)$-set D. Then for every edge $e = xy \in D$, we have $\alpha_D^e(x) \neq \emptyset$ and $\alpha_D^e(y) \neq \emptyset$.*

Proof. Suppose not, that for some edge $e = xy \in D$, either $\alpha_D^e(x) = \emptyset$ or $\alpha_D^e(y) = \emptyset$. Without loss of generality, let $\alpha_D^e(x) = \emptyset$. Let e' be an adjacent edge of e in G chosen incident with y if it is not a leaf, otherwise incident with x. Note that such an edge exists since G is connected of order at least three. In this case, the set $\{e'\} \cup D \setminus \{e\}$ is another $\gamma_{ev}(G)$-set, a contradiction to the uniqueness of D. Hence $\alpha_D^e(x) \neq \emptyset$ and likewise $\alpha_D^e(y) \neq \emptyset$. □

As an immediate consequence of Proposition 2 we have the following observation.

Observation 3. Let G be a connected graph of order at least three. If any pendant edge of G is in an $\gamma_{ev}(G)$-set, then G is not a UEVD-graph.

It is also noteworthy that the converse of Proposition 2 is not true. To see, simply consider the cycle C_4 that admits $\gamma_{ev}(C_4)$-sets of size one whereas each edge xy satisfies $\alpha_D^e(x) \neq \emptyset$ and $\alpha_D^e(y) \neq \emptyset$.
Recall that an *evd-set* D is said to be *independent* if no two edges of D have a common neighbor.

Proposition 3. *If G is a connected graph of order at least three with a unique $\gamma_{ev}(G)$-set D, then D is independent.*

Proof. Suppose that D contains two adjacent edges $e_1 = xy$ and $e_2 = xz$. By Observation 3, neither e_1 nor e_2 is a pendant edge. So let e be any edge incident with y. Clearly, $\{e\} \cup D - \{e_1\}$ is a $\gamma_{ev}(G)$-set different from D, a contradiction. □

The converse of Proposition 3 is not true in general. To see, consider the path P_6 that admits four $\gamma_{ev}(P_6)$-sets all of which are independent.

Proposition 4. *Let G be a connected graph of order at least three with a unique $\gamma_{ev}(G)$-set D. Then for every $e \notin D$, we have $\gamma_{ev}(G - e) \geq \gamma_{ev}(G)$.*

Proof. Suppose not, that $\gamma_{ev}(G-e) < \gamma_{ev}(G)$ for some $e \notin D$, and let D' be a $\gamma_{ev}(G-e)$-set. Hence the set D' *ev*-dominates all vertices of V_{G-e}, but since $V_{G-e} = V_G$, D' also *ev*-dominates V_G. This leads to a contradiction because of $|D'| < |D|$. Therefore $\gamma_{ev}(G-e) \geq \gamma_{ev}(G)$ for every $e \notin D$. □

Proposition 5. *Let G be a connected graph of order at least three with a unique $\gamma_{ev}(G)$-set D. Then for every $e \in D$, we have $\gamma_{ev}(G-e) > \gamma_{ev}(G)$.*

Proof. We first note that no edge of D is pendant, by Observation 3. Now, suppose that $\gamma_{ev}(G-e) \leq \gamma_{ev}(G)$ for some $e \in D$, and let D' be a $\gamma_{ev}(G-e)$-set. If $|D'| = \gamma_{ev}(G)$, then since $e \in D \setminus D'$ and D' *ev*-dominates V_{G-e} as well as V_G, we conclude that D' is a second $\gamma_{ev}(G)$-set, contradicting the uniqueness of D. Hence $|D'| < \gamma_{ev}(G)$. But then D' would be an *evd-set* smaller than D, a contradiction too. Therefore $\gamma_{ev}(G-e) > \gamma_{ev}(G)$ for every every $e \in D$. □

The converse of Proposition 5 is not true in general. For example, let G be the graph of order 10 obtained from a cycle C_8 whose vertices are labeled in order $x_1, x_2, \ldots, x_8, x_1$ by adding a two vertices y and z and the edges x_1x_5, yz, yx_3 and yx_7. Clearly, $X = \{yx_3, x_1x_5\}$ is a $\gamma_{ev}(G)$-set and $\gamma_{ev}(G-e) = 3 > \gamma_{ev}(G)$ for every $e \in X$. But X is not the only $\gamma_{ev}(G)$-set since $\{yx_7, x_1x_5\}$ is also a $\gamma_{ev}(G)$-set.

3. UEVD-trees

In this section, we investigate the trees T with unique $\gamma_{ev}(T)$-sets. In the first subsection, we establish three equivalent conditions for UEVD-trees, while in the second subsection we provide a constructive characterization of such trees.

3.1. Equivalent conditions for UEVD-trees

Theorem 1. *Let T be a tree of order at least three. Then the following conditions are equivalent:*

i) *T has a unique $\gamma_{ev}(T)$-set D.*

ii) *T has a $\gamma_{ev}(T)$-set D such that for every $e = xy \in D$, we have $\alpha^e_D(x) \neq \emptyset$ and $\alpha^e_D(y) \neq \emptyset$.*

iii) *T has a $\gamma_{ev}(T)$-set D containing no pendant edge such that $\gamma_{ev}(T-e) > \gamma_{ev}(T)$ for every $e \in D$.*

Proof. (i) ⇒ (ii) is true by Proposition 2 and (i) ⇒ (iii) is true by Proposition 5. Now, to prove the equivalence, we prove (iii) ⇒ (ii) and (ii)⇒(i).
(iii) ⇒ (ii). Let D be a $\gamma_{ev}(T)$-set that contains no pendant edge, and assume that $\gamma_{ev}(T-e) > \gamma_{ev}(T)$ for every $e \in D$. Suppose that there is an edge $e = xy \in D$ such that either $\alpha^e_D(x) = \emptyset$ or $\alpha^e_D(y) = \emptyset$, say $\alpha^e_D(x) = \emptyset$. By assumption, e is not a pendant edge. Let $e' \in E_T - D$ be an edge adjacent to e and incident with y. Then,

the set $(D \setminus \{e\}) \cup \{e'\}$ is a $\gamma_{ev}(T)$-set leading to $\gamma_{ev}(T-e) \leq \gamma_{ev}(T)$, a contradiction to our assumption.
(ii) $\Rightarrow$ (i). Let D be a $\gamma_{ev}(T)$-set such that for every $e = xy \in D$, we have $\alpha_D^e(x) \neq \emptyset$ and $\alpha_D^e(y) \neq \emptyset$. Clearly, if D contains two edges e_1 and e_2 incident with a common vertex, say u, then by definition $\alpha_D^{e_1}(u) = \alpha_D^{e_2}(u) = \emptyset$ yielding a contradiction. Therefore D is independent. Moreover, no edge $e = xy$ of D is pendant for otherwise either $\alpha_D^e(x) = \emptyset$ or $\alpha_D^e(y) = \emptyset$.
Now, to prove that D is the unique $\gamma_{ev}(T)$-set, we use an induction on the number of edges m of T. Clearly the base case is a path P_4 which has a unique $\gamma_{ev}(T)$-set. Assume that the result is true for all trees with sizes less than m. Now, let T be a tree with m edges. Let $e = xy$ be a non-pendant edge of T such that $e \notin D$. If such an edge does not exist, then T is a double star and certainly the unique edge in D is a unique $\gamma_{ev}(T)$-set. Hence we can assume that such an edge e exists. Consider the tree $T - e$ obtained from T by removing the edge e. Clearly each of the two components of T has order at least three, for otherwise the edge in the component of order two would be a pendant edge in T belonging to D, contradicting our earlier assumption. Let us denote by T_x the component of $T - e$ containing x, and likewise T_y is the component of $T - e$ containing y. Clearly, each of T_x and T_y has size less than m. Let $D_x = D \cap E_{T_x}$ and $D_y = D \cap E_{T_y}$. Then D_x is a $\gamma_{ev}(T_x)$-set and likewise D_y is a $\gamma_{ev}(T_y)$-set. In addition, since each edge $f = uv \in D_x$ still satisfies $\alpha_{D_x}^f(u) \neq \emptyset$ and $\alpha_D^f(v) \neq \emptyset$. By the induction hypothesis on T_x we have D_x is a unique $\gamma_{ev}(T_x)$-set and similarly D_y is a unique $\gamma_{ev}(T_y)$-set. Let r_x^* be the edge of D_x that *ev*-dominates x in T_x. Note that x might be an endvertex of r_x^* or not. Similarly, we can define r_y^* if necessary. Now assume that T has a second $\gamma_{ve}(T)$-set D', and let $D'_x = D' \cap E_{T_x}$ and $D'_y = D' \cap E_{T_y}$. If $e \notin D'$, then the unicity of D_x and D_y implies that $D'_x = D_x$ and $D'_y = D_y$. Therefore $D = D'$. Next suppose that $e \in D'$. In this case, it should be noted that $|D'| = |D'_x| + |D'_y| + 1$. Since $P(e, D') \neq \emptyset$ (by Proposition 1), either D'_x or D'_y is not an *evd-set* for T_x or T_y, respectively. Without loss of generality, assume that D'_x does not *ev*-dominate T_x. Notice that no edge incident with x in T_x belongs to D'_x. Hence let $e'_x \in E_{T_x}$ be any edge incident with x in T_x different from r_x^*. We note that such an edge e'_x can be chosen as desired. Indeed, if x is an endvertex of r_x^*, then r_x^* is not a pendant edge because of the unicity of D_x and thus e'_x can be chosen so that $e'_x \neq r_x^*$. Moreover, if x is not an endvertex of r_x^*, then e'_x is arbitrarily chosen. Therefore $D'_x \cup \{e'_x\}$ is an *evd-set* of T_x different from D_x, and since D_x is the unique $\gamma_{ev}(T_x)$-set, we must have $|D'_x \cup \{e'_x\}| > |D_x|$, that is $|D'_x| + 2 \geq |D_x|$. Similarly, if D'_y does not *ev*-dominate T_y, then $|D'_y| + 1 \geq |D_y|$ while if D'_y *ev*-dominates T_y, then $|D'_y| \geq |D_y|$. In either case, we may assume that $|D'_y| \geq |D_y|$. It follows that

$$|D'| = |D'_x| + |D'_y| + 1 \geq |D_x| - 2 + |D_y| + 1 > |D|,$$

a contradiction. Thus D is the only $\gamma_{ev}(T)$-set, which completes the proof. □

3.2. Characterization of UEVD-trees

The aim of this subsection is to provide a constructive characterization of the UEVD-trees. For this purpose, let $\mathcal{T}$ be the family of all trees that can be obtained from a sequence $T_1, T_2, ..., T_k$, $(k \geq 1)$, of trees T such that T_1 is the path P_4 with support vertices a and b, and if $k \geq 2$, then T_{i+1} can be obtained recursively from T_i by one of the operations defined below. Let $A(T_1) = \{ab\}$, $B(T_1) = V(P_4) - \{a, b\}$.

- Operation $\mathcal{O}_1$: Assume w is a support vertex of T_i. Then T_{i+1} is obtained from T_i by adding a new vertex v and the edge wv. Let $A(T_{i+1}) = A(T_i)$ and $B(T_{i+1}) = B(T_i) \cup \{v\}$.
- Operation $\mathcal{O}_2$: Assume w is a vertex of $B(T_i)$. Then T_{i+1} is obtained from T_i by adding a path $P_4 : u_1u_2u_3u_4$ and the edge u_1w. Let $A(T_{i+1}) = A(T_i) \cup \{u_2u_3\}$ and $B(T_{i+1}) = B(T_i) \cup \{u_1, u_4\}$.
- Operation $\mathcal{O}_3$: Assume w is a vertex of $B(T_i)$. Then T_{i+1} is obtained from T_i by adding a path $P_4 : u_1u_2u_3u_4$ and a new vertex u and the edges u_2u and uw. Let $A(T_{i+1}) = A(T_i) \cup \{u_2u_3\}$ and $B(T_{i+1}) = B(T_i) \cup \{u, u_1, u_4\}$.
- Operation $\mathcal{O}_4$: Assume w is a non-leaf vertex which is either a support vertex or adjacent to a support vertex of degree two in T_i. Then T_{i+1} is obtained from T_i by adding a path $P_4 : u_1u_2u_3u_4$ and the edge u_2w. Let $A(T_{i+1}) = A(T_i) \cup \{u_2u_3\}$ and $B(T_{i+1}) = B(T_i) \cup \{u_1, u_4\}$.
- Operation $\mathcal{O}_5$: Assume w is a vertex of T_i. Then T_{i+1} is obtained from T_i by adding t $(t \geq 1)$ paths $P_4 : u_1^j u_2^j u_3^j u_4^j$ and a new vertex u and the edges uw and $u_2^j u$ for every j. Let $A(T_{i+1}) = A(T_i) \cup \{u_2^j u_3^j : 1 \leq i \leq t\}$ and $B(T_{i+1}) = B(T_i) \cup \{u, u_1^j, u_4^j : 1 \leq i \leq t\}$.

Notice that from the way a tree $T \in \mathcal{T}$ is constructed, the set $A(T)$ is an edge-vertex dominating set of T. For a vertex v in a rooted tree T, we let $C(v)$ and $D(v)$ denote the set of *children* and *descendants*, respectively, of v. The *maximal subtree* at v is the subtree of T induced by $D(v) \cup \{v\}$, and is denoted by T_v. The *depth* of v is the largest distance from v to a vertex in $D(v)$.
In the rest of the paper, we shall prove:

Theorem 2. *A tree T is a UEVD-tree if and only if $T = P_2$ or $T \in \mathcal{T}$.*

We need the following lemmas.

Lemma 1. *If $T = P_2$ or $T \in \mathcal{T}$, then T has a unique $\gamma_{ev}(T)$-set.*

Proof. Clearly if $T = P_2$, then T has a unique $\gamma_{ev}(T)$-set. Hence assume that $T \in \mathcal{T}$. Then T can be constructed from a sequence $T_1, T_2, \ldots, T_k$ $(k \geq 1)$ of trees, where T_1 is a path P_4, and if $k \geq 2$, T_{i+1} can be obtained recursively from T_i by one of the

five operations defined above. We use the terminology of the construction for sets $A(T)$ and $B(T)$. If $k = 1$, then $T = P_4$ and clearly the edge of $A(T_1)$ is the unique $\gamma_{ev}(T_1)$-set. This establishes our basis case.
Assume that the result holds for all trees $T \in \mathcal{T}$ that can be constructed from a sequence of length at most $k-1$, and let $T' = T_{k-1}$. Applying our inductive hypothesis to $T' \in \mathcal{T}$ shows that $A(T')$ is the unique $\gamma_{ev}(T')$-set. Clearly, if T is obtained from T' using Operation $\mathcal{O}_1$, then $\gamma_{ev}(T) = \gamma_{ev}(T')$ and $A(T') = A(T)$ is the unique $\gamma_{ev}(T)$-set. Hence let us examine the following four cases.

Case 1. *T is obtained from T' using Operation $\mathcal{O}_2$.*
Certainly, $\gamma_{ev}(T) \leq \gamma_{ev}(T')+1$. The equality $\gamma_{ev}(T) = \gamma_{ev}(T')+1$ is obtained from the fact that there is a $\gamma_{ev}(T)$-set F containing the edge u_2u_3 and neither u_3u_4, u_1u_2 nor u_1w (if $u_1w \in F$, then it can replaced by wy, for some neighbor y of w in T'). Hence $A(T) = A(T') \cup \{u_2u_3\}$ is a $\gamma_{ev}(T)$-set. Now assume that T has another $\gamma_{ev}(T)$-set D different from $A(T)$, and recall that $w \in B(T')$. Clearly, $D \cap \{u_3u_4, u_3u_2\} \neq \emptyset$. Without loss of generality, assume that $u_3u_2 \in D$. If u_2u_1 or $u_1w \in D$, then for any edge f incident with w in T', the set $D' = \{f\} \cup D - \{u_2u_1, u_1w\}$ is also a $\gamma_{ev}(T)$-set for which $D' \cap E_{T'}$ is a $\gamma_{ev}(T')$-set that contains an edge incident with w, and thus becomes a second $\gamma_{ev}(T')$-set, a contradiction. Hence $u_2u_1, u_1w, u_3u_4 \notin D$, and thus $D - \{u_3u_2\}$ is again a $\gamma_{ev}(T')$-set different from $A(T')$, a contradiction. Therefore $A(T) = A(T') \cup \{u_2u_3\}$ is the unique $\gamma_{ev}(T)$-set.

Case 2. *T is obtained from T' using Operation $\mathcal{O}_3$.*
The inequality $\gamma_{ev}(T) \leq \gamma_{ev}(T') + 1$ follows from the fact that $A(T') \cup \{u_2u_3\}$ is an *evd-set* of T, and the equality $\gamma_{ve}(T) = \gamma_{ve}(T')+1$ follows from the fact that there is a $\gamma_{ev}(T)$-set that contains u_2u_3 and that does not contain the edges u_3u_4, u_1u_2, u_2u, uw. Hence $A(T) = A(T') \cup \{u_2u_3\}$ is a $\gamma_{ev}(T)$-set. Now assume that T has another $\gamma_{ev}(T)$-set D different from $A(T)$, and let $F = \{u_3u_4, u_2u_3, u_1u_2, u_2u, uw\}$. Clearly, $|D \cap F| \geq 1$. Now, if $|D \cap F| \geq 2$, then one can construct another $\gamma_{ev}(T)$-set D' that contains only the edge u_2u_3 and any the edge of F can be replaced by an edge incident with w in T'. Using the fact that $w \notin B(T')$, the set $D' \cap E_{T'}$ becomes a second $\gamma_{ev}(T')$-set, a contradiction. Hence $|D \cap F| = 1$, and thus $u_2u_3 \in D$. But then $D' \cap E_{T'}$ is also a second $\gamma_{ev}(T')$-set, a contradiction. Therefore $A(T) = A(T') \cup \{u_2u_3\}$ is the unique $\gamma_{ev}(T)$-set.

Case 3. *T is obtained from T' using Operation $\mathcal{O}_4$.*
Then $\gamma_{ev}(T) \leq \gamma_{ev}(T') + 1$ since $A(T') \cup \{u_2u_3\}$ is an *evd-set* of T. The equality follows from the fact that there is a $\gamma_{ev}(T)$-set that contains u_2u_3 and an edge with endvertices w and some neighbor of w in T'. Consequently, $A(T) = A(T') \cup \{u_2u_3\}$ is a $\gamma_{ev}(T)$-set. Now assume that T has a second $\gamma_{ev}(T)$-set D different from $A(T)$, and let $F = \{u_3u_4, u_2u_3, u_1u_2, u_2w\}$. Then $|D \cap F| \geq 1$. If $|D \cap F| \geq 2$, then we must have u_2u_3 and $u_2w \in D$. The minimality of D implies that w is a support vertex in T' with leaf neighbor w'. In this case, the set $D' = \{ww'\} \cup D - \{u_2w\}$ is a $\gamma_{ev}(T)$-set for which $D' \cap E_{T'}$ is a $\gamma_{ev}(T')$-set that contains a pendant edge, contradicting the unicity of $A(T')$. Hence $|D \cap F| = 1$, implying that $u_2u_3 \in D$. Since w is either a support vertex or adjacent to a support vertex of degree two in T', the set D must

contain an edge incident with w. In that case $D' \cap E_{T'}$ is $\gamma_{ev}(T')$-set different from $A(T')$, a contradiction. Therefore $A(T) = A(T') \cup \{u_2u_3\}$ is the unique $\gamma_{ev}(T)$-set.

Case 4. *T is obtained from T' using Operation $\mathcal{O}_5$.*
Then $\gamma_{ev}(T) \leq \gamma_{ev}(T') + t$ since $A(T) = A(T') \cup \{u_2^j u_3^j : 1 \leq j \leq t\}$. The equality follows from the fact that there is a $\gamma_{ev}(T)$-set that contains the edge $u_2^j u_3^j$ for every $j \in \{1, \ldots, t\}$ and neither any edge incident with u nor any edge of the t added paths P_4. Therefore $A(T) = A(T') \cup \{u_2^j u_3^j : 1 \leq j \leq t\}$ is a $\gamma_{ev}(T)$-set. Finally, as for the previous cases, it is easy to show that the uniqueness of $A(T')$ leads to the uniqueness of $A(T)$.
Through all situations, we conclude that $A(T)$ is the unique $\gamma_{ev}(T)$-set and T is UEVD-tree. □

Lemma 2. *If T is a nontrivial tree with a unique $\gamma_{ev}(T)$-set, then $T = P_2$ or $T \in \mathcal{T}$.*

Proof. If the number of vertices, n of T, is two, then $T = P_2$. Hence we assume that $n \geq 3$. To show that $T \in \mathcal{T}$ we use an induction on n. Since there is no tree T of order three with a unique unique $\gamma_{ev}(T)$-set, let $n \geq 4$. If $n = 4$, then $T = P_4$ and clearly $T \in \mathcal{T}$. This establishes the base case. Let $n \geq 5$ and assume that any tree T' of order $n' < n$ having a unique $\gamma_{ev}(T')$-set belongs to the family $\mathcal{T}$. Let T be a tree of order n with a unique $\gamma_{ev}(T)$-set D. Recall that by Observation 3, no pendant edge belongs to D and by Proposition 3, D is independent.
First, assume that T has a strong support vertex u, and let x and y be two leaves adjacent to u. Let $T' = T - x$. It is easy to see that $\gamma_{ev}(T) = \gamma_{ev}(T')$ and that the uniqueness of D implies that it is also the unique $\gamma_{ev}(T')$-set. By the inductive hypothesis on T', we have $T' \in \mathcal{T}$. Since the tree T can be obtained from T' by using Operation $\mathcal{O}_1$, we deduce that $T \in \mathcal{T}$. Therefore, in the sequel we will assume that every support vertex of T is weak, that is, adjacent to exactly one leaf. Since $n \geq 5$ and every support vertex is weak, we conclude that $\operatorname{diam}(T) \geq 4$.
Let $v_1, v_2, \ldots, v_k$ ($k \geq 5$) be a diametral path in T chosen so that $d_T(v_3)$ is as small as possible. Root T at v_k. Clearly, $d_T(v_2) = 2$, and $v_2v_3 \in D$. If v_3 has a child of degree 2, say y, other than v_2, then D must contain the pendant edge incident with y, which leads to a contradiction. Thus v_2 is the unique child of v_3 of degree 2. Hence either $d_T(v_3) = 2$ or $d_T(v_3) = 3$ and v_3 is a weak support vertex.
Assume first that $d_T(v_3) = 2$. By Proposition 2, $\alpha_D^{v_2v_3}(v_3) \neq \emptyset$ and thus v_4 is a private vertex of the edge v_2v_3. Then v_4 must have degree 2 for otherwise any child of v_4 would be an end-vertex of an edge belonging to D, contradicting $v_4 \in P(v_2v_3, D)$. Let $T' = T - T_{v_4}$. The unicity of D implies that $n' \geq 4$. Since $D - \{v_2v_3\}$ *ev*-dominates $V(T')$, $\gamma_{ev}(T') \leq \gamma_{ev}(T) - 1$. The equality follows from the fact that any $\gamma_{ev}(T')$-set can be extended to an *evd-set* of T by adding to it the edge v_2v_3. Therefore $\gamma_{ev}(T') = \gamma_{ev}(T) - 1$, and $D \cap E_{T'}$ is a $\gamma_{ev}(T')$-set. Now, if D' is a $\gamma_{ev}(T')$-set different from $D \cap E_{T'}$, then $D' \cup \{v_2v_3\}$ would be a $\gamma_{ev}(T)$-set different from D, a contradiction. Hence $D \cap E_{T'}$ is the unique $\gamma_{ev}(T')$-set for which we notably have no edge incident with v_5 in T' belonging to $D \cap E_{T'}$ (because of v_4 is a private vertex of

v_2v_3 with respect to D). By the inductive hypothesis on T', we have $T' \in \mathcal{T}$, where $v_5 \in B(T')$. Therefore $T \in \mathcal{T}$ because it can be obtained from T' by using Operation $\mathcal{O}_2$.
In the sequel, we can assume that v_3 is a support vertex of degree three. Let v_3' be the unique leaf neighbor of v_3. We consider the following two cases.

Case 1. v_4 is an endvertex of some edge belonging to D.
Let f be the edge of D incident with v_4. First, suppose that $f = v_4v_5$. Since by Proposition 2, $\alpha_D^f(v_4) \neq \emptyset$, we deduce that some child of v_4, say z, belongs to $\alpha_D^f(v_4)$. We claim that z is a leaf, and thus v_4 is a support vertex. Suppose not, and let z' be a child of z, and z'' the child (if any) of z'. Regardless of the existence or not of the vertex z'', D must contain the edge zz', which contradicts the fact that $z \in \alpha_D^f(v_4)$. Hence z is leaf. Second, assume that $f \neq v_4v_5$, and let z be a child of v_4 such that $f = zv_4$. Clearly, z is not a leaf (since D contains no pendant edge). A similar argument to that used above, it can be shown that z is a support vertex of degree two. Consequently, v_4 is either a support vertex or has a child which is a support vertex of degree two. Now, whatever the situation that occurs, let $T' = T - T_{v_3}$. By Proposition 2, $\alpha_D^f(v_4) \neq \emptyset$ we deduce that T' has order at least four. On the other hand, one can easily see that $\gamma_{ev}(T') = \gamma_{ev}(T) - 1$, and that the unicity of D implies that $D \cap E_{T'}$ is also the unique $\gamma_{ev}(T')$-set containing the edge f which is incident with v_4. By the inductive hypothesis on T', we have $T' \in \mathcal{T}$, where v_4 is either a support vertex of T' or adjacent to support vertex of degree two. Therefore $T \in \mathcal{T}$ because it can be obtained from T' by using Operation $\mathcal{O}_4$.

Case 2. v_4 is not an endvertex of any edge of D.
Clearly, v_4 cannot be a support vertex in T. Consider two subcases.

Subcase 2.1. $v_4 \in P(v_2v_3, D)$.
Hence no edge incident with v_5 belongs to D, in particular $v_4v_5 \notin D$. We claim that $d_T(v_4) = 2$. Suppose to the contrary that $d_T(v_4) \geq 3$, and let y be any child of v_4 different from v_3. According to the diametrical path, y has depth at most two and therefore D must contain an edge incident with y. But then v_4 is no longer a private neighbor of v_2v_3 with respect to D, a contradiction. Hence $d_T(v_4) = 2$.
Now, let $T' = T - T_{v_4}$. Since v_5 is not ev-dominated by v_2v_3, we deduce that $D \cap E_{T'} \neq \emptyset$. Moreover, the unicity of D requires that T' has order $n' \geq 4$. Also, it is easy to see that $\gamma_{ev}(T') = \gamma_{ev}(T) - 1$, and that the unicity of D implies that $D \cap E_{T'}$ is the unique $\gamma_{ev}(T')$-set in which v_5 is not an endvertex of any edge of $D \cap E_{T'}$. By the inductive hypothesis on T', we have $T' \in \mathcal{T}$, where $v_5 \in B(T')$. Therefore $T \in \mathcal{T}$ because it can be obtained from T' by using Operation $\mathcal{O}_3$.

Subcase 2.2. $v_4 \notin P(v_2v_3, D)$.
We claim that every subtree rooted at a child of v_4 (if any other than v_3) is isomorphic to T_{v_3}. To see, let y be a child of v_4 different from v_3. Since v_4 is not a support vertex, $d_T(y) \geq 2$. Recall that T has no strong support vertex. Now, since v_4 is not an endvertex of any edge of D, the vertex y cannot be a support vertex of degree two. Moreover, the choice of diametral path with the condition that $d_T(v_3)$ is a small as possible, vertex y cannot be in a path of length three starting from v_4 in which y and its

child are of degree two. Consequently, according the cases considered above, T_y must be a path P_4 in which y is a support vertex. Now, let $p = d_T(v_4)$ and $T' = T - T_{v_4}$. Clearly, by Proposition 2 and the fact that v_5 is not ev-dominated by an edge incident with v_4, the order of T' is $n' \geq 4$. Also, one can see that $\gamma_{ev}(T') = \gamma_{ev}(T) - p + 1$, and that $D \cap E_{T'}$ is the unique $\gamma_{ev}(T')$-set. By the inductive hypothesis on T', we have $T' \in \mathcal{T}$, and therefore $T \in \mathcal{T}$ because it is obtained from T' by using Operation $\mathcal{O}_5$. □

According to Lemmas 1 and 2, the proof of Theorem 2 is achieved.

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References

[1] M. Chellali and T.W. Haynes, *Trees with unique minimum paired-dominating sets*, Ars Combin. **73** (2004), 3–12.

[2] ———, *A characterization of trees with unique minimum double dominating sets*, Util. Math. **83** (2010), 233–242.

[3] M. Fischermann and L. Volkmann, *Unique minimum domination in trees*, Australas. J. Combin. **25** (2002), 117–124.

[4] G. Gunther, B. Hartnell, L.R. Markus, and D. Rall, *Trees with unique minimum paired-dominating sets*, Congr. Numer. **101** (1994), 55–63.

[5] T.W. Haynes and M.A. Henning, *Trees with unique minimum total dominating sets*, Discuss. Math. Graph Theory **22** (2002), no. 2, 233–246 https://doi.org/10.7151/dmgt.2349.

[6] ———, *Trees with unique minimum semitotal dominating sets*, Graphs Combin. **36** (2020), no. 3, 689–702 https://doi.org/10.1007/s00373–020–02145–0.

[7] ———, *Unique minimum semipaired dominating sets in trees*, Discuss. Math. Graph Theory **43** (2023), no. 1, 35–53 https://doi.org/10.7151/dmgt.2349.

[8] B. Krishnakumari, Y.B. Venkatakrishnan, and M. Krzywkowski, *On trees with total domination number equal to edge-vertex domination number plus one*, Proc. Math. Sci. **126** (2016), 153–157 https://doi.org/10.1007/s12044–016–0267–6.

[9] J.R. Lewis, *Vertex-edge and edge-vertex domination in graphs*, Ph.D. thesis, Clemson University, Clemson, 2007.

[10] J.W. Peters, *Theoretical and algorithmic results on domination and connectivity*, Ph.D. thesis, Clemson University, Clemson, 1986.

[11] Y.B. Venkatakrishnan and B. Krishnakumari, *An improved upper bound of edge–vertex domination number of a tree*, Information Processing Letters **134** (2018), 14–17
https://doi.org/10.1016/j.ipl.2018.01.012.

[12] W. Zhao, F. Wang, and H. Zhang, *Construction for trees with unique minimum dominating sets*, Int. J. Comput. Math. Comput. Sys. Theory **3** (2018), no. 3, 204–213
https://doi.org/10.1080/23799927.2018.1531930.