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Research Article

On coherent configuration of circular-arc graphs

Fatemeh Raei Barandagh[1,*] and Amir Rahnamai Barghi[2]

[1]Department of Mathematics Education, Farhangian University, P.O. Box 14665-889, Tehran, Iran
f.raei@cfu.ac.ir

[2]Department of Mathematics, K. N. Toosi University of Technology, Tehran, Iran
rahnama@kntu.ac.ir

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Abstract: For any graph, Weisfeiler and Leman assigned the smallest matrix algebra which contains the adjacency matrix of the graph. The coherent configuration underlying this algebra for a graph Γ is called the coherent configuration of Γ, denoted by $\mathcal{X}(\Gamma)$. In this paper, we study the coherent configuration of circular-arc graphs. We give a characterization of the circular-arc graphs Γ, where $\mathcal{X}(\Gamma)$ is a homogeneous coherent configuration. Moreover, all homogeneous coherent configurations which are obtained in this way are characterized as a subclass of Schurian coherent configurations.

Keywords: Coherent configuration, homogeneous, circular-arc graph, wreath product

AMS Subject classification: 05E30, 05C75

1. Introduction

The theory of coherent configuration was started by Higman in [7] as a tool for studying permutation groups, and it was independently started by Weisfeiler and Leman in [13] as the theory of stationary graphs for studying graph isomorphism problem. For each graph, Weisfeiler and Leman assigned the smallest matrix algebra which contains the adjacency matrix of the graph. The importance of this algebra lies in the well-known fact that the automorphism group of each graph is equal to the automorphism group of its matrix algebra. The coherent configuration underlying this algebra for a graph Γ is called *the coherent configuration of* Γ or the *scheme of* Γ, and denoted by $\mathcal{X}(\Gamma)$. In other words, $\mathcal{X}(\Gamma)$ is the smallest coherent configuration

* *Corresponding Author*

on the vertex set of Γ such that the edge set of Γ is the union of some basic relations of $\mathcal{X}(\Gamma)$ [5].

For any finite permutation group G one can associate a coherent configuration denoted by $\mathrm{Inv}(G)$. One of the most important concepts in the theory of coherent configurations is to characterize Schurian coherent configurations. A coherent configuration $\mathcal{X}$ is said to be *Schurian* if $\mathcal{X} = \mathrm{Inv}(G)$ for some permutation group G, see [14]. In general, the Schurity problem (even for a class consisting of only one coherent configuration) is quite difficult [5]. But there are non-trivial classes of coherent configurations such that the Schurity problem is completely solved for them, e.g. 1-regular coherent configurations and the coherent configurations of algebraic forests includes cographs, trees, interval graphs and rooted-directed path graphs [6].

In this paper, we study the Schurity problem for the coherent configurations of other interesting family of intersection graphs, namely, circular-arc graphs which are the intersection graphs of a family of arcs of a circle. Circular-arc graphs received considerable attention by Tucker [10–12], and Durán, Lin and McConnel [4, 8, 9]. Several characterizations and recognition algorithms have been formulated for circular-arc graphs [2, 8]. In this paper, we are interested in an algebraic description of circular-arc graphs.

Our main results give a characterization of circular-arc graphs Γ, where $\mathcal{X}(\Gamma)$ is a homogeneous coherent configuration, or in other words, an association scheme. The characterization is established in the terms of *wreath product* of graphs, also called the lexicographic product. The rest of this section is to state our results.

Theorem 1. *Let Γ be a circular-arc graph. $\mathcal{X}(\Gamma)$ is homogeneous if and only if Γ is isomorphic to the wreath product of a complete graph and an elementary circular-arc graph.*

Let n be a positive integer and let S be a subset of $\mathbb{Z}_n$ such that $S = \{\pm 1, \dots, \pm k\}$ for $0 \le 2k + 1 < n$. Then the Cayley graph $\mathrm{Cay}(\mathbb{Z}_n, S)$ is circular-arc (see Sec. 5), and we call it *elementary circular-arc graph*. For $k = 0$ it is empty graph and for $k = 1$ it is an undirected cycle. In fact, analysis duplicating the vertices of an elementary circular-arc graph shows that the wreath product of a complete graph and an elementary circular-arc graph is a circular-arc graph.

In the following theorem we give a characterization of homogeneous coherent configurations which are the coherent configuration of circular-arc graphs.

Theorem 2. *Let $\mathcal{X} = \mathcal{X}(\Gamma)$, where Γ is a circular-arc graph. Then $\mathcal{X}$ is homogeneous if and only if it is isomorphic to the wreath product of a rank 2 coherent configuration and $\mathcal{X}_0$, where $\mathcal{X}_0$ is one of the following:*

(i) a rank 2 coherent configuration,

(ii) the wreath product of two rank 2 coherent configurations,

(iii) the coherent configuration of a dihedral group.

We know that any rank 2 coherent configuration and any coherent configuration of a dihedral group are Schurian. Moreover, the wreath product of two Schurian coherent configurations is Schurian, see [14]. Thus we have the following corollary:

Corollary 1. *Let $\mathcal{X}$ be a homogeneous coherent configuration such that $\mathcal{X} = \mathcal{X}(\Gamma)$ for a circular-arc graph Γ. Then $\mathcal{X}$ is Schurian.*

The automorphism group of each graph is equal to the automorphism group of its coherent configuration, see [13]. Moreover, it is well-known that the automorphism group of the wreath product of two coherent configurations is equal to the wreath product of their automorphism groups. Denote by S_n the symmetric group acting on n points, by D_{2n} the dihedral group on n elements, and by $G \wr H$ the wreath product of two groups G and H. The following corollary is a consequence of Theorems 1 and 2.

Corollary 2. *Let Γ be a circular-arc graph on n vertices, such that $\mathcal{X}(\Gamma)$ is homogeneous. Then there is an even integer k, $k|n$, such that $\mathrm{Aut}(\Gamma)$ is isomorphic to $S_{\frac{n}{k}} \wr G$, where G is S_k or $S_2 \wr S_{\frac{k}{2}}$ or D_{2k}.*

This paper is organized as follows. In Section 2, we present some preliminaries on graph theory and coherent configurations. In Section 3, we first remind the concept of circular-arc graphs and then we introduce arc-function and reduced arc-function of a circular-arc graph. Moreover, we characterize non-empty regular circular-arc graphs without twins. Section 4 contains relationship between wreath product of graphs and wreath product of their coherent configurations. In Section 5, we define elementary circular-arc graphs, and then we characterize their coherent configurations. Finally, in Section 6 we give the proof of Theorem 1 and Theorem 2.

Notation. Throughout the paper, V denotes a finite set. The diagonal of the Cartesian product V^2 is denoted by 1_V.

For $r, s \subset V^2$ and $X, Y \subset V$ we have the following notations:

$$r^* = \{(u, v) \in V^2 : \ (v, u) \in r\},$$

$$r_{X,Y} = r \cap (X \times Y)\ ,\ r_X = r_{X,X},$$

$$r \cdot s = \{(v, u) \in V^2 : \ (v, w) \in r,\ (w, u) \in s \ \text{ for some } w \in V\},$$

$$r \otimes s = \{((v_1, v_2), (u_1, u_2)) \in V^2 \times V^2 : \ (v_1, u_1) \in r \ \text{ and } (v_2, u_2) \in s\},$$

Also for any $v \in V$, set $vr = \{u \in V : \ (v, u) \in r\}$ and $n_r(v) = |vr|$.

For $S \in 2^{V^2}$ denote by $S^{\cup}$ the set of all unions of the elements of S, and set $S^* = \{s^* : \ s \in S\}$ and $vS = \cup_{s \in S} vs$. For $T \in 2^{V^2}$, set

$$S \cdot T = \{s \cdot t : \ s \in S, t \in T\}.$$

For an integer n, let $\mathbb{Z}_n$ be the ring of integer numbers modulo n. Set

$$A\mathbb{Z}_n := \{\{i, i+1, \ldots, i+k\} : i, k \in \mathbb{Z}_n \text{ and } k \neq n-1\}.$$

For each set $\{i, i+1, \ldots, i+k\}$, the points i and $i+k$ are called the end-points of the set.

2. Preliminaries

2.1. Graphs.

All terminologies and definitions of graph theory have been adapted from [1]. In this paper, we consider finite and undirected simple graphs. We denote a complete graph on n vertices by K_n, and a cycle on n vertices by C_n.

Let $\Gamma = (V, R)$ be a graph with vertex set V and edge set R. Let E be an equivalence relation on V, then $\Gamma_{V/E}$ is a graph with vertex set V/E in which distinct vertices X and Y are adjacent if and only if at least one vertex in X is adjacent in Γ with some vertex in Y. For every subset X of V, the graph Γ_X is the subgraph of Γ induced by X.

Let Γ_i be a graph on V_i, for $i = 1, 2$. The graphs Γ_1 and Γ_2 are *isomorphic* if there is a bijection $f : V_1 \to V_2$, such that two vertices u and v in V_1 are adjacent in Γ_1 if and only if their images $f(u)$ and $f(v)$ are adjacent in Γ_2. Such a bijection is called an *isomorphism* between Γ_1 and Γ_2. The set of all isomorphism between Γ_1 and Γ_2 is denoted by $\mathrm{Iso}(\Gamma_1, \Gamma_2)$. An isomorphism from a graph to itself is called an *automorphism*. The set of all automorphisms of a graph Γ is the *automorphism group* of Γ, and is denoted by $\mathrm{Aut}(\Gamma)$.

The *lexicographic product* or *wreath product* of graphs Γ_1 and Γ_2 is the graph $\Gamma_1 \wr \Gamma_2$ with vertex set $V_1 \times V_2$ in which (u_1, u_2) is adjacent to (v_1, v_2) if and only if either u_1 and v_1 are adjacent in Γ_1 or $u_1 = v_1$ and also u_2 and v_2 are adjacent in Γ_2.

Let $\Gamma = (V, R)$ be a graph. Two vertices $u, v \in V$ are *twins* if u and v are adjacent in Γ and $vR \backslash \{u\} = uR \backslash \{v\}$, where the set of neighbors of a vertex v in the graph Γ is denoted by vR.

2.2. Coherent configurations.

In this part all terminologies and notations are based on [3, 5].

Definition 1. A pair $\mathcal{X} = (V, S)$, where V is a finite set and S is a partition of V^2, is called a *coherent configuration* on V if $1_V \in S^{\cup}$, $S^* = S$, and for any $r, s, t \in S$ the number

$$c_{rs}^t := |vr \cap us^*|$$

does not depend on the choice of $(v,u) \in t$. The coherent configuration $\mathcal{X}$ is called *homogeneous* if $1_V \in S$.

The elements of V, S, $S^\cup$ and the numbers c_{rs}^t are called the *points*, the *basic relations*, the *relations* and the *intersection numbers* of $\mathcal{X}$, respectively. The numbers $|V|$ and $|S|$ are called the *degree* and the *rank* of $\mathcal{X}$. A unique basic relation containing a pair $(v,u) \in V^2$ is denoted by $r_{\mathcal{X}}(v,u)$ or $r(v,u)$.

An equivalence relation on a subset of V belonging to $S^\cup$ is called an equivalence relation of $\mathcal{X}$. Any coherent configuration has *trivial* equivalence relations: 1_V and V^2. Let $e \in S^\cup$ be an equivalence relation. For a given $X \in V/e$ the *restriction* of $\mathcal{X}$ to X is

$$\mathcal{X}_X = (X, S_X),$$

where S_X is the set of all non-empty relations r_X with $r \in S$. The *quotient* of $\mathcal{X}$ modulo e is defined

$$\mathcal{X}_{V/e} = (V/e, S_{V/e}),$$

where $S_{V/e}$ is the set of all non-empty relations of the form $\{(X,Y) : \ s_{X,Y} \neq \emptyset\}$, with $s \in S$.
Two coherent configurations $\mathcal{X}_1$ and $\mathcal{X}_2$ are called *isomorphic* if there exists a bijection between their point sets in such a way that induces a bijection between their sets of basic relations. Such a bijection is called an *isomorphism* between $\mathcal{X}_1$ and $\mathcal{X}_2$. The set of all isomorphism between $\mathcal{X}_1$ and $\mathcal{X}_2$ is denoted by $\mathrm{Iso}(\mathcal{X}_1, \mathcal{X}_2)$. The group of all isomorphisms of $\mathcal{X}$ to itself contains a normal subgroup

$$\mathrm{Aut}(\mathcal{X}) = \{f \in \mathrm{Sym}(V) : s^f = s, s \in S\}$$

called the *automorphism group* of $\mathcal{X}$ where $s^f = \{(u^f, v^f) : (u,v) \in s\}$.

The *wreath product* $\mathcal{X}_1 \wr \mathcal{X}_2$ of $\mathcal{X}_1 = (V_1, S_1)$ and $\mathcal{X}_2 = (V_2, S_2)$ is a coherent configuration on $V_1 \otimes V_2$ with the following basic relations

$$V_1^2 \otimes r, \quad \text{for } r \in S_2 \backslash 1_{V_2} \quad \text{and} \quad s \otimes 1_{V_2}, \quad \text{for } s \in S_1.$$

2.3. The coherent configuration of a graph.

There is a natural partial order " $\leq$ " on the set of all coherent configurations on V. Namely, given two coherent configurations $\mathcal{X} = (V,S)$ and $\mathcal{X}' = (V,S')$ we set

$$\mathcal{X} \leq \mathcal{X}' \ \Leftrightarrow \ S^\cup \subseteq (S')^\cup.$$

In this case, $\mathcal{X}$ is called a *fusion* of $\mathcal{X}'$ and $\mathcal{X}'$ is called a *fission* (extension) of $\mathcal{X}$. The minimal and maximal elements with respect to " $\leq$ " are the *trivial* and the *complete* coherent configuration on V, respectively: the basic relations of the former one are the reflexive relation 1_V and (if $|V| > 1$) its complement in V^2, whereas the relations of the later one are all binary relations on V.

Let R be an arbitrary relation on the set V. Denote by $\mathcal{X}(R)$ the smallest coherent configuration with respect to " $\leq$ " such that R is a union of its basic relations. Let $\Gamma = (V, R)$ be a graph with vertex set V and edge set R. By the coherent configuration of Γ we mean $\mathcal{X}(\Gamma) = \mathcal{X}(R)$. For example, if Γ is a complete graph or empty graph with at least 2 vertices, then its coherent configuration is the trivial coherent configuration of rank 2. One can check that if $\mathcal{X}(\Gamma)$ is homogeneous, then Γ is a regular graph.
In general, it is quite difficult to find the coherent configuration of an arbitrary graph. In [6], the coherent configuration of a graph has been studied for some classes of graphs.

3. Circular-arc graphs

From [1], for a given family $\mathcal{F}$ of subsets of V, one may associate an *intersection graph*. This is the graph whose vertex set is $\mathcal{F}$, two different sets in $\mathcal{F}$ being adjacent if their intersection is non-empty. Circular-arc graph is the intersection graph of a family of arcs of a circle.

Lemma 1. *Let Γ be a graph on V with n vertices. Then Γ is a circular-arc graph if and only if there exists a function $f : V \to A\mathbb{Z}_m$, for some m, such that Γ is the intersection graph of the family $\mathrm{Im}(f) = \{f(v) : v \in V\}$. Moreover, this function can be chosen such that*

(1) any element of $\mathbb{Z}_m$ is the end-point of at least one set in $\mathrm{Im}(f)$,

(2) each set in $\mathrm{Im}(f)$ contains at least two elements.

Proof. To prove sufficient part, let Γ be a circular-arc graph. Then by the definition it is the intersection graph of some arcs of a circle C. Without loss of generality, we may assume that the end-points of any of these arcs are distinct. Let m be the number of these end-points and let $A = \{a_0, a_1, \ldots, a_{m-1}\}$ be the set of all of them. It is clear that $m \leq 2n$. Here the indices of the points a_i are the elements of $\mathbb{Z}_m$; they are chosen in such a way that the point a_i precedes the point a_{i+1} in the clockwise order of the circle C. Then for any vertex $v \in V$ there exist uniquely determined elements $i_v, j_v \in \mathbb{Z}_m$ such that

$$A_v := C_v \cap A = \{a_{i_v}, a_{i_v+1} \ldots, a_{j_v}\},$$

where C_v is a subset of C which is the arc corresponding to v. Moreover, it is easily seen that $C_u \cap C_v$ is not empty if and only if $i_v \in A_u$ or $j_v \in A_u$ or $i_u, j_u \in A_v$.

Therefore, the vertices u and v are adjacent if and only if the set $A_u \cap A_v$ is not empty. Now define a function $f : V \to A\mathbb{Z}_m$ by

$$f(v) = \{i_v, i_v + 1, \ldots, j_v\}.$$

Then Γ is the intersection graph of the family $\mathrm{Im}(f)$. Moreover, statements (1) and (2) immediately follow from the definition of f.
Conversely, let $m \leq 2n$ and $f : V \to A\mathbb{Z}_m$ be a function such that Γ is the intersection graph of $\mathrm{Im}(f)$. Consider a circle C and choose m distinct points on it. We may label these points by the elements of $\mathbb{Z}_m$ such that these points appear in C in clockwise order. Since $\mathrm{Im}(f) \subset A\mathbb{Z}_m$ for each vertex $v \in V$, there exist $i_v, j_v \in \mathbb{Z}_m$ such that $f(v) = \{i_v, i_v + 1, \ldots, j_v\}$. We correspond an arc $C_v \subset C$, from i_v to j_v in clockwise order to the vertex v. It is clear that the set $f(u) \cap f(v)$ is not empty if and only if $C_u \cap C_v$ is not empty. It follows that Γ is the intersection graph of the set $\{C_v : v \in V\}$. So it is a circular-arc graph. This completes the proof of the lemma. □

The function $f : V \to A\mathbb{Z}_m$ satisfying statements (1) and (2) and conditions of Lemma 1 is called the *arc-function* of the graph Γ and the number m is called the *length* of f.

Theorem 3. *Let $\Gamma = (V, R)$ be a non-empty circular-arc graph on n vertices. Suppose that for any two vertices u and v in V we have:*

$$v \in uR \quad \Rightarrow \quad uR \not\subset \{v\} \cup vR. \tag{1}$$

Then there exists an arc-function f of Γ such that the following statements hold:

(i) no set of $\mathrm{Im}(f)$ is a subset of another set of $\mathrm{Im}(f)$,

(ii) the length of f is equal to n,

(iii) any element $i \in \mathbb{Z}_n$ is the end-point of exactly two sets in $\mathrm{Im}(f)$.

Remark 1. The graph Γ satisfying condition (1) is a connected graph. Indeed, otherwise, it is easily seen that Γ is an interval graph. On the other hand, each interval graph is chordal, and so it has a vertex whose neighborhood is a complete graph (see [1]) which contradicts the condition (1).

Proof. By Lemma 1, there exists an arc-function f' of Γ of length $m' \leq 2n$. Denote by $\sim$ the binary relation on $\mathbb{Z}_{m'}$ defined by $i \sim j$ if and only if for any $v \in V$

$$i, j \in f'(v) \text{ or } i, j \notin f'(v).$$

One can check that $\sim$ is an equivalence relation, and its equivalence classes belong to $A\mathbb{Z}_{m'}$. By the definition of " $\sim$ " any element of $\mathrm{Im}(f')$ is a disjoint union of some classes. Let us define a function f such that for each $v \in V$, $f(v)$ is the set of $\sim$-classes contained in $f'(v)$. The equivalence classes of $\sim$ can be identified by $\mathbb{Z}_m$. By this identification, we have $f(v) \in A\mathbb{Z}_m$.
We claim that f is an arc-function of Γ. Indeed, from the definition of f it follows that for each two vertices u and v, the set $f(u) \cap f(v)$ is not empty if and only if the set $f'(u) \cap f'(v)$ is not empty. Moreover, statement (1) of Lemma 1 is obvious. Thus, it is sufficient to verify that statement (2) of Lemma 1 occurs. Suppose on the contrary, that there is a vertex $v \in V$ such that $f(v)$ contains exactly one element. Then $f'(v)$ is a class of the equivalence $\sim$. By condition (1) this implies that v is an isolated vertex in Γ, which is impossible by Remark 1. Thus f is an arc-function of Γ.
By Lemma 1, the graph Γ is isomorphic to the intersection graph of the family $\mathrm{Im}(f')$. Thus for two adjacent vertices u and v in V, if $f'(u) \subseteq f'(v)$ then any vertex in $V \setminus \{v\}$ which is adjacent to u in Γ is also adjacent to v. On the other hand, it is easy to see that $f'(u) \subseteq f'(v)$ is equivalent to $f(u) \subseteq f(v)$. Therefore, we have

$$f(u) \subseteq f(v) \quad \Rightarrow \quad uR \subset \{v\} \cup vR. \tag{2}$$

Hence, statement (i) follows from condition (1).
Statement (ii) is a consequence of statement (iii). First we will show that any element $i \in \mathbb{Z}_m$ is the end-point of exactly two sets in $\mathrm{Im}(f)$. Suppose on the contrary that there is an element $i \in \mathbb{Z}_m$ which is an end-point of at least three sets of $\mathrm{Im}(f)$. By statement (2) of Lemma 1, there are at least two sets $f(u)$ and $f(v)$ such that

$$i+1 \in f(v) \cap f(u) \;\; or \;\; i-1 \in f(v) \cap f(u).$$

It follows that in any case $f(v) \subseteq f(u)$ or $f(u) \subseteq f(v)$. From (2), this contradicts condition (1). Thus any element $i \in \mathbb{Z}_m$ is the end-point of at most two sets in $\mathrm{Im}(f)$. To complete the proof, suppose that there exists $i \in \mathbb{Z}_m$ which is an end-point of exactly one set, say $f(v)$, in $\mathrm{Im}(f)$. By statement (2) of Lemma 1, we have $i+1 \in f(v)$ or $i-1 \in f(v)$. Suppose that the former inclusion holds. Then by (2) we have

$$u \in vR \;\Rightarrow\; i \notin f(v) \cap f(u). \tag{3}$$

If $i+1$ is an end-point of $f(v)$ then, since Γ does not contain any isolated vertex, so there is a vertex $u \in V$ such that $f(u) \cap f(v)$ is not empty. From (3) it follows that $i+1$ is an end-point of $f(u)$. Let $w \in vR$, then the set $f(v) \cap f(w)$ is not empty. From (3) we have $i+1 \in f(w)$ and so $f(w) \cap f(u)$ is not empty. So, $w \in uR$. Therefore, in this case $vR \subseteq \{u\} \cup uR$, that contradicts the condition (1). Thus we suppose $i+1$ is not an end-point of $f(v)$. In this case, statement (1) of Lemma 1 implies that $i+1$ is an end-point of a set in $\mathrm{Im}(f)$, say $f(u)$. From Lemma 1, we have $f(u) \nsubseteq f(v)$ and it follows that $f(v)\backslash\{i\} \subseteq f(u)$. Now from (3), we have $vR \subseteq \{u\} \cup uR$, which contradicts the condition (1). If $i-1 \in f(v)$, by the same argument we have a contradiction again. Thus, any element $i \in \mathbb{Z}_m$ is the end-point of exactly two sets in $\mathrm{Im}(f)$. This completes the proof of the theorem. □

Any arc-function $f : V \to A\mathbb{Z}_n$, satisfying conditions (i), (ii) and (iii) of Theorem 3 is called the *reduced arc-function* of the graph Γ.

Corollary 3. *Let $\Gamma = (V, R)$ be a graph which satisfies the conditions of Theorem 3. Then $n_R(v) = 2|f(v)| - 2$ for each vertex $v \in V$, where f is the reduced arc-function of Γ.*

Proof. Let $v \in V$. From statement (iii) of Theorem 3 any $i \in f(v)$ is the end-point of exactly two elements in $\mathrm{Im}(f)$. Therefore, we get

$$n_R(v) \leq 2|f(v)| - 2. \tag{4}$$

In fact, by statement (i) of Theorem 3 for each $u \in vR$, exactly one of the end-points of $f(u)$ belongs to $f(v)$. Thus we have equality in (4), and we are done. □

Proposition 1. *Let $\Gamma = (V, R)$ be a non-empty circular-arc graph without twins. Then Γ is regular if and only if for any two vertices u and v in V we have:*

$$v \in uR \quad \Rightarrow \quad uR \not\subset \{v\} \cup vR. \tag{5}$$

Proof. Suppose that Γ is regular and u and v are two adjacent vertices of the graph Γ. Then $|\{v\} \cup vR| = |\{u\} \cup uR|$. However, $\{v\} \cup vR \neq \{u\} \cup uR$ because u and v are not twins. It follows that there exists a vertex in uR, different from v, which is not in vR; and there exists a vertex in vR, different from u, which is not in uR. Therefore, the condition (5) holds.

Conversely, suppose that Γ satisfies condition (5). By the same argument as Remark 1 the graph Γ is connected. Thus, it is sufficient to show that any two adjacent vertices u and v have the same degree. On the other hand, by Theorem 3 there is a reduced arc-function $f : V \to A\mathbb{Z}_n$, where $n = |V|$. So by Corollary 3, it is sufficient to show that $|f(v)| = |f(u)|$, or equivalently

$$|f(u)\backslash f(v)| = |f(v)\backslash f(u)|. \tag{6}$$

Note that the set $f(u) \cap f(v)$ is not empty because u and v are adjacent. Moreover, by hypothesis, u and v are not twins so $f(v) \neq f(u)$. We may assume that

$$f(u) \cup f(v) \neq \mathbb{Z}_n. \tag{7}$$

Indeed, otherwise, we would have $f(u) \cup f(v) = \mathbb{Z}_n$ and then from statement (i) of Theorem 3, any set in $\mathrm{Im}(f)$ different from both $f(u)$ and $f(v)$ has one end-point in $f(v)$ and one end-point in $f(u)$. This implies that any vertex in $V \backslash \{u, v\}$ is adjacent to both u and v, which is impossible because u and v are not twins.
Let $i \in f(u) \backslash f(v)$. Then by statement ($iii$) of Theorem 3, there are exactly two vertices $u_i, v_i \in V$, for which i is the end-point of both $f(u_i)$ and $f(v_i)$, or equivalently due to (7) we have

$$\{i\} = f(u_i) \cap f(v_i).$$

Moreover, by statement (i) of Theorem 3, neither $f(u_i)$ nor $f(v_i)$ is a subset of $f(u)$. Now, from (7) it follows that the end-points of $f(u_i)$ and $f(v_i)$ different from i are not in the set $f(u)$. Thus, exactly one of $f(u_i)$ or $f(v_i)$ contains the set $f(v) \cap f(u)$. Assume that $f(u) \cap f(v) \subset f(u_i)$. Since, by statement ($i$) of Theorem 3, $f(v)$ is not a subset of $f(u_i)$, it follows that the end-point of $f(u_i)$, different from i, is in the set $f(v) \backslash f(u)$, denote this end-point by j_i, (see Figure 1). So far we could

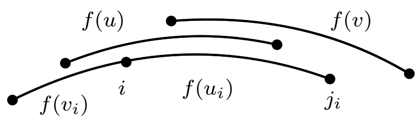

Figure 1. **Some arcs of reduced arc-function f of Γ**

define the mapping, $i \to j_i$, from $f(u) \backslash f(v)$ to $f(v) \backslash f(u)$. Now we claim that this mapping is a bijection. To do so, we first prove that it is injective. Suppose on the contrary that there are $i, i' \in f(u) \backslash f(v)$ such that $j_i = j_{i'}$. Then $f(u_i) \subset f(u_{i'})$ or $f(u_{i'}) \subset f(u_i)$. However, in both cases this contradicts statement (i) of Theorem 3. Now, let $j \in f(v) \backslash f(u)$ then in a similar way there is a corresponding element of $f(u) \backslash f(v)$, say i. By statement (i) of Theorem 3, it is clear that $j_i = j$. This shows that the mapping is surjective.
The same argument can be done for the case $f(u) \cap f(v) \subset f(v_i)$. Thus (6) follows and this proves the proposition. □

Corollary 4. *Let Γ be an m-regular circular-arc graph on n vertices. Suppose that $m \geq 1$ and the graph has no twins. Then for each reduced arc-function f of Γ and each $v \in V$, we have $|f(v)| = \frac{m+2}{2}$.*

Proof. Let $\Gamma = (V, R)$ be an m-regular circular-arc graph on n vertices. Then $m = n_R(v)$ for each vertex $v \in V$. Since $m \geq 1$, the graph Γ is non-empty and from hypothesis it has no twins, thus Γ satisfies the conditions of Theorem 3. So, it also satisfies the conditions of Corollary 3. Thus for reduced arc-function f we have, $n_R(v) = 2|f(v)| - 2$ for each vertex $v \in V$, and we are done.

□

4. Graphs and coherent configurations

In this section we prove some results on the coherent configuration of a graph. In particular we will study a relationship between the wreath product of two graphs and the wreath product of their coherent configurations.

Lemma 2. *Let $\Gamma = (V, R)$ be a graph. For each integer k, let*

$$R_k = \{(u, v) \in R : \quad |uR \cap vR| = k\}.$$

Then R_k is a union of some basic relations of $\mathcal{X}(\Gamma)$.

Proof. Let S be the set of basic relations of $\mathcal{X}(\Gamma)$. Then $R = \bigcup_{s\in S'} s$ where $S' \subset S$. It is sufficient to show that R_k contains any relation from S' whose intersection with R_k is not empty. To do this, let s be such a relation. Then there exists a pair $(u, v) \in s$ such that $|uR \cap vR| = k$. On the other hand, by the definition of intersection numbers we have $|uR \cap vR| = \sum_{r,t\in S'} c^s_{rt}$. Thus the number $|uR \cap vR|$ does not depend on the choice of $(u, v) \in s$. By definition of R_k this implies that $s \subset R_k$ as required. □

Theorem 4. *Let Γ be a graph on the vertex set V and let*

$$E = \{(u, v) \in V \times V \ : \ u \text{ and } v \text{ are twins or } u = v\}.$$

Then E is an equivalence relation of $\mathcal{X}(\Gamma)$. Moreover, if Γ is a graph such that with $\mathcal{X}(\Gamma)$ is homogeneous then, it is isomorphic to wreath product of the graph $\Gamma_{V/E}$ and a complete graph.

Proof. Let S be the set of basic relations of $\mathcal{X}(\Gamma)$ and let R be the edge set of Γ. Then there exists a subset S' of S such that

$$R = \bigcup_{s\in S'} s. \tag{8}$$

To prove the first statement, we need to check that any non-reflexive basic relation $r \in S$ such that $r \cap E \neq \emptyset$ is contained in E. To do this, let $(u, v) \in r$. We claim that $(u, v) \in E$, or equivalently u and v are twins.
First we show that

$$uR\backslash\{v\} \subseteq vR\backslash\{u\}. \tag{9}$$

If the set $uR\backslash\{v\}$ is empty, then (9) is clear. Now, we may assume that there exists an element w in V such that $w \in uR\backslash\{v\}$. It is enough to show that v is adjacent to w in Γ. By (8) there exists a basic relation $s \in S'$ so that $(u, w) \in s$. Denote by t the basic relation which contains (v, w). It is sufficient to show that $t \in S'$.

We have $w \in us \cap vt^*$, thus $|us \cap vt^*| = c^r_{st^*} \neq 0$. Since intersection number does not depend on the choice of $(u, v) \in r$, for $(u', v') \in r$ we have

$$|u's \cap v't^*| = c^r_{st^*} \neq 0. \tag{10}$$

On the other hand, by the choice of r there exists $(u', v') \in r \cap E$. So by (10), there exists a vertex $w' \in V$ such that

$$w' \in u's \cap v't^*. \tag{11}$$

Moreover, since $s \in S'$, we have $w' \in u'R\backslash\{v'\}$. On the other hand, $u'R\backslash\{v'\} = v'R\backslash\{u'\}$, since u' and v' are twins. It follows that $w' \in v'R\backslash\{u'\}$, so from (11) we conclude that w' is adjacent to v' in Γ and so from (8) we have $t \in S'$. The converse inclusion of (9) can be proved in a similar way. Thus u and v are twins and the first statement follows.

To prove the second statement, suppose that Γ is a graph such that $\mathcal{X}(\Gamma)$ is homogeneous. It is well-known fact that all classes of an equivalence relation of a homogeneous coherent configuration have the same size, say m, where m divides $n = |V|$. Moreover, for each $X \in V/E$ we have $u, v \in X$ if and only if u and v are twins. Thus for each $X \in V/E$ we have

$$\Gamma_X \simeq K_m. \tag{12}$$

Fix a class $X_0 \in V/E$. For each $X \in V/E$ choose an isomorphism $f_X \in \mathrm{Iso}(\Gamma_{X_0}, \Gamma_X)$. Then the mapping

$$f : V \to V/E \times X_0 \tag{13}$$

$$v \mapsto (X, f_X^{-1}(v)),$$

is a bijection, where X is a class of E containing v. In order to complete the proof, we show that the above bijection is a required isomorphism:

$$f \in \mathrm{Iso}(\Gamma, \Gamma_{V/E} \wr \Gamma_{X_0}). \tag{14}$$

Take two different vertices u and v in V, then $f(u) = (X, u_0)$ and $f(v) = (Y, v_0)$, where $X, Y \in V/E$, $u \in X$, $v \in Y$ and $u_0, v_0 \in X_0$. It is enough to show that u and v are adjacent in Γ if and only if $f(u)$ and $f(v)$ are adjacent in $\Gamma_{V/E} \wr \Gamma_{X_0}$.

First, we assume that u and v are not twins. Then $X \neq Y$. In this case, by definition of E, if u and v are adjacent in Γ then any vertices in X and any vertices in Y are adjacent to each other. Also, if X and Y are adjacent in $\Gamma_{V/E}$, by definition of $\Gamma_{V/E}$, there is a vertex in X and a vertex in Y which are adjacent. However, since all of the vertices in each class of V/E are twins, then u and v are adjacent in Γ. Thus u and v are adjacent in Γ if and only if X and Y are adjacent in $\Gamma_{V/E}$. Therefore, u and v are adjacent in Γ if and only if $f(u)$ and $f(v)$ are adjacent in $\Gamma_{V/E} \wr \Gamma_{X_0}$.

Now, we may assume that u and v are twins. Then $X = Y$. However, u and v are twins, so they are adjacent in Γ. Since, f_X is an isomorphism thus $f_X^{-1}(u) \neq f_X^{-1}(v)$, so $u_0 \neq v_0$. From (12), it follows that u_0 and v_0 are adjacent in Γ_X. Therefore, $f(u)$ and $f(v)$ are adjacent in $\Gamma_{V/E} \wr \Gamma_{X_0}$. Thus f is an isomorphism and (14) follows, as desired. $\square$

In the next theorem, we show that the coherent configuration of the wreath product of two graphs is smaller than the wreath product of their coherent configuration. In general, we do not have equality here. For example, the coherent configuration of the wreath product of two complete graphs is a coherent configuration of rank 2, but the wreath product of their coherent configurations has rank 3.

Theorem 5. *Let Γ_1 and Γ_2 be two graphs. Then $\mathcal{X} = \mathcal{X}(\Gamma_1 \wr \Gamma_2)$ is isomorphic to a fusion of $\mathcal{Y} = \mathcal{X}(\Gamma_1) \wr \mathcal{X}(\Gamma_2)$. Moreover, if Γ_1 is a complete graph and Γ_2 is a graph without twins such that its coherent configuration is homogeneous, then $\mathcal{X}$ and $\mathcal{Y}$ are isomorphic.*

Proof. Let $\Gamma_i = (V_i, R_i)$, $\mathcal{X}_i = \mathcal{X}(\Gamma_i)$ and S_i be the set of basic relations of $\mathcal{X}_i$ for $i = 1, 2$. Then there exists $S_i' \subset S_i$ such that

$$R_i = \bigcup_{s \in S_i'} s. \tag{15}$$

Let Γ be the wreath product of Γ_1 and Γ_2, and let R be the edge set of Γ. Then

$$\begin{aligned} R &= \{((k,i),(l,j)) \in (V_1 \times V_2)^2 : (i,j) \in R_2 \ \text{ or } \ (k,l) \in R_1 \text{ with } i = j\} \\ &= \{((k,i),(l,j)) \in (V_1 \times V_2)^2 : (i,j) \in R_2\} \cup \\ &\quad \{((k,i),(l,j)) \in (V_1 \times V_2)^2 : (k,l) \in R_1 \text{ with } i = j\}. \end{aligned}$$

So, by (15) we have

$$R = \{(V_1)^2 \otimes s : s \in S_2'\} \ \cup \ \{s \otimes 1_{V_2} : s \in S_1'\}. \tag{16}$$

Therefore, R is a union of some basic relations of $\mathcal{Y} = \mathcal{X}_1 \wr \mathcal{X}_2$. Thus we conclude that

$$\mathcal{X}(\Gamma) \leq \mathcal{Y}, \tag{17}$$

and so the first statement follows.
To prove the second statement, let Γ_1 be a complete graph on n vertices and let Γ_2 be a graph without twins such that $\mathcal{X}_2$ is homogeneous. Then, $\mathcal{X}_1 \wr \mathcal{X}_2$ is homogeneous

and from the first statement it follows that $\mathcal{X}(\Gamma)$ is homogeneous. Thus $1_{V_1} \otimes 1_{V_2}$ is a basic relation of $\mathcal{X}(\Gamma)$.

If Γ_2 is an empty graph, then it is easy to see that we have equality in (17). So we may suppose that Γ_2 is a non-empty graph. Since $\mathcal{X}(\Gamma_2)$ is homogeneous, so there exists a positive integer t such that Γ_2 is a t-regular graph. Let $t_0(i,j)$ be the number of common neighbors of two adjacent vertices i and j in Γ_2. Since Γ_2 is without twins, we have

$$t_0(i,j) < t-1. \tag{18}$$

Let $u=(k,i)$ and $v=(l,j)$ be two adjacent vertices in Γ. Since, for each $i \in V_2$ the graph $\Gamma_{V_1 \times i}$ is isomorphic to Γ_1, and for each two adjacent vertices i and j in Γ_2 the set $(V_1 \times i) \times (V_1 \times j)$ is a subset of R, thus we have

$$|uR \cap vR| = \begin{cases} (n-2)+tn, & i=j \\ 2(n-1)+t_0(i,j)n, & i \neq j. \end{cases} \tag{19}$$

Using (18), for each $i \neq j$ we have

$$2(n-1)+t_0(i,j)n < (n-2)+tn. \tag{20}$$

Define

$$E := \{(u,v) \in R : |uR \cap vR| = (n-2)+tn\}.$$

From (19) and (20) it follows that

$$E = \cup_{i \in V_2} (V_1 \times i)^2 \backslash (1_{V_1} \otimes 1_{V_2}).$$

Now, from Lemma 2, the set E is a union of some of the basic relations of $\mathcal{X}(\Gamma)$. On the other hand, $E = s \otimes 1_{V_2}$, where s is the non-reflexive basic relation of $\mathcal{X}_1$ of rank 2. However, $s \otimes 1_{V_2}$ is a basic relation of $\mathcal{X}_1 \wr \mathcal{X}_2$. Therefore, from (17) it is obvious that E is a basic relation of $\mathcal{X}(\Gamma)$. Hence,

$$F = E \cup (1_{V_1} \otimes 1_{V_2}) \tag{21}$$

is an equivalence relation of $\mathcal{X}(\Gamma)$.

The coherent configuration $\mathcal{X}_1 \wr \mathcal{X}_2$ is the minimal coherent configuration which contains an equivalence F such that for each class $X \in V/F$, $(\mathcal{X}_1 \wr \mathcal{X}_2)_X$ is isomorphic to $\mathcal{X}_1$ and $(\mathcal{X}_1 \wr \mathcal{X}_2)_{V/F}$ is isomorphic to $\mathcal{X}_2$. In order to prove equality in (17), it is sufficient to show that $\mathcal{X}(\Gamma)$ has the above property.

Let $X \in V/F$. Then by (21), $\mathcal{X}(\Gamma)_X$ is isomorphic to $\mathcal{X}_1$. Moreover, from (17) it follows that

$$\mathcal{X}(\Gamma)_{V/F} \leq \mathcal{X}_2. \tag{22}$$

However, the edge set of Γ_2 is a union of some of the basic relations of $\mathcal{X}(\Gamma)_{V/F}$. Thus we have equality in (22), and we are done. □

Remark 2. Let Γ and Γ' be two graphs with the same vertex set. Suppose that the edge set of Γ' is a union of some basic relations of $\mathcal{X}(\Gamma)$. Then $\mathcal{X}(\Gamma') \leq \mathcal{X}(\Gamma)$.

5. Elementary circular-arc graphs

Given integers n and k such that $0 \leq 2k+1 < n$, set $C_{n,k} = \mathrm{Cay}(\mathbb{Z}_n, S)$ where $S = \{\pm 1, \ldots, \pm k\}$. It immediately follows that $C_{n,k}$ is a $2k$-regular graph without twins. Note that $C_{n,0}$ is an empty graph and $C_{n,1}$ is an undirected cycle on n vertices. From definition, one can verify that $C_{n,k}$ is the graph with vertex set $V = \mathbb{Z}_n$ in which two vertices i and j are adjacent if and only if

$$\{i, \ldots, k+i\} \cap \{j, \ldots, k+j\} \neq \emptyset.$$

Suppose that $f : V \to A\mathbb{Z}_n$, such that $f(i) = \{i, \ldots, k+i\}$. Then the graph $C_{n,k}$ is the intersection graph of the family $\mathrm{Im}(f)$. Thus, by Lemma 1 we conclude that $C_{n,k}$ is a circular-arc graph, and we call it an *elementary circular-arc graph*.

Example 1. For $n = 2k+2$, two different vertices i and j are adjacent in $C_{n,k}$ if and only if $j \neq i+k+1$. Thus $C_{n,k}$ is isomorphic to a graph on n vertices which is obtained from a complete graph by removing the edges of a perfect matching. It is easy to check that the coherent configuration of this graph is isomorphic to $\mathcal{X}_1 \wr \mathcal{X}_2$, where $\mathcal{X}_1$ is a rank 2 coherent configuration on 2 points and $\mathcal{X}_2$ is a rank 2 coherent configurations on $k+1$ points.

Theorem 6. *A regular circular-arc graph without twins is elementary.*

Proof. Let Γ be a circular-arc graph with the vertex set V where $n = |V|$. Suppose that it has no twins. Then by Proposition 1 and Theorem 3, there exists a reduced arc-function f of Γ, such that for each vertex $v \in V$ we have $f(v) = \{i_v, \ldots, j_v\}$. Define a bijection from V to $\mathbb{Z}_n$, the vertex set of $C_{n,k}$, such that $v \to i_v$. From Corollary 3 we conclude that Γ is a $2k$-regular graph. Then $k = |f(v)| - 1$ for any vertex v by Corollary 4. Hence $j_v = i_v + k$. By Lemma 1, two vertices u and v in V are adjacent if and only if $f(u) \cap f(v) \neq \emptyset$. It follows that i_u and i_v are adjacent if and only if $\{i_u, \ldots, k+i_u\} \cap \{i_v, \ldots, k+i_v\} \neq \emptyset$. Therefore, the bijection defined above gives the required isomorphism. □

Theorem 7. *Let n and k be two positive integers such that $2k+2 < n$. Then the coherent configuration of the graph $C_{n,k}$ is isomorphic to a coherent configuration of a dihedral group.*

Proof. Let R be the edge set of $C_{n,k}$. For two vertices $i, j \in \mathbb{Z}_n$, we define $d(i,j)$ be the distance of i and j in the graph $C_{n,1}$. Suppose that i and j are two adjacent vertices in $C_{n,k}$. Then by definition of $C_{n,k}$, we have $d(i,j) \leq k$. Without loss of generality we may assume that the vertices $\{i+1, i+2, \ldots, j-1\}$ are between i and j. Thus they are adjacent to both of i and j in the graph $C_{n,k}$. Moreover, the vertices

$\{j-k, j-k+1\ldots, i-1\}$ and $\{j+1, j+2, \ldots, i+k\}$ are adjacent to both of i and j too. Thus the latter three sets are subsets of $iR \cap jR$, which are of size $d(i,j)-1$, $k-d(i,j)$ and $k-d(i,j)$ respectively. Moreover, since $n > 2k+2$, they are disjoint. On the other hand, $B_i := \{i-k, i-k+1, \ldots, j-k-1\}$ is the set of all other vertices which are adjacent to i, and $B_j := \{i+k+1, i+k+2, \ldots, j+k\}$ is the set of all other vertices which are adjacent to j, (see Figure 2).

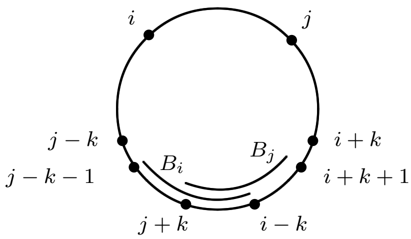

Figure 2. **Some vertices of the graph $C_{n,1}$ and the sets B_i and B_j**

It is clear that B_i and B_j are disjoint from the above three subsets. In addition, we have $|B_i| = |B_j| = d(i,j)$. Moreover, since $n > 2k+2$, the vertex $j-k-1$ is not in B_j and the vertex $i+k+1$ is not in B_i. It follows that $B_i \neq B_j$, and

$$|B_i \cap B_j| < d(i,j). \tag{23}$$

On the other hand, $B_i \cap B_j \subset iR \cap jR$ and thus

$$|iR \cap jR| = (d(i,j)-1) + 2(k-d(i,j)) + |B_i \cap B_j| = 2k - d(i,j) - 1 + |B_i \cap B_j|. \tag{24}$$

Now, set

$$R_{2k-2} := \{(i,j) \in R : \quad |iR \cap jR| = 2k-2\}.$$

Then R_{2k-2} is a symmetric relation. Moreover, from (24) we see that

$$(i,j) \in R_{2k-2} \;\Leftrightarrow\; |B_i \cap B_j| = d(i,j) - 1. \tag{25}$$

If $d(i,j) = 1$, then by (23) we have $|B_i \cap B_j| = 0$. Thus, from (25) we see that

$$d(i,j) = 1 \;\Rightarrow\; (i,j) \in R_{2k-2}. \tag{26}$$

If $1 < d(i,j) \leq k$, then $j-k-2 \in B_i \backslash B_j$ and $i+k+2 \in B_j \backslash B_i$. It follows that $|B_i \cap B_j| < d(i,j) - 1$. Thus, from (25) we have $(i,j) \notin R_{2k-2}$. Then using (26) we have $(i,j) \in R_{2k-2}$ if and only if $d(i,j) = 1$.

It follows that the graph $(\mathbb{Z}_n, R_{2k-2})$ is isomorphic to an undirected cycle on n points, say C_n. By Remark 2, we conclude that R_{2k-2} is union of some basic relations of $\mathcal{X}(C_{n,k})$. Thus by Lemma 2, we have $\mathcal{X}(C_n) \leq \mathcal{X}(C_{n,k})$.

It is well-known that $\mathcal{X}(C_n) = \mathrm{Inv}(D_{2n})$, where D_{2n} is a dihedral group on n elements. So, in order to complete the proof of the theorem it is enough to show that $\mathcal{X}(C_{n,k}) \leq \mathcal{X}(C_n)$. Equivalently, it is suffices to verify that $\mathrm{Aut}(\mathcal{X}(C_n)) \leq \mathrm{Aut}(\mathcal{X}(C_{n,k}))$. Since the automorphism group of a graph is equal to the automorphism group of its coherent configuration, it is sufficient to show that $\mathrm{Aut}(C_n) \leq \mathrm{Aut}(C_{n,k})$.
Since $\mathrm{Aut}(C_n)$ is the dihedral group D_{2n}, and D_{2n} is generated by automorphisms σ and δ where $i^\sigma = i+1$ and $i^\delta = n-i$ for each $i \in \mathbb{Z}_n$. It is enough to show that $\sigma, \delta \in \mathrm{Aut}(C_{n,k})$. Let $i, j \in \mathbb{Z}_n$. Two vertices i and j are adjacent in $C_{n,k}$ if and only if

$$\{i, \ldots, k+i\} \cap \{j, \ldots, k+j\} \neq \emptyset \Leftrightarrow$$

$$\{i+1, \ldots, k+i+1\} \cap \{j+1, \ldots, k+j+1\} \neq \emptyset \Leftrightarrow$$

$$\{i^\sigma, \ldots, k+i^\sigma\} \cap \{j^\sigma, \ldots, k+j^\sigma\} \neq \emptyset. \tag{27}$$

Moreover, we have (27) if and only if i^σ and j^σ are adjacent in $C_{n,k}$. Thus $\sigma \in \mathrm{Aut}(C_{n,k})$. In a similar way we can show that $\delta \in \mathrm{Aut}(C_{n,k})$, this completes the proof. □

6. Proof of the main theorems

Proof of Theorem 1.

We first prove the necessity condition of the theorem. Let $\Gamma = (V, R)$ be a circular-arc graph such that $\mathcal{X}(\Gamma)$ is homogeneous and $|V| = n$. Denote by E the equivalence relation on V defined in Theorem 4. Then by this theorem Γ is isomorphic to wreath product of a complete graph and the graph $\Gamma_{V/E}$. In particular, it is easy to see that $\Gamma_{V/E}$ is also a circular-arc graph and has no twins. So, to complete the proof it is enough to show that $\Gamma_{V/E}$ is an elementary circular-arc graph.
If $\Gamma_{V/E}$ is empty, then it is isomorphic to $C_{m,0}$ with $m = |V/E|$, and we are done. We suppose that $\Gamma_{V/E}$ is non-empty. The graph Γ is regular, because $\mathcal{X}(\Gamma)$ is homogeneous. By the definition of E this implies that $\Gamma_{V/E}$ is regular too. From Corollary 4, the graph $\Gamma_{V/E}$ is $2k$-regular for some integer $k > 0$. Therefore, from Theorem 6, the latter graph is isomorphic to the elementary circular-arc graph $C_{m,k}$. Thus Γ is isomorphic to the wreath product of a complete graph and an elementary circular-arc graph.
Conversely, let Γ_1 be a complete graph and let Γ_2 be an elementary circular-arc graph. If Γ_2 is an empty graph then $\mathcal{X}(\Gamma_2)$ is homogeneous. If it is a non-empty elementary circular-arc graph then from Example 1 and Theorem 7 we conclude that $\mathcal{X}(\Gamma_2)$ is homogeneous. On the other hand, the wreath product of two homogeneous coherent configuration is homogeneous. Thus, from Theorem 5, $\mathcal{X}(\Gamma_1 \wr \Gamma_2)$ is homogeneous. This completes the proof of the theorem. □

Proof of Theorem 2.

We first assume that $\mathcal{X}$ is homogeneous and there is a circular-arc graph Γ such that $\mathcal{X}(\Gamma) = \mathcal{X}$. From Theorem 1, Γ is isomorphic to the wreath product of a complete graph and an elementary circular-arc graph. Let the elementary circular-arc graph be an empty graph, then its coherent configuration is of rank 2. Otherwise, from Example 1 and Theorem 7 the coherent configuration of an elementary circular-arc graph is isomorphic to the wreath product of a rank 2 coherent configuration on 2 points and a rank 2 coherent configuration, or it is isomorphic to a coherent configuration of a dihedral group. Therefore, in any case the coherent configuration of an elementary circular-arc graph is homogeneous. Moreover, any elementary circular-arc graph is without twins. Hence, from Theorem 5, it follows that $\mathcal{X}(\Gamma)$ is isomorphic to the wreath product of a rank 2 coherent configuration and the coherent configuration of an elementary circular-arc graph which is of rank 2 or the wreath product of two rank 2 or the coherent configuration of a dihedral group.
Conversely, assume that $\mathcal{X} = \mathcal{X}(\Gamma)$ such that Γ is a circular-arc graph. Also, assume that $\mathcal{X}$ is isomorphic to the wreath product of a rank 2 coherent configuration and a coherent configuration which is of rank 2 or the wreath product of two rank 2 or the coherent configuration of a dihedral group. Since any rank 2 coherent configuration and any coherent configuration of a dihedral group are homogeneous, it is enough to note that the wreath product of two homogeneous coherent configuration is homogeneous. Thus $\mathcal{X}$ is homogeneous and the proof is complete. □

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