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Research Article

Antipodal number of Cartesian products of complete graphs with cycles

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Abstract: Let G be a simple connected graph with diameter d, and $k \in [1, d]$ be an integer. A radio k-coloring of graph G is a mapping $g : V(G) \to \{0\} \cup \mathbb{N}$ satisfying $|g(u) - g(v)| \geq 1 + k - d(u, v)$ for any pair of distinct vertices u and v of the graph G, where $d(u, v)$ denotes distance between vertices u and v in G. The number $\max\{g(u) : u \in V(G)\}$ is known as the span of g and is denoted by $rc_k(g)$. The radio k-chromatic number of graph G, denoted by $rc_k(G)$, is defined as $\min\{rc_k(g) : g \text{ is a radio } k\text{-coloring of } G\}$. For $k = d-1$, the radio k-coloring of graph G is called an antipodal coloring. So $rc_{d-1}(G)$ is called the antipodal number of G and is denoted by $ac(G)$. Here, we study antipodal coloring of the Cartesian product of the complete graph K_r and cycle C_s, $K_r \square C_s$, for $r \geq 4$ and $s \geq 3$. We determine the antipodal number of $K_r \square C_s$, for even $r \geq 4$ with $s \equiv 1 \pmod 4$; and for any $r \geq 4$ with $s = 4t + 2$, t odd. Also, for the remaining values of r and s, we give lower and upper bounds for $ac(K_r \square C_s)$.

Keywords: Radio k-coloring, Antipodal coloring, Antipodal number, Cartesian Product, Complete graph, Cycle.

AMS Subject classification: 05C15, 05C78, 05C90

1. Introduction

An important motivation for coloring of graphs is the frequency assignment problem. In this problem, radio transmitters are assigned frequencies in a manner that prevents interference. One of the natural reasons for the interference is that when nearer transmitters receive closer frequencies. Frequency bands are defined simultaneously for all wireless networks based on usage type and relevance. The primary goal is,

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therefore, to allocate radio frequencies for transmitters at various places with the least possible spread and without causing interference, as demonstrated in [3]. So the frequency assignment problem can be considered as an allocation of numbers (color numbers) to vertices of G so that color numbers of nearer vertices have desired separation, where vertices of G are the transmitters and two of them are adjacent if they have close proximity. So for a simple and connected graph G having diameter d and an integer $k \in [1, d]$, a *radio k-coloring* of G is a mapping $g : V(G) \to \{0\} \cup \mathbb{N}$ satisfying $|g(u) - g(v)| \geq 1 + k - d(u, v)$, for any pair of distinct vertices u and v of the graph G, where $d(u, v)$ denotes distance between vertices u and v in G. The largest positive integer (or color number) allocated by g, termed as the *span of* g, and is denoted by $span(g)$ or $rc_k(g)$. The notation $rc_k(G)$ stands for radio k-chromatic number of the graph G, and is defined by

$$rc_k(G) = min\{rc_k(g) : g \text{ is a radio } k\text{-coloring of G}\}$$

Throughout the paper, we consider simple and connected graphs only. Chartrand et al. [1] proposed the idea of radio k-coloring of graphs. It may be noted that for $k = 1$, the radio k-coloring problem coincides with the usual proper coloring of graphs. For particular values of k, there are specific names of a radio k-coloring, such as, if $k = d$, then this coloring is simply known as radio coloring; and if $k = d - 1$ then it is known as an *antipodal coloring*. So $rc_d(G)$ and $rc_{d-1}(G)$ are respectively called the *radio number* (also represented by $rn(G)$) and *antipodal number* (also represented by $ac(G)$) of graph G. A radio k-coloring g of a graph G is called a *minimal radio k-coloring* if $rc_k(g) = rc_k(G)$.

The definition below is useful in the paper.

Definition 1. For a graph G of order r, and a radio k-coloring g of G, let $v_1, v_2, \ldots, v_r$ be an arrangement of vertices in such a way that $g(v_j) \leq g(v_{j+1})$, for all $j = 1, 2, \ldots, r - 1$. Then ϵ_j, $j = 2, 3, \ldots, r$, is defined as $\epsilon_j = g(v_j) - g(v_{j-1}) - (1 + k - d(v_j, v_{j-1}))$. We note that ϵ_j are non-negative integers.

The lemma presented below gives information about the radio k-chromatic number of any arbitrary graph.

Lemma 1. *[9] Consider a graph G with order r. Then for any radio k-coloring g of G, we have*

$$rc_k(g) = (r - 1)(k + 1) - \sum_{j=2}^{r} d(v_j, v_{j-1}) + \sum_{j=2}^{r} \epsilon_j \tag{1.1}$$

where the vertices v_j, $j = 1, 2, \ldots, r$, are arranged in the order given in Definition 1.

Remark 1. In equation (1.1), for a particular k and a fixed graph G, the term $(k + 1)(r - 1)$ is a constant. So a radio k-coloring g of graph G becomes a minimal coloring if $\sum_{j=2}^{r} d(v_{j-1}, v_j)$ and $\sum_{j=2}^{r} \epsilon_j$ attain respectively the maximum and minimum value simultaneously among all possible radio k-colorings of G.

The notation $G_1 \square G_2$ stands for Cartesian product of any two graphs G_1 and G_2. The vertex set of $G_1 \square G_2$ is $V(G_1) \times V(G_2)$, and any two arbitrary vertices (u_1, v_1) and (u_2, v_2) are adjacent in $G_1 \square G_2$ if and only if $u_1 = u_2$ and $\{v_1, v_2\}$ is an edge in G_2, or $v_1 = v_2$ and $\{u_1, u_2\}$ is an edge in G_1. A complete graph K_r is a simple and connected graph with r vertices so that there is an edge between every two distinct vertices in it. Kchikech et al. [5] obtained some upper bounds for $rc_k(G)$, whenever G is the Cartesian product of two graphs. The same authors found lower and upper bounds for $rc_k(P_n \square P_n)$ when $k \geq 2n-3$. Kim et al. [8] determined the radio number for Cartesian product of a complete graph with a path. The radio number of $GP(n,1)$ is obtained by Kola and Panigrahi [10]. The radio number of toroidal grid $C_s \square C_s$ is determined by Morris-Rivera et al. [13]. Saha and Panigrahi [15] determined $rn(C_s \square C_r)$ whenever sr is even. For an arbitrary graph G, Kola and Panigrahi [11] have given a lower bound of $rc_k(G)$. Additionally, they have proved that for some values of r, the above lower bound agrees with $rn(C_r \square P_2)$. Chartrand et al. [2] have obtained the antipodal number of some classes of paths, and for an arbitrary graph G, they have also found bounds for $ac(G)$ in terms of the antipodal number of paths. The same authors had conjectured about the antipodal number of paths and this has been proved by Khennoufa and Togni [6]. Khennoufa and Togni [7] have determined $ac(Q_n)$, where Q_n is the n-dimensional hypercube. Juan and Liu [4] obtained the antipodal number of a cycle. Saha and Panigrahi [14] obtained the antipodal number of powers of cycles.
In this research article, we compute the antipodal number of $K_r \square C_s$, for an even integer $r \geq 4$ with $s \equiv 1 \pmod 4$; and for any $r \geq 4$ with $s = 4t+2$, t odd. Then, for the rest of the values of r and s, we give lower and upper bounds for $ac(K_r \square C_s)$.

2. Antipodal number of $K_r \square C_s$

To determine a lower bound for $ac(K_r \square C_s)$, we need the theorem below, given by Kola and Panigrahi [11].

Theorem 1. *[11] Let G be a simple and connected graph having order r with diameter d. If $d(x,y) + d(x,z) + d(y,z) \leq M$, for every triplet of vertices x, y and z of G, then*

$$rc_k(G) \geq \begin{cases} \frac{(r-1)(3(k+1)-M)}{4} & \text{if } r \text{ is odd and } M \not\equiv k \pmod 2, \\ \frac{(r-1)(3(k+1)-M+1)}{4} & \text{if } r \text{ is odd and } M \equiv k \pmod 2, \\ \frac{(r-2)(3(k+1)-M)}{4} + k - d + 1 & \text{if } r \text{ is even and } M \not\equiv k \pmod 2, \\ \frac{(r-2)(3(k+1)-M+1)}{4} + k - d + 1 & \text{if } r \text{ is even and } M \equiv k \pmod 2. \end{cases}$$

Definition 2. The smallest integer M for which $d(x,y) + d(x,z) + d(y,z) \leq M$ for every triplet of vertices x, y and z of a graph G, is known as the *triameter* of the graph G, and denoted by $tr(G)$.

Lemma 2. *[11] For connected graphs G_1 and G_2, we have $tr(G_1 \square G_2) = tr(G_1) + tr(G_2)$.*

Since $tr(K_r) = 3$ and $tr(C_s) = s$ (Saha and Panigrahi [14]), the following lemma is immediate from Lemma 2.

Lemma 3. *For $r \geq 3$ and $s \geq 3$, the triameter of $K_r \square C_s$ is $s+3$.*

The lemma below gives the diameter of the Cartesian product between two graphs in terms of the diameter of the individual graphs.

Lemma 4. *[15] If G_1 and G_2 are simple and connected graphs then $diam(G_1 \square G_2) = diam(G_1) + diam(G_2)$.*

Since the diameter of cycle C_s is $\lfloor \frac{s}{2} \rfloor$ and the diameter of complete graph K_r is 1, the following lemma is immediate from Lemma 4.

Lemma 5. *The diameter of $K_r \square C_s$ is given by*

$$diam(K_r \square C_s) = \begin{cases} \frac{s+1}{2} & \text{if } s \text{ odd,} \\ \frac{s}{2}+1 & \text{if } s \text{ even.} \end{cases}$$

The theorem below gives a lower bound for $ac(K_r \square C_s)$.

Theorem 2. *For the even positive integer r,*

$$ac(K_r \square C_s) \geq \begin{cases} \frac{1}{8}(s^2r - 2s) & \text{if } s \equiv 0 \pmod 4, \\ \frac{1}{8}(s^2r - sr - 2s + 2) & \text{if } s \equiv 1 \pmod 4, \\ \frac{1}{8}(s^2r + 2sr - 2s - 4) & \text{if } s \equiv 2 \pmod 4, \\ \frac{1}{8}(s^2r - 3sr - 2s + 6) & \text{if } s \equiv 3 \pmod 4. \end{cases}$$

Proof. By Lemma 3, the triameter of $K_r \square C_s$ is $M = s+3$. Depending on values of s, we consider four cases.

Case I. $s \equiv 0 \pmod 4$.
Here diam$(K_r \square C_s)$= $\frac{s}{2}+1$, by Lemma 5. In this case $M = s+3$ is odd integer and $\frac{s}{2}$ is even integer. So $M \not\equiv \frac{s}{2} \pmod 2$. Then by Theorem 1, we get

$$ac(K_r \square C_s) \geq \frac{(sr-2)\left[3\left(\frac{s}{2}+1\right) - s - 3\right]}{4} + d - 1 - d + 1 = \frac{1}{8}(s^2r - 2s).$$

Case II. $s \equiv 1 \pmod 4$.
In this case diam$(K_r \square C_s)$= $\frac{s+1}{2}$, by Lemma 5. Also both $\frac{s-1}{2}$ and $M = s+3$ are even integers, so we get $M \equiv \frac{s-1}{2} \pmod 2$. Then by Theorem 1, we have

$$ac(K_r \square C_s) \geq \frac{(sr-2)\left[3\left(\frac{s+1}{2}\right) - s - 3 + 1\right]}{4} + d - 1 - d + 1 = \frac{1}{8}(s^2r - sr - 2s + 2).$$

Case III. $s \equiv 2 \pmod 4$.
In this case diam($K_r \square C_s$)$= \frac{s}{2} + 1$, by Lemma 5. Also both $\frac{s}{2}$ and $M = s + 3$ are odd integers, so we get $M \equiv \frac{s}{2} \pmod 2$. Then by Theorem 1, we have

$$ac(K_r \square C_s) \geq \frac{(sr-2)\left[3\left(\frac{s}{2}+1\right) - s - 3 + 1\right]}{4} + d - 1 - d + 1 = \frac{1}{8}(s^2r + 2sr - 2s - 4).$$

Case IV. $s \equiv 3 \pmod 4$.
In this case diam($K_r \square C_s$)$= \frac{s+1}{2}$, by Lemma 5. Here $\frac{s-1}{2}$ is an odd and $M = s + 3$ is an even integer. So $M \not\equiv \frac{s}{2} \pmod 2$. Then by Theorem 1, we get

$$ac(K_r \square C_s) \geq \frac{(sr-2)\left[3\left(\frac{s+1}{2}\right) - s - 3\right]}{4} + d - 1 - d + 1 = \frac{1}{8}(s^2r - 3sr - 2s + 6).$$

$\square$

Theorem 3. *For odd positive integer r,*

$$ac(K_r \square C_s) \geq \begin{cases} \frac{1}{8}(s^2r - 2s) & \text{if } s \equiv 0 \pmod 4, \\ \frac{1}{8}(s^2r - sr - s + 1) & \text{if } s \equiv 1 \pmod 4, \\ \frac{1}{8}(s^2r + 2sr - 2s - 4) & \text{if } s \equiv 2 \pmod 4, \\ \frac{1}{8}(s^2r - 3sr - s + 3) & \text{if } s \equiv 3 \pmod 4. \end{cases}$$

Proof. By Lemma 3, the triameter of $K_r \square C_s$ is $M = s + 3$. Depending on values of s, we consider four cases.

Case I. $s \equiv 0 \pmod 4$.
Here diam($K_r \square C_s$)$= \frac{s}{2} + 1$, by Lemma 5. In this case $M = s + 3$ is odd integer and $\frac{s}{2}$ is even integer. So $M \not\equiv \frac{s}{2} \pmod 2$. Then by Theorem 1, we get

$$ac(K_r \square C_s) \geq \frac{(sr-2)\left[3\left(\frac{s}{2}+1\right) - s - 3\right]}{4} + d - 1 - d + 1 = \frac{1}{8}(s^2r - 2s).$$

Case II. $s \equiv 1 \pmod 4$.
In this case diam($K_r \square C_s$)$= \frac{s+1}{2}$, by Lemma 5. Also both $\frac{s-1}{2}$ and $M = s + 3$ are even integers, so we get $M \equiv \frac{s-1}{2} \pmod 2$. Then by Theorem 1, we have

$$\begin{aligned} ac(K_r \square C_s) &\geq \frac{(sr-1)\left[3\left(\frac{s+1}{2}\right) - s - 3 + 1\right]}{4} \\ &= \frac{1}{8}(s^2r - sr - s + 1). \end{aligned}$$

Case III. $s \equiv 2 \pmod 4$.
In this case diam$(K_r \square C_s)$= $\frac{s}{2}+1$, by Lemma 5. Also both $\frac{s}{2}$ and $M = s+3$ are odd integers, so we get $M \equiv \frac{s}{2} \pmod 2$. Then by Theorem 1, we have

$$ac(K_r \square C_s) \geq \frac{(sr-2)\left[3\left(\frac{s}{2}+1\right)-s-3+1\right]}{4}+d-1-d+1=\frac{1}{8}(s^2r+2sr-2s-4).$$

Case IV. $s \equiv 3 \pmod 4$.
In this case diam$(K_r \square C_s)$= $\frac{s+1}{2}$, by Lemma 5. Here $\frac{s-1}{2}$ is an odd and $M = s+3$ is an even integer. So $M \not\equiv \frac{s}{2} \pmod 2$. Then by Theorem 1, we get

$$ac(K_r \square C_s) \geq \frac{(sr-1)\left[3\left(\frac{s+1}{2}\right)-s-3\right]}{4}=\frac{1}{8}(s^2r-3sr-s+3).$$

□

The following lemma gives an ordering of vertices of $K_r \square C_s$, for r even and $s \equiv 1 \pmod 4$, with some distance separation among them. This ordering of vertices is useful to give antipodal coloring.

Lemma 6. *For the even integer r, $r \geq 4$, and $s \equiv 1 \pmod 4$, there is an ordering $v_1, v_2, \ldots, v_{sr}$ of vertices of $K_r \square C_s$ such that $d(v_{2j}, v_{2j-1}) = \frac{s+1}{2}$ for $j = 1, 2, \ldots, \frac{sr}{2}$, and $d(v_{2j+1}, v_{2j}) = \frac{s+3}{4}$ for $j = 1, 2, \ldots, \frac{sr}{2}-1$. Moreover, $d(v_j, v_{j-2}) = \frac{s+3}{4}$ for $j = 3, 4, \ldots, sr$.*

Proof. In $K_r \square C_s$, there are r copies of cycle C_s, say, $C_s^{(0)}, C_s^{(1)}, \ldots, C_s^{(r-1)}$. For $i = 0, 1, \ldots, r-1$, let $x_0^{(i)}, x_2^{(i)}, \ldots, x_{s-1}^{(i)}$ be the vertices of $C_s^{(i)}$. Now we define an ordering of the vertices of $K_r \square C_s$ as

$$v_{2j+1} = x^{((2j)(\bmod r))}_{\left(j\left(\frac{s-1}{4}\right)\right)(\bmod s)}$$

and

$$v_{2j+2} = x^{((2j+1)(\bmod r))}_{\left(j\left(\frac{s-1}{4}\right)+\frac{s-1}{2}\right)(\bmod s)}$$

for all $j \in \left\{0, 1, \ldots, \frac{sr}{2}-1\right\}$. From the above ordering of the vertices of $K_r \square C_s$, v_j and v_{j-2} lie on different cycles with $d\left(v_j, v_{j-2}\right) = \frac{s-1}{4}+1$ for all $j = 3, 4, \ldots, sr$. Also, $d(v_j, v_{j+1}) = \frac{s-1}{2}+1$ for all $j = 1, 3, \ldots, sr-1$, and $d(v_{j+1}, v_{j+2}) = \frac{s-1}{4}+1$ for all $j = 1, 3, \ldots, sr-3$. Since s and $\frac{s-1}{4}$ are co-prime. So all the vertices of $C_s^{(0)}, C_s^{(2)}, \ldots, C_s^{(r-2)}$ are covered by the ordering of the vertices $v_1, v_3, \ldots, v_{sr-1}$, and the vertices of $C_s^{(1)}, C_s^{(3)}, \ldots, C_s^{(r-1)}$ are covered by the ordering of the vertices $v_2, v_4, \ldots, v_{sr}$. Therefore, we obtain an ordering of the vertices of $K_r \square C_s$ as $v_1, v_2, \ldots, v_{sr}$ so that the sequence $\left\{d\left(v_j, v_{j+1}\right)\right\}_{j=1}^{sr-1}$ is an alternating sequence of $\frac{s+1}{2}$ and $\frac{s+3}{4}$ with $d\left(v_j, v_{j-2}\right) = \frac{s+3}{4}$ for all $j = 3, 4, \ldots, sr$. □

Theorem 4. *For even integer $r \geq 4$, and $s \equiv 1 \pmod 4$, the antipodal number of $K_r \Box C_s$ is given by*

$$ac(K_r \Box C_s) = \frac{1}{8}(s^2 r - sr - 2s + 2).$$

Proof. We consider the ordering $v_1, v_2, \ldots, v_{sr}$ of vertices of $K_r \Box C_s$ as given in Lemma 6. Now, we define a mapping $g : V(K_r \Box C_s) \to \mathbb{N} \cup \{0\}$ such that $g(v_1) = 0$ and $g(v_j) = g(v_{j-1}) + \left(\frac{s+1}{2}\right) - d(v_j, v_{j-1})$, $2 \leq j \leq sr$. We show that g is an antipodal coloring of $K_r \Box C_s$. Clearly, for $4 \leq l \leq sr$, $l+1 \leq j \leq sr$, the pair of vertices v_j and v_{j-l} satisfy the antipodal coloring condition. So we need to show that g satisfies antipodal coloring condition for the pair of vertices v_j and v_{j-2}, $3 \leq j \leq sr$; and v_j and v_{j-3}, $4 \leq j \leq sr$. When j is odd, $d(v_j, v_{j-1}) = \frac{s+3}{4}$, $d(v_{j-1}, v_{j-2}) = \frac{s+1}{2}$ and $d(v_j, v_{j-2}) = \frac{s+3}{4}$. Therefore

$$\begin{aligned}
g(v_j) - g(v_{j-2}) &= g(v_j) - g(v_{j-1}) + g(v_{j-1}) - g(v_{j-2}) \\
&= \left(\frac{s+1}{2}\right) - d(v_j, v_{j-1}) + \left(\frac{s+1}{2}\right) - d(v_{j-1}, v_{j-2}) \\
&= \left(\frac{s+1}{2}\right) - \left(\frac{s+3}{4}\right) \\
&= 1 + \frac{s-1}{2} - d(v_j, v_{j-2}).
\end{aligned}$$

$$\begin{aligned}
g(v_j) - g(v_{j-3}) &= g(v_j) - g(v_{j-2}) + g(v_{j-2}) - g(v_{j-3}) \\
&= \frac{s-1}{4} + \left(\frac{s+1}{2}\right) - d(v_{j-2}, v_{j-3}) \\
&= \left(1 + \frac{s-1}{2}\right) - 1 \\
&\geq 1 + \frac{s-1}{2} - d(v_j, v_{j-3}),
\end{aligned}$$

as $d(v_j, v_{j-3}) \geq 1$.

When j is even, $d(v_j, v_{j-1}) = \frac{s+1}{2}$, $d(v_{j-1}, v_{j-2}) = \frac{s+3}{4}$ and $d(v_j, v_{j-2}) = \frac{s+3}{4}$. Therefore

$$\begin{aligned}
g(v_j) - g(v_{j-2}) &= g(v_j) - g(v_{j-1}) + g(v_{j-1}) - g(v_{j-2}) \\
&= \left(\frac{s+1}{2}\right) - d(v_j, v_{j-1}) + \left(\frac{s+1}{2}\right) - d(v_{j-1}, v_{j-2}) \\
&= \left(\frac{s+1}{2}\right) - \left(\frac{s+3}{4}\right) \\
&= 1 + \frac{s-1}{2} - d(v_j, v_{j-2})
\end{aligned}$$

$$\begin{aligned}
g(v_j) - g(v_{j-3}) &= g(v_j) - g(v_{j-2}) + g(v_{j-2}) - g(v_{j-3}) \\
&= \frac{s-1}{4} + \left(\frac{s+1}{2}\right) - d(v_{j-2}, v_{j-3}) \\
&= \left(1 + \frac{s-1}{2}\right) - \left(\frac{s+3}{4}\right) \\
&\geq 1 + \frac{s-1}{2} - d(v_j, v_{j-3}).
\end{aligned}$$

For $j = 4, 5, \ldots, sr$, the vertices v_j, v_{j-2} and v_{j-3} all lie on different cycles, we get $d(v_j, v_{j-3}) \geq d(v_{j-2}, v_{j-3}) - d(v_j, v_{j-2}) \geq \frac{s+1}{2} - \frac{s+3}{4} + 1 = \frac{s+3}{4}$. Hence g is an antipodal coloring. From the definition of the mapping g and ordering of the vertices v_j, we get

$$\sum_{j=2}^{sr} d(v_j, v_{j-1}) = \frac{sr}{2}\left(\frac{s+1}{2}\right) + \left(\frac{sr}{2} - 1\right)\left(\frac{s+3}{4}\right) \text{ and } \sum_{j=2}^{sr} \epsilon_j = 0.$$

From Lemma 1 we get,

$$\begin{aligned} span(g) &= g(v_{sr}) \\ &= (sr-1)\left(\frac{s+1}{2}\right) - \frac{sr}{2}\left(\frac{s+1}{2}\right) - \left(\frac{sr}{2} - 1\right)\left(\frac{s+3}{4}\right) \\ &= \frac{1}{8}(s^2r - sr - 2s + 2). \end{aligned}$$

So $ac(K_r \square C_s) \leq span(g) = \frac{1}{8}(s^2r - sr - 2s + 2)$. This upper bound matches the lower bound found in Case II of Theorem 2, and so the result follows. $\square$

The example below illustrates Lemma 6 and Theorem 4.

Example 1. Here we consider the graph $K_6 \square C_5$. So $r = 6$ and $s = 5$. Figure 1 represents ordering of vertices given in Lemma 6, and Figure 2 gives the antipodal coloring obtained in Theorem 4 with span 14.

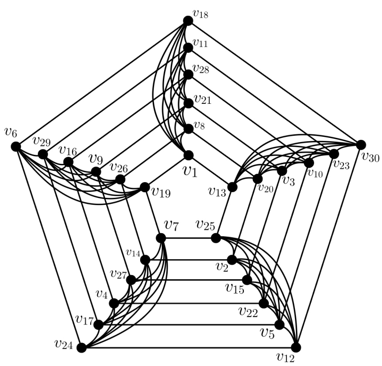

Figure 1. **Ordering of vertices $K_6 \square C_5$ as described in Lemma 6**

In the lemma below, we give an ordering of the vertices of $K_r \square C_s$, for any positive integer $r \geq 4$ and $s = 4t + 2$ with t odd. This ordering of vertices will be useful to determine a minimal antipodal coloring.

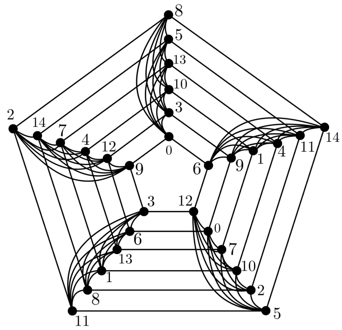

Figure 2. **The antipodal coloring of $K_6\Box C_5$ as described in Theorem 4 with span 14.**

Lemma 7. *For any positive integer $r \geq 4$ and $s = 4t+2$ with t odd, there is an arrangement $v_1, v_2, \ldots, v_{sr}$ of vertices of $K_r\Box C_s$ such that $d(v_{2j}, v_{2j-1}) = \frac{s}{2}+1$ for $j = 1, 2, \ldots, \frac{sr}{2}$, and $d(v_{2j+1}, v_{2j}) = \frac{s+2}{4}$ for $j = 1, 2, \ldots, \frac{sr}{2}-1$. Moreover, $d(v_j, v_{j-2}) = \frac{s+2}{4}+1$ for $j = 3, 4, \ldots, sr$.*

Proof. In $K_r\Box C_s$, there are s copies of cycle K_r, say, $K_r^{(0)}, K_r^{(1)}, \ldots, K_r^{(s-1)}$. For $i = 0, 1, \ldots, s-1$, let $x_0^{(i)}, x_2^{(i)}, \ldots, x_{r-1}^{(i)}$ be the vertices of $K_r^{(i)}$. Now we define an ordering of the vertices of $K_r\Box C_s$ as

$$v_{2j+1} = x_{j(\bmod r)}^{\left(j\left(\frac{s+2}{4}\right)\right)(\bmod s)}$$

and

$$v_{2j+2} = x_{(j+3)(\bmod r)}^{\left(j\left(\frac{s+2}{4}\right)+\frac{s}{2}\right)(\bmod s)}$$

for all $j \in \left\{0, 1, \ldots, \frac{sr}{2}-1\right\}$. From the above ordering of the vertices of $K_r\Box C_s$, v_j and v_{j-2} lie on different cycles with $d\left(v_j, v_{j-2}\right) = \frac{s+2}{4}+1$ for all $j = 3, 4, \ldots, sr$. Also, $d(v_j, v_{j+1}) = \frac{s}{2}+1$ for all $j = 1, 3, \ldots, sr-1$, and $d(v_{j+1}, v_{j+2}) = \frac{s+2}{4}$ for all $j = 1, 3, \ldots, sr-3$. Since s and $\frac{s+2}{4}$ are co-prime, all the vertices of $K_r^{(0)}, K_r^{(2)}, \ldots, K_r^{(s-2)}$ are covered by the ordering of the vertices $v_1, v_3, \ldots, v_{sr-1}$, and the vertices of $K_r^{(1)}, K_r^{(3)}, \ldots, K_r^{(s-1)}$ are covered by the ordering of the vertices $v_2, v_4, \ldots, v_{sr}$. Thus we obtain an ordering of the vertices of $K_r\Box C_s$ as $v_1, v_2, \ldots, v_{sr}$ so that the sequence $\{d\left(v_j, v_{j+1}\right)\}_{j=1}^{sr-1}$ is an alternating sequence of $\frac{s}{2}+1$ and $\frac{s+2}{4}$ with $d\left(v_j, v_{j-2}\right) = \frac{s+2}{4}+1$ for all $j = 3, 4, \ldots, sr$. $\square$

Theorem 5. *For any positive integer $r \geq 4$ and $s = 4t+2$ with t odd, the antipodal number of the graph $K_r\Box C_s$ is given by*

$$ac(K_r\Box C_s) = \frac{1}{8}(s^2r + 2sr - 2s - 4).$$

Proof. We consider the ordering $v_1, v_2, \ldots, v_{sr}$ of vertices of $K_r \square C_s$ as given in Lemma 7. Now, we define a mapping $g : V(K_r \square C_s) \to \mathbb{N} \cup \{0\}$ such that $g(v_1) = 0$ and $g(v_j) = g(v_{j-1}) + \left(\frac{s}{2} + 1\right) - d(v_j, v_{j-1})$, $2 \leq j \leq sr$. We can show that g is an antipodal coloring of $K_r \square C_s$. By Lemma 1,

$$\begin{aligned} span(g) &= g(v_{sr}) \\ &= (sr-1)\left(\frac{s}{2}+1\right) - \frac{sr}{2}\left(\frac{s}{2}+1\right) - \left(\frac{sr}{2}-1\right)\left(\frac{s+2}{4}\right) \\ &= \frac{1}{8}(s^2r + 2sr - 2s - 4). \end{aligned}$$

So $ac(K_r \square C_s) \leq \frac{1}{8}(s^2r + 2sr - 2s - 4)$. This upper bound coincides with the lower bound obtained in Case III of Theorems 2 and 3, and hence the result follows. $\square$

Example 2. Here we consider the graph $K_5 \square C_6$. So $r = 5$ and $s = 6$. Figure 3 represents ordering of vertices given in Lemma 7, and Figure 4 gives the antipodal coloring obtained in Theorem 5 with span 28.

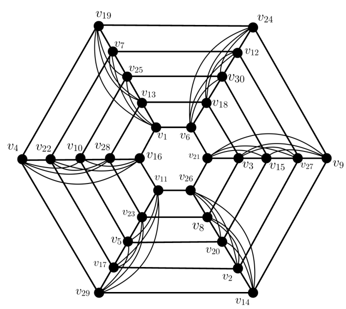

Figure 3. **Ordering of vertices $K_5 \square C_6$ as described in Lemma 7**

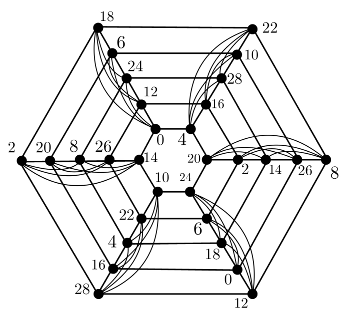

Figure 4. **The antipodal coloring of $K_5 \square C_6$ as described in Theorem 5 with span 28.**

We give an upper bound of $ac(K_r \square C_s)$, for the remaining values of r and s, applying the following result by Kchikech et al. [5], which holds true for radio and antipodal numbers only.

Theorem 6. *[5] For two graphs G_1 and G_2 of order $r \geq 2$ and s respectively, and for any integer $k \geq diam(G_1 \square G_2) - 1$,*

$$rc_k(G_1 \square G_2) \leq s\,(rc_k(G_1)) + (s-1)k - \sum(G_2),$$

where $\sum(G_2) = \max_{\pi} \sum_{j=0}^{s-2} d_{G_2}(\pi(j+1), \pi(j))$, and π is a permutation on vertex set $V(G_2) = \{0, 1, \ldots, s-1\}$.

The following result is due to Liu and Zhu [12].

Lemma 8. *[12] For cycle C_s, $s \geq 3$, the radio number $rn(C_s)$ is*

$$rn(C_s) = \begin{cases} \frac{1}{8}(s^2+6s-8) & \textit{if } s \equiv 0 \pmod 4, \\ \frac{1}{8}(s^2+2s-3) & \textit{if } s \equiv 1 \pmod 4, \\ \frac{1}{8}(s^2+4s-4) & \textit{if } s \equiv 2 \pmod 4, \\ \frac{1}{8}(s^2+4s-5) & \textit{if } s \equiv 3 \pmod 4. \end{cases}$$

Since the distance between every pair of distinct vertices of the complete graph K_r is 1, we get $\sum(K_r) = r-1$. Then we obtain the result of Theorem 7 below by applying Theorem 6 and Lemma 8.

Theorem 7. *For positive integer $r \geq 2$ and $s \geq 3$,*

$$ac(K_r \square C_s) = ac(C_s \square K_r) \leq \begin{cases} \frac{1}{8}(s^2r+10sr-16r-4s+8) & \textit{if } s \equiv 0 \pmod 4, \\ \frac{1}{8}(s^2r+6sr-15r-4s+12) & \textit{if } s \equiv 1 \pmod 4, \\ \frac{1}{8}(s^2r+8sr-12r-4s+8) & \textit{if } s \equiv 2 \pmod 4, \\ \frac{1}{8}(s^2r+8sr-17r-4s+12) & \textit{if } s \equiv 3 \pmod 4. \end{cases}$$

One sees that bounds given in Theorems 2 and 3 are sharp for many values of r and s as proved in Theorems 4 and 5. The example below gives some light on the sharpness of bounds for remaining values of r and s.

Example 3. We obtain that antipodal number of $K_2 \square C_5$ and $K_2 \square C_6$ are 8 and 15 respectively, see Figure 5 and 6. However from Theorems 2 and 3 one gets that the lower bounds for $ac(K_2 \square C_5)$ and $ac(K_2 \square C_6)$ are 4 and 10 respectively. We find that antipodal number of $K_3 \square C_6$ is 16, as given in Figure 7. This number coincides with the lower bound given in Theorem 3 for $ac(K_3 \square C_6)$.

We have also checked that lower bound obtained in Theorem 3 is sharp for $r = 3$ and several values of s when $s \equiv 2 \pmod 4$. Hence we state the open problem below.

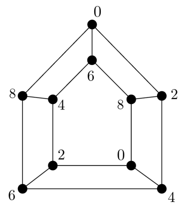

Figure 5. **Minimal antipodal coloring of $K_2\Box C_5$ with span 8**

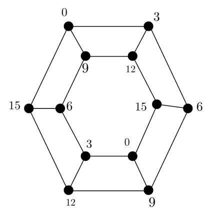

Figure 6. **Minimal antipodal coloring of $K_2\Box C_6$ with span 15**

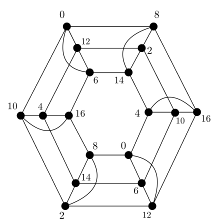

Figure 7. **Minimal antipodal coloring of $K_3\Box C_6$ with span 16**

Open Problem 1. Lower bound obtained in Theorem 3 is sharp for $r = 3$ and $s \equiv 2 \pmod 4$.

Example 4. From Theorem 7, we obtain the upper bound of $ac(K_2\Box C_5)$ and $ac(K_2\Box C_6)$ as 9 and 16 respectively. However we have seen in Example 3 that the antipodal numbers of $K_2\Box C_5$ and $K_2\Box C_6$ are 8 and 15 respectively. Hence the bounds given in Theorem 7 are very close to the exact numbers for some values of r and s.

Conflict of interest. The authors declare that they have no conflict of interest.

Data Availability. Data sharing is not applicable to this article as no datasets were generated or analyzed during the current study.

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