Communications in Combinatorics and Optimization
Vol. 9, No. 2 (2024), pp. 185-196
https://doi.org/10.22049/cco.2023.28629.1639

CCO
Commun. Comb. Optim.

Research Article

Independence number and connectivity of maximal connected domination vertex critical graphs

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Received: 3 May 2023; Accepted: 16 October 2023
Published Online: 20 October 2023

Abstract: A k-CEC graph is a graph G which has connected domination number $\gamma_c(G) = k$ and $\gamma_c(G+uv) < k$ for every $uv \in E(\overline{G})$. A k-CVC graph G is a 2-connected graph with $\gamma_c(G) = k$ and $\gamma_c(G - v) < k$ for any $v \in V(G)$. A graph is said to be maximal k-CVC if it is both k-CEC and k-CVC. Let δ, κ, and α be the minimum degree, connectivity, and independence number of G, respectively. In this work, we prove that for a maximal 3-CVC graph, if $\alpha = \kappa$, then $\kappa = \delta$. We additionally consider the class of maximal 3-CVC graphs with $\alpha < \kappa$ and $\kappa < \delta$, and prove that every 3-connected maximal 3-CVC graph when $\kappa < \delta$ is Hamiltonian connected.

Keywords: connected domination, independence number, connectivity.

AMS Subject classification: 05C69, 05C40

1. Introduction

The basic graph theoretic terminology throughout this paper follow that of Bondy and Murty [3], and all graphs in this paper are simple and connected. Let G be a finite graph with vertex set $V(G)$ and edge set $E(G)$. For $S \subseteq V(G)$, $G[S]$ denotes the subgraph of G *induced* by S. The *open neighborhood* $N_G(v)$ of a vertex v in G is the set of vertices that is adjacent to v. The *closed neighborhood* $N_G[v]$ of a vertex v in G is $\{v\} \cup N_G(v)$. The *degree* $deg_G(v)$ of a vertex v in G is $|N_G(v)|$. Let $\delta(G)$

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be the *minimum degree* of a graph G. $N_G(v) \cap S$ is denoted by $N_S(v)$ where S is a vertex subset of G. A connected graph without cycles is a *tree*. A tree with n vertices of degree 1 and exactly one vertex of degree n is a *star* $K_{1,n}$. An *independent set* is a set whose all pairs of vertices are non-adjacent. The *independence number* of G, $\alpha(G)$, is the maximum cardinality of an independent set of G.

For a connected graph G, a *cut set* is a vertex subset $S \subseteq V(G)$ such that $G - S$ is disconnected. The *connectivity* $\kappa(G)$ is the minimum cardinality of a vertex cut set of a graph G. If $S = \{a\}$ is a minimum cut set of G, then G has a *cut vertex* a and $\kappa(G) = 1$. A graph G is said to be *s-connected* if $\kappa(G) \geq s$. When there is no ambiguity, we shorten $\delta(G), \alpha(G)$, and $\kappa(G)$ to δ, α, and κ, respectively.

A path that visits every vertex of a graph exactly once is called a *Hamiltonian path*. If every pair of vertices of a graph are joined by a Hamiltonian path, then the graph is *Hamiltonian-connected*. It is an exercise to check that Hamiltonian connectivity exists only when the graphs are ℓ-connected for $\ell \geq 3$. For a graph G, the *Mycielskian* $\mu(G)$ of G is the graph with vertex set $V(G) \cup V' \cup \{x\}$, where $V' = \{u' | u \in V(G)\}$ and with edge set $E(G) \cup \{uv' | uv \in E(G)\} \cup \{v'x | v' \in V'\}$.

Let D and X be subsets of $V(G)$, then we say that D *dominates* X, or $D \succ X$, if every vertex in $X \setminus D$ is adjacent to a vertex in D. Furthermore, we write $a \succ X$ when $D = \{a\}$. In particular, if $X = V(G)$, then D is called a *dominating set* of G and we write $D \succ G$ instead of $D \succ V(G)$. A dominating set D of a graph G is called a *connected dominating set* of G if $G[D]$ is connected. A connected dominating set D of G is denoted by $D \succ_c G$. Let γ_c*-set* denote a smallest connected dominating set. The *connected domination number* of G is the cardinality of a γ_c-set of G and it is denoted by $\gamma_c(G)$. Let D be a subset of $V(G)$, then D is called a *total dominating set* of a graph G if every vertex in G is adjacent to a vertex in D. The *total domination number* is the minimum cardinality of a total dominating set of G and is denoted by $\gamma_t(G)$.

A graph G is *k-connected domination edge critical*, *k-CEC*, if $\gamma_c(G) = k$ but $\gamma_c(G + xy) < k$ for any $xy \notin E(G)$. If $\gamma_c(G) = k$ but $\gamma_c(G - x) < k$ for any $x \in V(G)$, then G is *k-connected domination vertex critical*, *k-CVC*. A *maximal k-CVC* graph is a k-CVC graph having largest possible number of edges. Thus, a maximal k-CVC graph is both edge and vertex critical. It can be observed that connected domination is defined on connected graph. From here on, we assume that k-CVC graphs are 2-connected. A *k-total domination edge critical*, *k-TEC*, graph can be defined similarly.

The aim of this paper is to study how the connectivity and the independence number are related if the graphs are maximal 3-CVC. For related results in the graphs whose domination number decreases after adding any edge (k-DEC graphs), Zhang and Tian [11] proved that every 3-DEC graph satisfies $\alpha \leq \kappa + 2$ and proved further that $\kappa = \delta$ if the equality holds. Kaemawichanurat [8] showed that every 3-CEC graph satisfies $\alpha \leq \kappa + 2$. Furthermore, for any 3-CEC graph, if $\kappa + 1 \leq \alpha \leq \kappa + 2$, then $\kappa = \delta$ with only one exception.
In this paper, we prove that if G is a maximal 3-CVC graph with the condition $\alpha = \kappa$, then $\kappa = \delta$. We provide a class of maximal 3-CVC graphs with $\alpha < \kappa < \delta$ so that the condition $\alpha = \kappa$ is needed. We finish by showing that all 3-connected maximal

3-CVC graphs are Hamiltonian-connected if $\kappa < \delta$.

2. Preliminaries

We state the results that used in establishing our theorems. The first theorem was proved by Chvátal and Erdös [5] which is Hamiltonian property of graphs when independence number and connectivity are given.

Theorem 1. *[5] Let G be an ℓ-connected graph with the independence number α. If $\alpha < \ell$, then G is Hamiltonian-connected.*

Chen et al. [4] provided properties of 3-CEC graphs as detailed in Lemmas 1 and 2.

Lemma 1. *[4] Let G be a 3-CEC graph and $ab \in E(\bar{G})$. If D_{ab} is a γ_c-set of $G + ab$. Then*

(1) $|D_{ab}| = 2$,

(2) $\{a, b\} \cap D_{ab} \neq \emptyset$,

(3) if $a \in D_{ab}$ and $b \notin D_{ab}$, then $D_{ab} \cap N_G(b) = \emptyset$.

Lemma 2. *[4] Let G be a 3-CEC graph having A an independent set containing $|A| = m \geq 3$ vertices. Then we can rename the vertices in A as $v_1, v_2, \dots, v_m$ in which there is a corresponding path $u_1, u_2, \dots, u_{m-1}$ in $G - A$ so that, for all $1 \leq i \leq m-1$, $\{v_i, u_i\} \succ_c G + v_i v_{i+1}$.*

In Lemma 3, Ananchuen et al. [2] gave basic properties of 3-CVC graphs.

Lemma 3. *[2] Let G be a 3-CVC graph containing a vertex x. If D_x is a γ_c-set of $G - x$, then*

(1) $|D_x| = 2$ and

(2) $D_x \cap N_G[x] = \emptyset$.

Simmons [10] showed that 3-TEC graphs have $\alpha \leq \delta + 2$. Ananchuen [1] observed that a 3-CEC graph is also 3-TEC and vice versa. Thus every 3-CEC graph satisfies $\alpha \leq \delta + 2$. For 3-CEC graphs, the result that $\alpha = \delta + 2$ was established by Kaemawichanurat et al. [9]. These results can be combined into the following theorem.

Theorem 2. *[10] If G is a 3-CEC graph with $\delta \geq 2$, then $\alpha \leq \delta + 2$. Furthermore, if $\alpha = \delta + 2$, then there is the unique vertex $a \in V(G)$ so that $deg(a) = \delta$ and the subgraph $G[N[a]]$ is complete.*

We previously established [7] some results on maximal 3-CVC graphs.

Lemma 4. *[7] Suppose that G is a maximal 3-CVC graph having a cut set $S \subseteq V(G)$ and let $C_1, C_2, \ldots, C_r$ be the components that are obtained from $G - S$. Further, we let $x \in V(G)$. If $x \in V(C_i) \cup S$ which $|V(C_i)| > 1$ or $r \geq 3$, then*

(1) $D_x \cap S \neq \emptyset$ and

(2) S is not dominated by x.

Lemma 5. *[7] Suppose that G is a maximal 3-CVC graph having a cut set $S \subseteq V(G)$ and let $C_1, C_2, \ldots, C_r$ be the components that are obtained from $G - S$. Further, for some $i \in \{1, 2, \ldots, r\}$, we let $x \in V(C_i)$. Then*

(1) Let $y \in V(C_j)$ for some $j \in \{1, 2, \ldots, r\}$ such that $\{x, y\}$ does not dominate G. If $r \geq 3$ or $|V(C_i)|, |V(C_j)| > 1$, then $|D_{xy} \cap \{x, y\}| = 1$ and $|D_{xy} \cap S| = 1$.

(2) If $c \in D_x$ is an isolated vertex in S, then $r = 2$ and $\{u\} = V(C_j)$ for some $j \in \{1, 2\}$, where $\{u\} = D_x - \{c\}$.

In [7], we further characterized all maximal 3-CVC graphs whose smallest cut set contains no edges.

Theorem 3. *[7] If G is a maximal 3-CVC graph having a smallest cut set S. If S is independent, then G is isomorphic to $G_3 = \mu(K_s)$.*

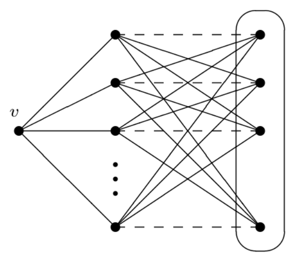

Figure 1. **A graph** $G_3 = \mu(K_s)$

In previous work [6], we established an upper bound for the independence number of maximal 3-CVC graphs in terms of the minimum degree.

Theorem 4. *[6] Let G be a maximal 3-CVC graph. Then $\alpha \leq \delta$.*

3. Connectivity of Maximal 3-CVC Graphs

In this section, we use Theorem 4 to prove that every maximal 3-CVC graph satisfies $\alpha \leq \kappa$. We further construct examples of such graphs for which $\alpha = \kappa$. In [7], we completely characterized all maximal 3-CVC graphs having connectivity at most three. Thus, we focus on $|S| = \kappa \geq 4$. Let $C_1, \ldots, C_m$ be the component of $G - S$. In particular, we let $H_1 = \cup_{i=1}^{\lfloor \frac{m}{2} \rfloor} V(C_i)$ and $H_2 = \cup_{i=\lfloor \frac{m}{2} \rfloor+1}^{m} V(C_i)$. Let I be a maximum independent set of G, $I_i = I \cap H_i$ and $|I_i| = \alpha_i$ for $i \in \{1, 2\}$. Then $I = I_1 \cup I_2 \cup (S \cap I)$. Let $|I_1 \cup I_2| = p$.

Theorem 5. *If G is a 3-CVC graph having independence number α and connectivity κ, then $\alpha \leq \kappa$*

Proof. For contradiction, assume that $\kappa + 1 \leq \alpha$. So $|S| + 1 \leq \alpha_1 + \alpha_2 + |S \cap I|$. Hence

$$|S - I| + 1 = |S| - |S \cap I| + 1 \leq \alpha_1 + \alpha_2 \tag{3.1}$$

Claim 1. $|V(C_i)| > 1$ for all $1 \leq i \leq r$, and $|H_i| > 1$.

Suppose that $V(C_i) = \{c\}$ for some $i \in \{1, 2, \ldots, r\}$. So by Theorem 4, $N_G(c) \subseteq S$. Then we have

$$\delta \leq \deg_G(c) < |S| + 1 = \kappa + 1 \leq \alpha \leq \delta,$$

a contradiction, thus establishing Claim 1.

Let $p = \alpha_1 + \alpha_2$ and $\{a_1, a_2, \ldots, a_p\} = \cup_{i=1}^{2} I_i$. If $p = 1$, then, by (3.1), $|S - I| = 0$. This implies that $S \cap I = S$ which implies that the set S is independent. Note that G is G_3 by Theorem 3. Hence, $N_{G_3}(x)$ in the graph G_3 is a minimum cut set which $G_3 - N_{G_3}(x)$ has a component containing exactly one vertex x. This contradicts Claim 1. Thus, $p > 1$.

Claim 2. $|D_{ab} \cap \{a, b\}| = 1$ and $|D_{ab} \cap (S - I)| = 1$ for any $a, b \in \cup_{i=1}^{2} I_i$.

Since $|S| \geq 4$ and $2 \leq p = \alpha_1 + \alpha_2$, if $p \geq 3$, then $\cup_{i=1}^{2} I_i - \{a, b\} \neq \emptyset$. If $p = 2$, then, by (3.1), $|S| - |S \cap I| + 1 \leq 2$. Because $|S| \geq 4$, we get $|S \cap I| \geq 3$, specifically, $S \cap I \neq \emptyset$. Thus $(S \cap I) \cup (\cup_{i=1}^{2} I_i - \{a, b\}) \neq \emptyset$ inplying that $\{a, b\}$ does not dominate G. By Lemma 5(1) and Claim 1, $|D_{ab} \cap \{a, b\}| = 1$ and $|D_{ab} \cap S| = 1$. Renaming vertices if necessary, we let $a \in D_{ab}$ and $\{a'\} = D_{ab} \cap S$. Since $(G + ab)[D_{ab}]$ is connected, $a' \in S - I$. This proves Claim 2.

Assume that $p = 2$. We consider the graph $G + a_1a_2$. By Claim 2, $|D_{a_1a_2} \cap (S - I)| = 1$. Since $D_{a_1a_2} \cap (S - I) \subseteq S - I$, by (3.1),

$$1 \leq |S - I| \leq \alpha_1 + \alpha_2 - 1 = p - 1 = 1.$$

Therefore, $D_{a_1a_2} \cap (S - I) = S - I$. If $p \geq 3$, then Lemma 2 yields that the vertices $a_1, a_2, \ldots, a_p$ can be renamed as $x_1, x_2, \ldots, x_p$ and there is a corresponding path $y_1, y_2, \ldots, y_{p-1}$ for which $\{x_i, y_i\} \succ_c G + x_ix_{i+1}$ for $i \in \{1, 2, \ldots, p - 1\}$. Since

$\{x_1, x_2, \ldots, x_p\} \subseteq \cup_{i=1}^{2} I_i$, it follows by Claim 2 that $\{y_1, y_2, \ldots, y_{p-1}\} \subseteq S - I$. So, the equation (3.1) gives $p - 1 \leq |S - I| \leq \alpha_1 + \alpha_2 - 1 = p - 1$. In both cases $p = 2$ and $p \geq 3$, we have that $\{y_1, y_2, \ldots, y_{p-1}\} = S - I$.

When $p = 2$, then it can be checked that the subgraph $G[\{y_1\}]$ is complete. When $p \geq 3$. Consider $G + x_i x_j$ for $2 \leq i \neq j \leq p$. By Claim 2, $|D_{x_i x_j} \cap \{x_i, x_j\}| = 1$ and $|D_{x_i x_j} \cap (S - I)| = 1$. Renaming vertices if necessary, w let $x_i \in D_{x_i x_j}$. As $S - I = \{y_1, y_2, \ldots, y_{p-1}\}$, by Lemma 1(3), $D_{x_i x_j} \cap (S - I) = \{y_{j-1}\}$. Since $x_i y_{i-1} \notin E(G)$, $y_{i-1} y_{j-1} \in E(G)$. Therefore, $G[\{y_1, y_2, \ldots, y_{p-1}\}]$ is a clique. Since $\{x_1, x_2, \ldots, x_p\} \subseteq I$, $y_i \succ (S \cap I)$ for $1 \leq i \leq p - 1$. Hence $y_i \succ S$. This contradicts Lemma 4(2). Therefore, $\alpha \leq \kappa$. □

By Theorem 3, the graph $G_3 = \mu(K_s)$ has $N_{G_3}(x)$ as a minimum cut set as well as a maximum independent set. Therefore $\alpha(G_3) = \kappa(G_3)$. Hence, the bound in Theorem 5 is sharp. In particular, for maximal 3-CVC graphs satisfying $\alpha = \kappa$, we have that $|S - I| + |S \cap I| = |S| = \alpha_1 + \alpha_2 + |S \cap I|$. So

$$|S - I| = \alpha_1 + \alpha_2 = p. \tag{3.2}$$

Renaming if necessary, we let $\alpha_1 \leq \alpha_2$. We will prove that if a maximal 3-CVC graph G satisfies $\alpha = \kappa$, then, any minimum cut set S, the graph $G - S$ has a component containing exactly one vertex. We may assume with a contradiction that $G - S$ has no singleton component. Thus, $|H_i| > 1$ for all $1 \leq i \leq 2$.

Lemma 6. *For a maximal 3-CVC graph G, if $|V(C_i)| > 1$ for all $1 \leq i \leq m$ and $\alpha = \kappa$, then $p \geq 3$.*

Proof. Suppose that $|H_i| > 1$ for all $1 \leq i \leq 2$. Firstly, assume that $p = 0$. So $S = S \cap I$. Theorem 3 implies that G is G_3. hence, G_3 has $N_{G_3}(x)$ as a minimum cut set and $G - N_{G_3}(x)$ has x as a singleton component, a contradiction. We discuss 2 cases.

Case 1. $p = 1$.

By (3.2), $|S - I| = 1$. We let $\{a_1\} = \cup_{i=1}^{2} I_i$, $\{v\} = S - I$, and $\{a_2, a_3, \ldots, a_\alpha\} = S \cap I$. Therefore $\alpha_1 = 0$ and $\alpha_2 = 1$. Therefore $a_1 \in H_2$. As $|S| \geq 4$, we have that $|S \cap I| \geq 3$. By Lemma 2, we can rename the vertices in $\{a_2, a_3, \ldots, a_\alpha\}$ as $x_1, x_2, \ldots, x_{\alpha-1}$ for which there is a corresponding path $P = y_1, y_2, \ldots, y_{\alpha-2}$ such that $\{x_i, y_i\} \succ_c G + x_i x_{i+1}$ for $i \in \{1, \ldots, \alpha - 2\}$. Note that $y_i \neq a_1$ because every vertex y_i is adjacent to a vertex of I for $1 \leq i \leq \alpha - 2$. To dominate a_1, $y_i \in H_2 \cup \{v\}$. We consider 2 subcases.

Subcase 1.1. The vertex v is not in the path P.

Thus $V(P) \subseteq H_2$, and hence $x_i \succ H_1$ for $1 \leq i \leq \alpha - 2$. Because $N_{H_1}(v) \neq \emptyset$, it follows that S is a minimum cut set. Let $u \in N_{H_1}(v)$. Thus $u \succ \{x_1, x_2, \ldots, x_{\alpha-2}, v\}$. By Lemma 4(2) we get that $u x_{\alpha-1} \notin E(G)$. For $G + u y_{\alpha-2}$. Since $u x_{\alpha-1}, y_{\alpha-2} x_{\alpha-1} \notin E(G)$. Lemma 5(1) implies that $|D_{u y_{\alpha-2}} \cap \{u, y_{\alpha-2}\}| = 1$ and $|D_{u y_{\alpha-2}} \cap S| = 1$. Hence, $y_{\alpha-2} \in D_{u y_{\alpha-2}}$ or $u \in D_{u y_{\alpha-2}}$. When $y_{\alpha-2} \in D_{u y_{\alpha-2}}$, by Lemma 1(3),

$\{x_1, x_2, \ldots, x_{\alpha-2}, v\} \cap D_{uy_{\alpha-2}} = \emptyset$. Hence $x_{\alpha-1} \in D_{uy_{\alpha-2}}$. But note that $G[D_{uy_{\alpha-2}}]$ is not connected. Hence $u \in D_{uy_{\alpha-2}}$. Since $(G+uy_{\alpha-2})[D_{uy_{\alpha-2}}]$ is connected, $x_{\alpha-1} \notin D_{uy_{\alpha-2}}$. If $x_i \in D_{uy_{\alpha-2}}$ for all $1 \leq i \leq \alpha-2$, then no vertex in $D_{uy_{\alpha-2}}$ is adjacent to $x_{\alpha-1}$. Thus $v \in D_{uy_{\alpha-2}}$, and therefore $va_1 \in E(G)$. Consider $G + ua_1$. Since $ux_{\alpha-1}, a_1x_{\alpha-1} \notin E(G)$, by Lemma 5(1), $|D_{ua_1} \cap \{u, a_1\}| = 1$ and $|D_{ua_1} \cap S| = 1$. Hence either $u \in D_{ua_1}$ or $a_1 \in D_{ua_1}$. In the case $u \in D_{ua_1}$, $v \notin D_{ua_1}$ because of Lemma 1(3). Since $(G + ua_1)[D_{ua_1}]$ is connected, $x_{\alpha-1} \notin D_{ua_1}$. To dominate $x_{\alpha-1}$, $D_{ua_1} \cap \{x_1, x_2, \ldots, x_{\alpha-2}\} \neq \emptyset$. So $D_{ua_1} \cap S = \emptyset$, a contradiction. Hence $a_1 \in D_{ua_1}$. Lemma 1(3) implies that $v \notin D_{ua_1}$. Since $(G + ua_1)[D_{ua_1}]$ is connected, $\{x_1, x_2, \ldots, x_{\alpha-1}\} \cap D_{ua_1} = \emptyset$. Note that $D_{ua_1} \cap S = \emptyset$, a contradiction. Therefore, Subcase 1.1 cannot occur.

Subcase 1.2. The vertex v is in the path P.

In this case, $y_j = v$ for some $j \in \{1, 2, \ldots, \alpha-2\}$. Hence $x_i \succ H_1$ for $i \neq j$, and $\alpha-1$ and $va_1 \in E(G)$. Because $a_1, x_{\alpha-1} \in I$, it follows that a_1 is not adjacent to $x_{\alpha-1}$. If $x_{\alpha-1}$ is not adjacent to the vertex $w \in H_1$, then consider $G+wa_1$. Lemma 5(1) yields that $|D_{wa_1} \cap \{w, a_1\}| = 1$ and $|D_{wa_1} \cap S| = 1$. Thus either $w \in D_{wa_1}$ or $a_1 \in D_{wa_1}$. In both cases, $x_{\alpha-1} \notin D_{wa_1}$ because $(G + wa_1)[D_{wa_1}]$ is connected. If $w \in D_{wa_1}$, then Lemma 1(3) gives $v \notin D_{wa_1}$. To dominate $x_{\alpha-1}$, $\{x_1, x_2, \ldots, x_{\alpha-2}\} \cap D_{wa_1} = \emptyset$. So $D_{wa_1} \cap S = \emptyset$, a contradiction. Hence $a_1 \in D_{wa_1}$. By the connectedness of $(G + wa_1)[D_{wa_1}]$, $D_{wa_1} \cap \{x_1, x_2, \ldots, x_{\alpha-1}\} = \emptyset$. To dominate x_{j+1}, $v \notin D_{wa_1}$. We then have $D_{wa_1} \cap S = \emptyset$, a contradiction. Thus $x_{\alpha-1} \succ H_1$. Clearly $x_i \succ H_1$ for $i \neq j$. Note that S is a minimum cut set. Thus $N_{H_1}(v) \neq \emptyset$. Let $u' \in N_{H_1}(v)$. Lemma 4(2) implies that $u' \succ S - \{x_j\}$. For $G + u'a_1$. By using the same arguments of $G + ua_1$, we get a contradiction. Therefore Case 1 cannot exist.

Case 2. $p = 2$.

Suppose $\{a_1, a_2\} = \cup_{i=1}^{2} I_i$. By (3.2), we have that $|S - I| = p = 2$. As $|S| \geq 4$, we have $|S \cap I| \geq 2$, specifically, $S \cap I \neq \emptyset$ and $\{a_1, a_2\}$ does not dominate G. Consider $G + a_1a_2$. Lemma 5(1) gives that $|D_{a_1a_2} \cap \{a_1, a_2\}| = 1$ and $|D_{a_1a_2} \cap S| = 1$. Without loss of generality, assume $a_1 \in D_{a_1a_2}$. By the connectedness of $(G + a_1a_2)[D_{a_1a_2}]$, $|(S - I) \cap D_{a_1a_2}| = 1$. Let $\{u\} = (S - I) \cap D_{a_1a_2}$. Thus $ua_1 \in E(G)$, $ua_2 \notin E(G)$, and $u \succ S \cap I$. If we let $v \in S - (I \cup \{u\})$, then by Lemma 4(2), we have that $uv \notin E(G)$. Thus $a_1v \in E(G)$

Subcase 2.1. $\alpha_1 = 1$ and $\alpha_2 = 1$.

Renaming vertices if necessary, suppose that $a_1 \in I_1$ and $a_2 \in I_2$. Since $|S \cap I| \geq 2$, there exist $a_3, a_4 \in S \cap I$. Consider $G + a_3a_4$. Lemma 1(2) gives that $D_{a_3a_4} \cap \{a_3, a_4\} \neq \emptyset$. To dominate a_1, $D_{a_3a_4} \neq \{a_3, a_4\}$. Without loss of generality, let $a_3 \in D_{a_3a_4}$. Lemma 1(1) implies that $|D_{a_3a_4} - \{a_3\}| = 1$. Let $y \in D_{a_3a_4} - \{a_3\}$. To dominate $\{a_1, a_2\}$, $y \notin \cup_{i=1}^{2} H_i$. By the connectedness of $(G + a_3a_4)[D_{a_3a_4}]$, $y \in \{v, u\}$. Since $uv \notin E(G)$, then $a_3u, a_3v \in E(G)$. Consider $G - a_3$. Lemma 3(2) implies that $D_{a_3} \cap \{u, v\} = \emptyset$, and Lemma 4(1) yields that $D_{a_3} \cap S \neq \emptyset$. Hence there exists $z \in D_{a_3} \cap (S \cap I)$. Lemma 3(1) implies that $|D_{a_3} - \{z\}| = 1$. We may let $\{z'\} = D_{a_3} - \{z\}$. As $z \in S \cap I$, we have z is not adjacent to a_1. Hence $z' \in H_1$ to dominate a_1. Therefore D_{a_3} does not dominate a_2 contradicting D_{a_3} is a dominating set of $G - a_3$. Subcase 2.1 cannot occur.

Subcase 2.2. $\alpha_1 = 0$ and $\alpha_2 = 2$.

Hence $u \succ H_1$. Let $b_1 \in H_1$. Clearly $\{a_1, b_1\}$ does not dominate G. Consider $G + a_1b_1$. Lemma 5(1) gives that $|D_{a_1b_1} \cap S| = 1$ and either $b_1 \in D_{a_1b_1}$ or $a_1 \in D_{a_1b_1}$. In the first case, $\{u, v\} \cap D_{a_1b_1} = \emptyset$ by Lemma 1(3). To dominate a_2, $D_{a_1b_1} \cap (S \cap I) = \emptyset$. Hence, $D_{a_1b_1} \cap S = \emptyset$, a contradiction. Therefore, $a_1 \in D_{a_1b_1}$. To dominate $H_1 - b_1$ and by the connectedness of $(G + a_1b_1)[D_{a_1b_1}]$, $(D_{a_1b_1} - \{a_1\}) \subseteq \{u, v\}$. Lemma 1(3) implies that $v \in D_{a_1b_1}$. Thus $v \succ H_1 - b_1$. Let $b_2 \in H_1 - \{b_1\}$. Therefore $b_2 \succ \{u, v\}$. Consider $G + a_1b_2$. Lemma 5(1) implies that we have $|D_{a_1b_1} \cap S| = 1$ and either $a_1 \in D_{a_1b_2}$ or $b_2 \in D_{a_1b_2}$. In the first case, $\{u, v\} \cap D_{a_1b_2} = \emptyset$ by Lemma 1(3). By the connectedness of $(G + a_1b_2)[D_{a_1b_2}]$, $(S \cap I) \cap D_{a_1b_2} = \emptyset$. Thus $D_{a_1b_2} \cap S = \emptyset$, a contradiction. Therefore, $b_2 \in D_{a_1b_2}$. To dominate a_2, $(S \cap I) \cap D_{a_1b_2} = \emptyset$. Lemma 1(3) yields that $D_{a_1b_2} \cap \{u, v\} = \emptyset$. Therefore $D_{a_1b_2} \cap S = \emptyset$, a contradiction and so Case 2 cannot occur. Thus $p \geq 3$. $\square$

By Lemma 6, we have that $p \geq 3$. By Lemma 2, the vertices in $\cup_{i=1}^{2} I_i$ can be ordered as $x_1, x_2, \ldots, x_p$ and there exists a path $y_1, y_2, \ldots, y_{p-1}$ with $\{x_i, y_i\} \succ_c G + x_ix_{i+1}$ for $i = 1, 2, \ldots, p - 1$.

Lemma 7. *$y_i \succ S \cap I$ and $y_i \in S - I$ for all $1 \leq i \leq p - 1$.*

Proof. Since $\{x_i, y_i\} \succ_c G + x_ix_{i+1}$ for $i = 1, 2, \ldots, p - 1$ and $x_i \in I$, $y_i \succ S \cap I$. By the connectedness of $(G + x_ix_{i+1})[D_{x_ix_{i+1}}]$ and by Lemma 5(1), $y_i \in S - I$. $\square$

Lemma 7 implies that $\{y_1, y_2, ..., y_{p-1}\} \subseteq S - I$. By (3.2), $|(S - I) - \{y_1, y_2, \ldots, y_{p-1}\}| = 1$. Let $\{y_p\} = (S - I) - \{y_1, y_2, \ldots, y_{p-1}\}$.

Lemma 8. *For $i, j \in \{2, 3, \ldots, p\}$, if $y_px_i, y_px_j \in E(G)$, then $y_{i-1}y_{j-1} \in E(G)$.*

Proof. Consider $G + x_ix_j$. Lemma 5(1) yields that $|D_{x_ix_j} \cap \{x_i, x_j\}| = 1$ and $|D_{x_ix_j} \cap S| = 1$. Without loss of generality, let $x_i \in D_{x_ix_j}$ and $\{a\} = D_{x_ix_j} \cap S$. By the connectedness of $(G + x_ix_j)[D_{x_ix_j}]$, $a \in S - I$. Since $x_j \succ (S - I) - \{y_{j-1}\}$, it follows by Lemma 1(3) that $a = y_{j-1}$. Since $y_{i-1}x_i \notin E(G)$, $y_{j-1}y_{i-1} \in E(G)$. $\square$

Lemma 9. *$\alpha_1, \alpha_2 > 0$.*

Proof. By the assumption that $\alpha_1 \leq \alpha_2$, we can suppose for contradiction that $\alpha_1 = 0$. Clearly $\{x_1, x_2, ..., x_p\} \subseteq H_2$ and $y_i \succ H_1$ for all $1 \leq i \leq p-1$. Note that S is a minimum cut set, so $N_{H_1}(y_p) \neq \emptyset$. Let $b \in N_{H_1}(y_p)$. Therefore $b \succ S - I$. Consider $G + x_1b$. Lemma 5(1) yields that $|D_{x_1b} \cap S| = 1$ and either $b \in D_{x_1b}$ or $x_1 \in D_{x_1b}$. Suppose that $b \in D_{x_1b}$. To dominate x_2, $D_{x_1b} \cap (S - I) \neq \emptyset$. Lemmas 2 and 1(3) then imply that $D_{x_1b} \cap (S - I) = \{y_p\}$. So $y_p \succ \{x_2, x_3, \ldots, x_p\}$. Lemma 8 gives, further, that $G[y_1, y_2, \ldots, y_{p-1}]$ is a clique. Lemma 7 then yields that $y_i \succ S \cap I$ for $i = 1, 2, \ldots, p - 1$. By Lemma 4(2), $y_iy_p \notin E(G)$ for $i = 1, 2, ..., p - 1$. Therefore

$y_1y_p \notin E(G)$. Because $\{x_1, y_1\} \succ_c G+x_1x_2$, $x_1y_p \in E(G)$, contradicting Lemma 1(3). Therefore $x_1 \in D_{x_1b}$. By the connectedness of $(G+x_1b)[D_{x_1b}]$, $D_{x_1b} \cap (S \cap I) = \emptyset$. Lemma 1(3) implies that $D_{x_1b} \cap (S-I) = \emptyset$. Thus $D_{x_1b} \cap S = \emptyset$, contradicting Lemma 5(1). □

Theorem 6. *Let G be a maximal 3-CVC graph having S a minimum cut set. If $\alpha = \kappa$, then $G-S$ has at least one component with exactly one vertex.*

Proof. Assume that G is a maximal 3-CVC graph with $\alpha = \kappa$. By (3.2), $|S-I| = \alpha_1 + \alpha_2$. Suppose that $G-S$ has no singleton component, specifically $|H_i| > 1$ for $i = 1, 2$. Let $\alpha_1 + \alpha_2 = p$. Lemma 6 implies that $p \geq 3$, and Lemma 9 gives that $0 < \alpha_1 \leq \alpha_2$. We also define $x_1, x_2, \ldots, x_p$, a path $y_1, y_2, \ldots, y_{p-1}$ and a vertex y_p as in the previous lemmas.

We may assume that there exist x_i, x_j for $i, j \in \{2, 3, \ldots, p\}$ such that $y_p \in D_{x_ix_j}$. Lemma 1(1) and 1(2) then imply that either $D_{x_ix_j} = \{x_i, y_p\}$ or $D_{x_ix_j} = \{x_j, y_p\}$. Without loss of generality, let $D_{x_ix_j} = \{x_j, y_p\}$. Thus $y_p \succ \{x_1, x_2, \ldots, x_p\} - \{x_i\}$. Since $\{x_i, y_i\} \succ_c G+x_ix_{i+1}$, $y_iy_p \in E(G)$. Lemma 8 yields that $G[\{y_1, y_2, \ldots, y_{p-1}\} - \{y_{i-1}\}]$ is a clique. Since $y_iy_{i-1} \in E(G)$, $y_i \succ S-I$. Lemma 7 implies that $y_i \succ S \cap I$. Therefore $y_i \succ S$, contradicting Lemma 4(2). Hence, $y_p \notin D_{x_ix_j}$ for any $i, j \in \{2, 3, \ldots, p\}$. By using the same arguments as in the proof of Lemma 8, the subgraph $G[\{y_1, y_2, \ldots, y_{p-1}\}]$ is complete. As $y_i \succ S \cap I$, by Lemma 4(2), we must have $y_iy_p \notin E(G)$ for $i \in \{1, 2, \ldots, p-1\}$. Since $\{x_i, y_i\} \succ_c G + x_ix_{i+1}$ for $i \in \{1, 2, \ldots, p-1\}$, $x_iy_p \in E(G)$. So $x_1 \succ S-I$. By Lemma 4(2), $S \cap I \neq \emptyset$, since otherwise $x_1 \succ S$. Let $x_1 \in H_i$ for some $i \in \{1, 2\}$. Then, we consider $G - x_1$. Since $|H_j| > 1$ for $j = 1, 2$, neither $D_{x_1} \subseteq H_1$ nor $D_{x_1} \subseteq H_2$. Lemma 4(1) gives, further, that $D_{x_1} \cap S \neq \emptyset$. Lemma 3(2) implies that $D_{x_1} \cap (S-I) = \emptyset$. Thus $D_{x_1} \cap (S \cap I) \neq \emptyset$. Let $u_1 \in D_{x_1} \cap (S \cap I)$. By Lemma 3(1), $|D_{x_1} - \{u_1\}| = 1$. Let $\{w\} = D_{x_1} - \{u_1\}$. If $w \in H_i$, then $u_1 \succ H_{3-i}$. Since $u_1 \in I$, $\alpha_{3-i} = 0$, contradicting Lemma 9. So $w \in H_{3-i}$ and $u_1 \succ H_i - x_1$. Since $u_1 \in I$, $I_i = \{x_1\}$. It follows that $\{x_2, x_3, \ldots, x_p\} \subseteq H_{3-i}$.

Claim 1. For all $u \in S \cap I$, u does not dominate $S-I$.

Assume that $u \succ S-I$. For $G-u$, Lemma 4(1) implies that $D_u \cap S \neq \emptyset$. By Lemma 3(2), we have that $D_u \cap (S-I) = \emptyset$. Hence there exists $u' \in D_u \cap (S \cap I)$. Lemma 3(1) gives that $|D_u - \{u'\}| = 1$. Let $\{z\} = D_u - \{u'\}$. To dominate x_1, $z \in H_i$. Clearly D_u does not dominate I_{3-i}, so we have a contradiction. This proves Claim 1.

Claim 1 and Lemma 7 imply that y_p is not adjacent to any vertex in $S \cap I$. Therefore, y_p is an isolated vertex in S.

Claim 2. $y_1 \succ H_i$.

Suppose y_1 is not adjacent to $b_1 \in H_i$. Consider $G + b_1x_2$. We see that $b_1y_1, x_2y_1 \notin E(G)$. Lemma 5(1) gives that $|D_{b_1x_2} \cap S| = 1$ and either $b_1 \in D_{b_1x_2}$ or $x_2 \in D_{b_1x_2}$. If $b_1 \in D_{b_1x_2}$, then $(S - \{y_1, y_p\}) \cap D_{b_1x_2} = \emptyset$ to dominate I_{3-i}. Since $y_px_2 \in E(G)$, by Lemma 1(3), $y_p \notin D_{b_1x_2}$. By the connectedness of $(G+b_1x_2)[D_{b_1x_2}]$,

$y_1 \notin D_{b_1x_2}$. Therefore $D_{b_1x_2} \cap S = \emptyset$, a contradiction. Hence $x_2 \in D_{b_1x_2}$. To dominate $I_{3-i} \cup (S \cap I)$, $D_{b_1x_2} \cap \{y_2, y_3, \ldots, y_p\} = \emptyset$. By the connectedness of $(G+b_1x_2)[D_{b_1x_2}]$, $((S \cap I) \cup \{y_1\}) \cap D_{b_1x_2} = \emptyset$. Therefore, $D_{b_1x_2} \cap S = \emptyset$, a contradiction, establishing Claim 2.

Let $b_1 \in H_i - \{x_1\}$. Recall that $u_1 \succ H_i - x_1$. Clearly $b_1u_1 \in E(G)$. By Claim 2 and Lemma 2, $b_1 \succ \{y_1, y_2, \ldots, y_{p-1}\} \cup \{u_1\}$. Consider $G - b_1$. Lemma 4(1) implies that $D_{b_1} \cap S \neq \emptyset$. Lemma 3(2) gives that $D_{b_1} \cap (\{y_1, y_2, \ldots, y_{p-1}\} \cup \{u_1\}) = \emptyset$. If there is $u_2 \in D_{b_1} \cap ((S \cap I) - \{u_1\})$, then, by Lemma 3(1), let $\{y'\} = D_{b_1} - \{u_2\}$. To dominate x_1, $y' \in H_i$. Thus D_{b_1} does not dominate x_2, a contradiction. Therefore, $\{y_p\} = D_{b_1} \cap S$. Note that y_p is an isolated vertex in S, so by Lemma 5(2), at least one of C_i is a singleton component, a contradiction. □

Theorem 6 leads to the following corollary.

Corollary 1. *If G is a maximal 3-CVC graph and $\alpha = \kappa$, then $\kappa = \delta$.*

Proof. Theorem 6 implies that $G - S$ has a component containning exactly one vertex. Renaming if necessary, we let $V(C_i) = \{c\}$. Hence $N_G(c) \subseteq S$. Thus, $\delta \leq \deg_G(c) \leq |S| = \kappa \leq \delta$. □

Now we give the construction of the class $\mathcal{G}_4(s)$ of maximal 3-CVC graphs with $\alpha < \kappa$ and $\kappa < \delta$ in order to show that the condition $\alpha = \kappa$ is needed in Corollary 1. We may let R, T, W, and Z be disjoint sets of vertices where $R = \{r_1, r_2, \ldots, r_s\}$, $T = \{t_1, t_2, \ldots, t_s\}$, $W = \{w_1, w_2, \ldots, w_s\}$, $Z = \{z_1, z_2, \ldots, z_s\}$, and $s \geq 3$. Note that we can construct a graph G in the class $\mathcal{G}_4(s)$ from R, T, W, and Z by adding edges depending on the join operations:

- for $1 \leq i \leq s$, $r_i \vee R \cup T \cup W - \{r_i, t_i\}$,
- $t_i \vee R \cup W \cup Z - \{w_i, r_i\}$,
- $w_i \vee R \cup T \cup Z - \{t_i, z_i\}$,
- $z_i \vee Z \cup T \cup W = \{z_i, w_i\}$ and
- adding edges so that the vertices in R and Z form cliques.

It can be checked that, for $1 \leq i \leq s$, $N_G(r_i) = R \cup T \cup W - \{r_i, t_i\}$, $N_G(t_i) = R \cup W \cup Z - \{w_i, r_i\}$, $N_G(w_i) = R \cup T \cup Z - \{t_i, z_i\}$, and $N_G(z_i) = Z \cup T \cup W = \{z_i, w_i\}$. Note that the sets T and W are independent. Figure 2 shows a graph G, where the double lines joining between two sets mean that every vertex in one set is joined to all vertices in the other set.

Lemma 10. *If $G \in \mathcal{G}_4(s)$, then G is a maximal 3-CVC graph.*

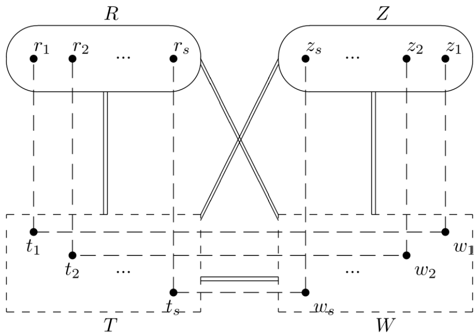

Figure 2. **A graph G in the class $\mathcal{G}_4(s)$**

Proof. Note that $\{r_1, t_2, w_2\} \succ_c G$. Thus $\gamma_c(G) \leq 3$. Let $u, v \in V(G)$ such that $\{u, v\} \succ_c G$. Suppose that $i \in \{1, ..., s\}$, and let $u = r_i$. To dominate the set Z, we have that $v \notin R$. For $v \in T$, we have, by connected, $v \neq t_i$. Hence $\{u, v\}$ does not dominate t_i. To dominate Z, we have that $v \notin W$. Hence $v \in Z$ implying that the subgraph $G[\{u, v\}]$ is disconnected, a contradiction. Thus, $\{u, v\} \cap R = \emptyset$. Note that, by symmetry, $\{u, v\} \cap Z = \emptyset$. Thus $\{u, v\} \subseteq T \cup W$. Renaming vertices if necessary, assume that $u = t_i$. Then, by connected, $v \in W - \{w_i\}$. Therefore $\{u, v\}$ does not dominate w_i. Thus $\gamma_c(G) = 3$.

To consider the criticality, we let $u, v \in V(G)$ such that $uv \notin E(G)$. For $1 \leq i \leq s$, if $\{u, v\} = \{r_i, t_i\}$, then $D_{uv} = \{r_i, t_i\}$. If $\{u, v\} = \{t_i, w_i\}$, then $D_{uv} = \{t_i, w_i\}$. If $\{u, v\} = \{w_i, z_i\}$, then $D_{uv} = \{w_i, z_i\}$. For $1 \leq i \neq j \leq s$, if $\{u, v\} = \{t_i, t_j\}$, then $D_{uv} = \{t_i, r_j\}$. If $\{u, v\} = \{w_i, w_j\}$, then $D_{uv} = \{w_i, z_j\}$. If $\{u, v\} = \{r_i, z_l\}$ where $l \in \{1, 2, \ldots, s\}$, then $D_{uv} = \{r_i, z_l\}$. Thus G is a 3-CEC graph. Let $v \in V(G)$. For $1 \leq i \neq j \leq s$, if $u = r_i$, then $D_v = \{t_i, z_j\}$. If $v = t_i$, then $D_v = \{t_j, r_i\}$. If $v = w_i$, then $D_v = \{z_i, w_j\}$. Finally, if $v = z_i$, then $D_v = \{w_i, r_j\}$. Therefore G is a maximal 3-CVC graph. □

Note that G has T as a maximum independent set and has $T \cup W$ as a minimum cut set. Hence $\alpha = s < 2s = \kappa$. Furthermore, for all $v \in V(G)$, G is a regular graph with $\deg_G(v) = 3s - 2$. Because $s \geq 3$, it follows that $\delta = 3s - 2 > 2s = \kappa$. Thus, $\alpha = \kappa$ is needed to prove Corollary 1.

Finally, we consider the Hamiltonian property of maximal 3-CVC graphs. Using Theorem 1, we obtain that:

Corollary 2. *Let G be a 3-connected maximal 3-CVC graph G. If $\kappa < \delta$, then G is Hamiltonian-connected.*

Proof. Let $\kappa < \delta$. Theorem 5 and Corollary 1 then yield that $\alpha < \kappa$. Hence Theorem 1 implies that G is Hamiltonian-connected. □

Therefore, to prove that every 3-connected maximal 3-CVC graph is Hamiltonian-connected, we need only prove the following conjecture.

Conjecture 7. For any 3-connected maximal 3-CVC graph G, if $\alpha = \kappa = \delta$, then G is Hamiltonian-connected.

Conflict of interest. The authors declare that they have no conflict of interest.

Data Availability. Data sharing is not applicable to this article as no datasets were generated or analyzed during the current study.

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