Communications in Combinatorics and Optimization
Vol. 9, No. 1 (2024), pp. 159-168
https://doi.org/10.22049/cco.2023.28186.1469

CCO
Commun. Comb. Optim.

Research Article

Quasi total double Roman domination in trees

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Received: 10 June 2023; Accepted: 11 October 2023
Published Online: 15 October 2023

Abstract: A quasi total double Roman dominating function (QTDRD-function) on a graph $G = (V(G), E(G))$ is a function $f : V(G) \longrightarrow \{0, 1, 2, 3\}$ having the property that (i) if $f(v) = 0$, then vertex v must have at least two neighbors assigned 2 under f or one neighbor w with $f(w) = 3$; (ii) if $f(v) = 1$, then vertex v has at least one neighbor w with $f(w) \geq 2$, and (iii) if x is an isolated vertex in the subgraph induced by the set of vertices assigned non-zero values, then $f(x) = 2$. The weight of a QTDRD-function f is the sum of its function values over the whole vertices, and the quasi total double Roman domination number $\gamma_{qtdR}(G)$ equals the minimum weight of a QTDRD-function on G. In this paper, we show that for any tree T of order $n \geq 4$, $\gamma_{qtdR}(T) \leq n + \frac{s(T)}{2}$, where $s(T)$ is the number of support vertices of T, that improves a known bound.

Keywords: quasi total double Roman domination, total double Roman domination, double Roman domination number, Roman domination number

AMS Subject classification: 05C69

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1. Introduction

All graphs considered in this article are finite, undirected, simple and without isolated vertices. Let $G = (V, E) = (V(G), E(G))$ be a graph of order $|V(G)| = n$. For any vertex $v \in V(G)$, the *open neighbourhood* of v is the set $N(v) = \{u \in V \mid uv \in E(G)\}$ and the *closed neighbourhood* of v is the set $N[v] = N(v) \cup \{v\}$. For a set $S \subseteq V$, the open neighbourhood of S is $N(S) = \bigcup_{v\in S} N(v)$ and the closed neighbourhood of S is $N[S] = N(S) \cup S$. We denote the *degree* of a vertex v in a graph G by $\deg_G(v)$, or simply by $\deg(v)$ if the graph G is clear from the context.

As usual a *path* and *star* on n vertices are denoted by P_n and $K_{1,n-1}$, and $DS_{p,q}$ denotes the *double star* of order $p + q + 2$. A vertex of degree one is called a *leaf* and its neighbor a *support vertex*. A support vertex is said to be *strong* if it has at least two leaf neighbors. A *tree* is an acyclic connected graph. For any integers $r \geq 1$ and $t \geq 0$, let $F_{r,t}$ be a tree obtained from a star $K_{1,r+t}$ by subdividing r edges exactly once. We say $F_{r,t}$ is a *wounded spider* if $t \geq 1$ and $r \geq 0$ and it is a *healthy spider* if $t = 0$ and $r \geq 2$. The *center vertex* of $F_{r,t}$ is also called the *head vertex* and the vertex at distance two from the head is called the *foot vertex*. A *path* joining two vertices u and v is called a (u, v)*-path*. The *diameter* of a connected graph G, denoted by $\text{diam}(G)$, is the length of a shortest path between the most distanced vertices in G. A *diametral path* of a graph G is a shortest path whose length equals $\text{diam}(G)$. *A rooted tree* T distinguishes one vertex r called the root. For a vertex v in a rooted tree T, the *maximal subtree* at v is subtree of T induced by v and its descendants, and is denoted by T_v. The *depth* of v is the largest distance from v to a descendant of v.

Roman domination is a variation of domination that was formally introduced in graph theory, by Cockayne et al. [6] in 2004. Since then, the topic has been widely studied. For more details on Roman domination and its variants, we refer the reader to the book chapters [3, 5] and survey [4]. It is worth mentioning that the quasi total version for Roman dominating functions has been introduced by Cabrera Martínez et al. [2] and has been further studied in [7, 12, 15].

In 2016, Beeler el at. defined a new variant of Roman domination in [1], namely double Roman dominating functions. A function $f : V(G) \to \{0, 1, 2, 3\}$ is a *double Roman dominating function* (DRD-function) on a graph G if the following conditions hold: (i) If $f(v) = 0$, then v must have one neighbor assigned 3 or two neighbors each assigned 2, and (ii) If $f(v) = 1$, then v must have at least one neighbor w with $f(w) \geq 2$. The *double Roman domination number* $\gamma_{dR}(G)$ equals the minimum weight of a DRD-function on G. A DRD-function of G with weight $\gamma_{dR}(G)$ is called a γ_{dR}-function of G. For a DRD-function f, let V_i be the set of vertices assigned the value i, where $i \in \{0, 1, 2, 3\}$. In that case, the function f will simply be referred to as $f = (V_0, V_1, V_2, V_3)$.

In 2020, Hao et al. [8] considered DRD-functions f such that the subgraph of G induced by the set $\{v \in V \mid f(v) \geq 1\}$ has no isolated vertices, and call such functions *total double Roman dominating functions*, TDRD-functions. The *total double Roman domination number* $\gamma_{tdR}(G)$ is the minimum weight of a TDRD-function on G. For more details, see also [9, 13, 14].

Recently, Kosari et al. [10, 11] defined the quasi total version for double Roman dominating functions. A *quasi total double Roman dominating function* (QTDRD-function) on a graph G is a DRD-function with the additional condition that if x is an isolated vertex in the subgraph induced by the set of vertices labeled with 1, 2 or 3, then $f(x) = 2$. The minimum weight of a QTDRD-function on G is called the *quasi total double Roman domination number* of G and is denoted by $\gamma_{qtdR}(G)$.

In this paper, we are interested in the study of quasi total double Roman domination number of trees and we prove that for any tree T of order $n \geq 4$, $\gamma_{qtdR}(T) \leq n + \frac{s(T)}{2}$, where $s(T)$ denotes the number of support vertices of T.

2. An upper bound for trees

In this section, we show that for any tree T with order $n \geq 4$, $\gamma_{qtdR}(T) \leq n + \frac{s(T)}{2}$, where $s(T)$ is the number of support vertices of T. We start with a simple observation and some examples.

Observation 1. ([11]) If v is a strong support vertex of a graph G different from stars, then there exists a $\gamma_{qtdR}(G)$-function f that assigns 3 to v and 0 to every leaf neighbor of v.

Example 1. Let $P^k_{2,3}$ be a tree obtained from a path $P := v_1v_2\dots v_k$ ($k \geq 4$) by adding a new vertex v and a path uw and adding the edges v_2v and v_3u. If k is odd, the assigning a 3 to v_2, a 2 to w and v_{2i+1} for $i \in \{1,\dots,\frac{k-1}{2}\}$ and a 0 to the other vertices provides a QTDRD-function on $P^k_{2,3}$ with weight $k+4$. If k is even, the assigning a 3 to v_2, a 1 to v_k, a 2 to w and v_{2i+1} for $i \in \{1,\dots,\frac{k-2}{2}\}$ and a 0 to the other vertices provides a QTDRD-function on $P^k_{2,3}$ with weight $k+4$. Thus $\gamma_{qtdR}(P^k_{2,3}) \leq n(P^k_{2,3}) + 1$.

Example 2. Let $P^{k\prime}_{2,3}$ be a tree obtained from $P^k_{2,3}$ by adding a new vertex v' and adding the edge $v_{k-1}v'$. If k is odd, the assigning a 3 to v_2 and v_{k-1}, a 2 to w and v_{2i+1} for $i \in \{1,\dots,\frac{k-3}{2}\}$ and a 0 to the other vertices provides a QTDRD-function on $P^{k\prime}_{2,3}$ with weight $k+5$. If k is even, the assigning a 3 to v_2 and v_{k-1}, a 1 to v_{k-2}, a 2 to w and v_{2i+1} for $i \in \{1,\dots,\frac{k-4}{2}\}$ and a 0 to the other vertices provides a QTDRD-function on $P^{k\prime}_{2,3}$ with weight $k+5$. Consequently, $\gamma_{qtdR}(P^{k\prime}_{2,3}) \leq n(P^{k\prime}_{2,3}) + 1$.

Example 3. Let $F^k_{r,t}$ be a tree obtained from $F_{r,t}$ centered at v by adding a path $v_1v_2\dots v_k$ and adding the edge v_1v. If $t \geq 1$, then let w be a leaf neighbor of v. If k is odd and $t = 0$, then assigning a 2 to v, each leaf of $F_{r,t}$ and v_{2i} for $i \in \{1,\dots,\frac{k-1}{2}\}$, a 1 to v_k and a 0 to the other vertices provides a QTDRD-function on $F^k_{r,t}$ with weight $n(F^k_{r,t}) + 1$. If k is odd and $t = 1$, then assigning a 2 to v, each leaf of $F_{r,t}$ at distance two from v and v_{2i} for $i \in \{1,\dots,\frac{k-1}{2}\}$, a 1 to w and v_k and a 0 to the other vertices provides a QTDRD-function on $F^k_{r,t}$ with weight $n(F^k_{r,t}) + 1$. If k is odd and $t \geq 2$, then assigning a 3 to v, each leaf of $F_{r,t}$ at distance two from v and v_{2i} for $i \in \{1,\dots,\frac{k-1}{2}\}$, a 1 to w and v_k and a 0 to the other vertices provides a QTDRD-function on $F^k_{r,t}$ with weight at most $n(F^k_{r,t}) + 1$.
If k is even and $t = 0$, the assigning a 2 to v and each leaf of $F_{r,t}$ and v_{2i} for $i \in \{1,\dots,\frac{k-1}{2}\}$ and a 0 to the other vertices provides a QTDRD-function on $F^k_{r,t}$ with weight $n(F^k_{r,t}) + 1$. If

k is even and $t = 1$, the assigning a 2 to v, each leaf of $F_{r,t}$ at distance two from v and v_{2i} for $i \in \{1, \ldots, \frac{k-1}{2}\}$, a 1 to w and a 0 to the other vertices provides a QTDRD-function on $F^k_{r,t}$ with weight at most $n(F^k_{r,t}) + 1$. Finally, if k is even and $t \geq 2$, the assigning a 3 to v, each leaf of $F_{r,t}$ at distance two from v and v_{2i} for $i \in \{1, \ldots, \frac{k-1}{2}\}$, a 1 to w and a 0 to the other vertices provides a QTDRD-function on $F^k_{r,t}$ with weight at most $n(F^k_{r,t}) + 1$. Thus, in either case we have $\gamma_{qtdR}(F^k_{r,t}) \leq n(F^k_{r,t}) + \frac{s(F^k_{r,t})}{2}$.

Example 4. Let $F^{k\prime}_{r,t}$ be a tree obtained from $F^k_{r,t}$ by adding a new vertex z and the edge $v_{k-1}z$. As in the above examples, it can be seen that $F^{k\prime}_{r,t}$ has a QTDRD-function with weight $n(F^{k\prime}_{r,t}) + 1$.

Theorem 2. *Let T be a tree of order $n \geq 4$. Then $\gamma_{qtdR}(T) \leq n + \frac{s(T)}{2}$.*

Proof. Let T be a tree of order $n \geq 4$. We will proceed by induction on the order n. If $n = 4$, then $T \in \{P_4, K_{1,3}\}$ and clearly $\gamma_{qtdR}(T) \leq 4 + \frac{s(T)}{2}$. This proves the base case. Let $n \geq 5$ and assume that if T' is a tree of order n', where $n' < n$ and $n' \geq 4$, then $\gamma_{qtdR}(T') \leq n' + \frac{s(T')}{2}$. If T is a star, then the function that assigns 3 to the central vertex, 1 to one of leaves and 0 to other leaves of the star, is a QTDRD-function of T of weight 4, and so $\gamma_{qtdR}(T) = 4 < n + \frac{s(T)}{2}$. Hence, we may assume that T is not a star and thus diam $(T) \geq 3$. If diam $(T) = 3$, then T is a double star $T \cong DS_{r,s}$, where $r \geq s \geq 1$ and $r \geq 2$. Let x and y be the two support vertices of T, where x has r leaf neighbors and y has s leaf neighbors. Then the function that assigns 3 to x and y and 0 to remaining vertices of T is a QTDRD-function of T of weight 6, leading to $\gamma_{qtdR}(T) = 6 \leq n + \frac{s(T)}{2}$. Hence, we can assume that diam $(T) \geq 4$, for otherwise the desired result follows.
If T has a support vertex v with at least three leaf neighbors, then consider the tree T' obtained from T' by removing one leaf neighbor of v, say u. Observe that v remains a strong support vertex in T' and that $s(T') = s(T)$. By Observation 1, v is assigned 3 under some γ_{qtdR}-function f on T', and such a γ_{qtdR}-function can be extended to a QTDRD-function of T by assigning a 0 to u, leading to $\gamma_{qtdR}(T) \leq \gamma_{qtdR}(T') \leq (n-1) + \frac{s(T')}{2} < n + \frac{s(T)}{2}$. Therefore, we can assume that every support vertex in T is adjacent to one or two leaves.

Let $u_1u_2 \ldots u_k$ be a diametral path of T chosen such that $\deg_T(u_2)$ is as large as possible. Note that u_2 is a support vertex and thus $\deg_T(u_2) \in \{2, 3\}$. Root T at u_k, and consider the following cases.

Case 1. $\deg_T(u_2) = 3$.
Thus u_2 has exactly two leaf neighbors. Suppose first that $\deg_T(u_3) = 2$ and let $T' = T - T_{u_3}$, that is T' is a tree obtained from T by deleting the vertex u_3 and its descendants. We note that T' has order $n' \geq 2$, because diam $(T) \geq 4$. If $n' = 2$, then T is a tree obtained from the path $u_1 \ldots u_5$ by adding a vertex z and an edge u_2z. In this case, it is not hard to see that $\gamma_{qtdR}(T) = 7 = n + \frac{s(T)}{2}$. If $n' = 3$, then T is isomorphic to one of the trees T_1 or T_3 illustrated in Figure 1. In each case, it is easy to see that $\gamma_{qtdR}(T) \leq n + \frac{s(T)}{2}$. Hence we may assume that $n' \geq 4$. Since any

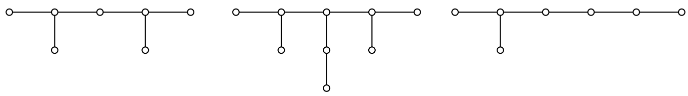

Figure 1.

$\gamma_{qtdR}(T')$-function can be extended to a QTDRD-function of T by assigning a 3 to u_2, a 1 to u_3 and a 0 to the leaf neighbors of u_2, by applying the induction hypothesis on T', we have $\gamma_{qtdR}(T) \leq \gamma_{qtdR}(T') + 4 \leq (n-4) + \frac{s(T)}{2} + 4 = n + \frac{s(T)}{2}$ as desired. Let us assume in the next that $\deg_T(u_3) \geq 3$. Let $T' = T - T_{u_2}$, and note that T' has order $n' \geq 4$, because diam $(T) \geq 4$ and $\deg_T(u_3) \geq 3$. Applying the induction hypothesis on T', we have $\gamma_{qtdR}(T') \leq (n-3) + \frac{s(T')}{2} = (n-3) + \frac{s(T)-1}{2}$. Now, if there exists a $\gamma_{qtdR}(T')$-function f' such that $f'(u_3) \neq 0$, then f' can be extended to a QTDRD-function of T by assigning 3 to u_2 and 0 to its two leaf neighbors, yielding $\gamma_{qtdR}(T) \leq \gamma_{qtdR}(T') + 3 < n + \frac{s(T)}{2}$. Henceforth, we may assume that every γ_{qtdR}-function of T' assigns 0 to u_3. According the choice of u_2 on the diametral path, let s be the number of children of u_3, with degree 3, other than u_2, r be the number of children of u_3 with degree 2 and t be the number of leaf neighbors of u_3 in T. Observe that if $t \geq 2$ (resp. $r \geq 2$), then u_3 would be assigned a 3 (resp. 2) under some γ_{qtdR}-function of T', contradicting our earlier assumption. Hence $t \leq 1$ and $r \leq 1$. Similarly, if $s \geq 1$, then u_3 could be assigned at least 1 under some γ_{qtdR}-function of T', contradicting our earlier assumption again. Hence $s = 0$. We distinguish the following subcases.

Subcase 1.1. $t = 1$.
Let u' denote the leaf neighbor of u_3, and let f' be a γ_{qtdR}-function of T'. By our earlier assumption we have $f'(u_3) = 0$ and thus $f'(u') = 2$. Consider the following situations.

(a) $r = 1$.
Then $f'(V(T'_{u_3})) = 5$. In this case, form f from $\gamma_{qtdR}(T')$-function f', by letting $f(x) = f'(x)$ for all $x \in T - T_{u_3}, f(u_2) = f(u_3) = 3, f(z) = 2$ for the leaf neighbor of the child of v_3 with degree 2 and $f(z) = 0$ for the remaining vertices of T_{v_3}. Then f is a QTDRD-function of T, yielding

$$\gamma_{qtdR}(T) \leq f'(V(T - T_{u_3})) + 8 = \gamma_{qtdR}(T') + 3 \leq (n-3) + \frac{s(T)-1}{2} + 3 < n + \frac{s(T)}{2}.$$

(b) $r = 0$ and $\deg_T(u_4) \geq 3$.
Let $T'' = T - T_{u_3}$. Then $s(T'') = s(T) - 2$ and T'' has order $n'' \geq 3$ because diam $(T) \geq 4$ and $\deg_T(u_4) \geq 3$. If $n'' = 3$, then T is isomorphic to the tree T_4 depicted in Figure 2 and it is easy to see that $\gamma_{qtdR}(T_3) = 9 < n + \frac{s(T)}{2}$, as desired. Hence, we may assume that $n'' \geq 4$. Applying the induction hypothesis

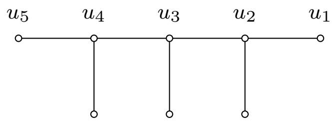

Figure 2. **Tree** T_4

on T'', we have $\gamma_{qtdR}(T'') \leq (n-5) + \frac{s(T)-2}{2}$. Since any $\gamma_{qtdR}(T'')$-function can be extended to a QTDRD-function of T by assigning 3 to the vertices u_2 and u_3, and 0 to each leaf at T_{u_3}, we get $\gamma_{qtdR}(T) \leq \gamma_{qtdR}(T'')+6 \leq (n-5)+\frac{s(T)-2}{2}+6 = n + \frac{s(T)}{2}$.

(c) $r = 0$ and $\deg(u_4) = 2$.
Let T''' be a tree obtained from T by deleting u_4 and its descendants, that is $T''' = T - T_{u_4}$. Then $s(T''') \leq s(T) - 1$ and T''' has order $n''' \geq 1$ because diam $(T) \geq 4$. If $n''' = 1$, then T is isomorphic to the tree T_5 depicted in Figure 3 and we have $\gamma_{qtdR}(T_5) = 8 < n+\frac{s(T)}{2}$. If $n''' = 2$, then T is isomorphic to the tree T_6 depicted in Figure 4 and we have $\gamma_{qtdR}(T_6) = 9 < n+\frac{s(T)}{2}$. If $n''' = 3$, then T is isomorphic to one of the trees T_7 or T_8 depicted in Figure 5 and it is easy to see that $\gamma_{qtdR}(T) = 10 < n + \frac{s(T)}{2}$, as desired. Thus, we may suppose that $n''' \geq 4$. Applying the induction hypothesis on T''', we have $\gamma_{qtdR}(T''') \leq (n-6)+\frac{s(T)-1}{2}$. Since any $\gamma_{qtdR}(T''')$-function can be extended to a QTDRD-function of T by assigning 3 to the vertices v_2 and v_3, and 0 to each other vertices of T_{v_3}, we get $\gamma_{qtdR}(T) \leq \gamma_{qtdR}(T''') + 6 \leq (n-6) + \frac{s(T)-1}{2} + 6 < n + \frac{s(T)}{2}$.

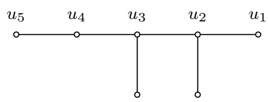

Figure 3. **Tree** T_5

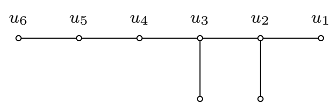

Figure 4. **Tree** T_6

Subcase 1.2. Assume that $t = 0$, $r = 1$ and $\deg(u_4) \geq 3$.
Let $T^1 = T - T_{u_3}$. Since diam $(T) \geq 4$ and $\deg(u_4) \geq 3$, T^1 has order $n_1 \geq 3$. If $n_1 = 3$, then T is isomorphic to the tree T_2 of the Figure 1 and it can be seen that $\gamma_{qtdR}(T) < n+\frac{s(T)}{2}$. Consequently, we can assume in the next that $n_1 \geq 4$. Using the

Figure 5. **Two trees discussed in situation (c)**

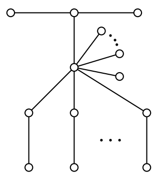

Figure 6. **Family $\mathcal{F}$**

induction hypothesis on T^1, we have $\gamma_{qtdR}(T^1) \leq n_1 + \frac{s(T^1)}{2} = (n-6) + \frac{s(T)-2}{2}$. Let f_1 be a $\gamma_{qtdR}(T^1)$-function. Then we extend f_1 to a QTDRD-function of T of weight $\gamma_{qtdR}(T^1) + 7$ by assigning 3 to the two children of u_3, 1 to u_3 and 0 to all leaves of T_{v_3}. This leads to $\gamma_{qtdR}(T) \leq \gamma_{qtdR}(T^1) + 7 \leq (n-6) + \frac{s(T)-2}{2} + 7 = n + \frac{s(T)}{2}$.

Subcase 1.3. Assume that $t = 0$, $r = 1$ and $\deg(u_4) = 2$.
If $\deg(u_i) = 2$ for $i \in \{4, 5, \ldots, k-1\}$, then $T = P^k_{2,3}$ and Example 1 implies that $\gamma_{qtdR}(T) \leq n + 1 < n + \frac{s(T)}{2}$. Hence we assume that $\deg(u_i) \geq 3$ for some $i \in \{5, \ldots, k-1\}$. Let $m \geq 4$ be the smallest integer with $\deg(u_m) = 2$ and $\deg(u_{m+1}) \geq 3$. If $m = k-1$, then we must have $T = P^{k\prime}_{2,3}$, since every support vertex in T is adjacent to one or two leaves and Example 2 leads to $\gamma_{qtdR}(T) \leq n + 1 < n + \frac{s(T)}{2}$. Thus we assume that $m < k-2$. Let $T^2 = T - T_{u_m}$. Clearly T^2 has order $n_2 \geq 4$. Applying the induction hypothesis on T^2, we have $\gamma_{qtdR}(T^2) \leq n_2 + \frac{s(T^2)}{2} = (n-m-3) + \frac{s(T)-2}{2}$. Let f_1 be a $\gamma_{qtdR}(T^2)$-function and f_2 be a $\gamma_{qtdR}(T_{u_m})$-function. Then the function h defined on $V(T)$ by $h(x) = f_1(x)$ for $x \in V(T^2)$ and $h(x) = f_2(x)$ for $x \in V(T_{u_m})$, is a QTDRD-function on T. Using Example 1, we get $\gamma_{qtdR}(T) \leq \gamma_{qtdR}(T^2) + \gamma_{qtdR}(T_{u_m}) \leq (n-m-3) + \frac{s(T)-2}{2} + (m+4) = n + \frac{s(T)}{2}$.

Case 2. $\deg_T(u_2) = 2$.
By the choice of the diametral path, each child of u_3 has degree at most two. Let r be the number of children of v_3 with degree 2 and t be the number of leaves adjacent to v_3. Note that $r \geq 1$ and $t \geq 0$. First let $r = 1$ and $t = 0$, that is $\deg_T(u_3) = 2$, and let T' be the tree obtained from T by deleting u_1. Since $\operatorname{diam}(T) \geq 4$, T' has order $n' \geq 4$. Applying the induction hypothesis on T', we have $\gamma_{qtdR}(T') \leq (n-1) + \frac{s(T)}{2}$. Let f' be a $\gamma_{qtdR}(T')$-function such that $f(u_2)$ is minimized. If $f'(v_2) \geq 2$, then we can extend f' to a QTDRD-function of T by assigning a 1 to u_1 and this leads to $\gamma_{qtdR}(T) \leq \gamma_{qtdR}(T') + 1 \leq n + \frac{s(T)}{2}$, as desired. If $f'(u_2) = 0$, then we must have $f(u_3) = 3$ and $f(u_4) \geq 1$ and by reassigning a 2 to u_3 and assigning a 2 to u_1, we obtain a QTDRD-function of T and thus $\gamma_{qtdR}(T) \leq \gamma_{qtdR}(T') + 1 \leq n + \frac{s(T)}{2}$, as desired. Hence we assume that $f(u_2) = 1$. Then by the choice of f' we must have

$f(u_3) = 2$. By reassigning a 0 to u_2 and assigning a 2 to u_1, we obtain a QTDRD-function of T. Consequently, $\gamma_{qtdR}(T) \leq \gamma_{qtdR}(T') + 1 \leq n + \frac{s(T)}{2}$.
Assume next that $r + t \geq 2$. We distinguish two situations.

Subcase 2.1. $\deg(u_4) \geq 3$.
Let T' be the tree obtained from T by deleting u_3 and all its descendants, that is $T' - T_{u_3}$. Since diam $(T) \geq 4$ and $\deg(u_4) \geq 3$, T' has order $n' \geq 3$. If $n' = 3$, then T is a tree belonging to the families $\mathcal{F}$ of trees illustrated in Figure 6 and so $n = t + 2r + 4$. It can be seen that

$$\gamma_{qtdR}(T) = \begin{cases} 2r+5, & \text{if} \quad t = 0 \\ 2r+6, & \text{if} \quad t \geq 1, \end{cases}$$

and thus $\gamma_{qtdR}(T) < n + \frac{s(T)}{2}$. Therefore, we may assume in the next that $n' \geq 4$. Applying the induction hypothesis on T', we have $\gamma_{qtdR}(T') \leq n' + \frac{s(T')}{2}$. Now, if $t \geq 2$, then any $\gamma_{qtdR}(T')$-function f', can be extended to a QTDRD-function of T of weight $\gamma_{qtdR}(T') + 4 + 2r$ by assigning 3 to u_3, 2 to every leaf neighbor of T_{u_3} not adjacent to u_3, 1 to one leaf neighbor of u_3 and 0 to the remaining vertices of T_{u_3}. It follows that

$$\gamma_{qtdR}(T) \leq (n - 2r - t - 1) + \frac{s(T')}{2} + 4 + 2r = n - t + 3 + \frac{s(T) - r - 1}{2} \leq n + \frac{s(T)}{2},$$

as desired. Assume now that $t = 1$. Then any $\gamma_{qtdR}(T')$-function f' can be extended to a QTDRD-function of T of weight $\gamma_{qtdR}(T') + 3 + 2r$ by assigning 2 to u_3 and all leaves of T_{u_3} that are not adjacent to v_3, 1 to the unique leaf neighbor of u_3 and 0 to the remaining vertices of T_{u_3}. It follows that

$$\gamma_{qtdR}(T) \leq (n - 2r - 2) + \frac{s(T')}{2} + 3 + 2r = n + 1 + \frac{s(T) - r - 1}{2} \leq n + \frac{s(T)}{2},$$

as desired.
Now, let $t = 0$. Form f from any $\gamma_{qtdR}(T')$ by assigning 2 to u_3 and to each leaf at distance two from u_3 in T_{u_3} and 0 to the children of u_3. Using the induction hypothesis on T', it follows that

$$\gamma_{qtdR}(T) \leq \gamma_{qtdR}(T') + 2r + 2 \leq (n - 2r - 1) + \frac{s(T) - r}{2} + 2r + 2 \leq n + \frac{s(T)}{2},$$

as desired.

Subcase 2.2. $\deg_T(u_4) = 2$.
If $\deg_T(u_i) = 2$ for each $4 \leq i \leq k - 1$, then $T = F^k_{r,t}$ and the result follows from Example 3. Hence we assume that $\deg_T(v_i) \geq 3$ for some $5 \leq i \leq k - 1$. Let $m \geq 4$ be the smallest integer such that $\deg_T(u_m) = 2$ and $\deg_T(u_{m+1}) \geq 3$. Let $T' = T - T_{u_m}$. Clearly, T' has order $n' \geq 3$. If $n' = 3$, then $T = F^{k\prime}_{r,t}$ and the result follows from Example 4. Therefore, we may assume in the next that $n' \geq 4$. Applying the induction hypothesis on T', we have $\gamma_{qtdR}(T') \leq n' + \frac{s(T')}{2}$. Let f' be a $\gamma_{qtdR}(T')$-function and f'' be a $\gamma_{qtdR}(T_{u_m})$-function and define h on $V(T)$ bt

$h(x) = f'(x)$ for $x \in V(T')$ and $h(x) = f''(x)$ for $x \in V(T_{u_m})$. It is easy to see that h is a QTDRD-function of T and thus

$$\begin{aligned}\gamma_{qtdR}(T) &\leq \gamma_{qtdR}(T') + \gamma_{qtdR}(T_{v_m})\\ &\leq (n - |V(T_{v_m})|) + \frac{s(T')}{2} + |V(T_{v_m})| + 1\\ &\leq n + 1 + \frac{s(T)-2}{2} = n + \frac{s(T)}{2}.\end{aligned}$$

This completes the proof. □

Let $\mathcal{T}$ be a family of trees which is obtained from k paths $P_4 = x_i^1 x_i^2 x_i^3 x_i^4$ $(k \geq 1)$ by adding $k-1$ edges between x_i^2s such that the resulting graph is connected (see Figure 7 for $k = 3$). The proof of the following theorems can be found in [11].

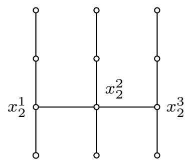

Figure 7. **A tree T in the family $\mathcal{T}$**

Theorem 3 **([11]).** *For $n \geq 2$, $\gamma_{qtdR}(P_n) = n + 1$.*

Theorem 4 **([11]).** *For any tree T of order $n \geq 4$, $\gamma_{qtdR}(T) \leq \frac{5}{4}n$, with equality if and only if $T \in \mathcal{T}$.*

These theorems show that the bound of Theorem 2 attains by any path of order at least three and any tree T in the family $\mathcal{T}$.

Acknowledgements. The authors are grateful to anonymous referees for their constructive suggestions that improved the paper.

Conflict of interest. The authors declare that they have no conflict of interest.

Data Availability. Data sharing is not applicable to this article as no datasets were generated or analyzed during the current study.

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