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Research Article

Cliques in the extended zero-divisor graph of finite commutative rings

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Abstract: Let R be a finite commutative ring with or without unity and $\Gamma_e(R)$ be its extended zero-divisor graph with vertex set $Z^*(R) = Z(R) \setminus \{0\}$ and two distinct vertices x, y are adjacent if and only if $x.y = 0$ or $x + y \in Z^*(R)$. In this paper, we characterize finite commutative rings whose extended zero-divisor graph have clique number 1 or 2. We completely characterize the rings of the form $R \cong R_1 \times R_2$, where R_1 and R_2 are local, having clique number 3, 4 or 5. Further we determine the rings of the form $R \cong R_1 \times R_2 \times R_3$, where R_1,R_2 and R_3 are local rings, to have clique number equal to six.

Keywords: Zero-divisor graph; Extended zero-divisor graph; Finite commutative rings; clique number.

AMS Subject classification: 13A15, 05C25, 05C30

1. Introduction

All graphs considered in this article are connected, simple and finite. A graph is denoted by $G = G(V(G), E(G))$, where $V(G)$ is the vertex set and $E(G)$ is the edge set of G. The order and the size of G are the cardinalities of $V(G)$ and $E(G)$, respectively. A complete graph on n vertices is denoted by K_n and the complete bipartite graph is denoted by $K_{n,m}$, where n and m are a cardinalities of partite subsets. We denote the adjacency relation between two vertices x and y by $x \sim y$. For $S \subset V$, the graph $G[S]$ is called an induced subgraph of G, with vertex set S and

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whose edge set consists of all the edges of E having vertices in S. A clique of a graph G is defined as the complete subgraph of G. The cardinality of the largest clique is called the clique number and is denoted by $\omega(G)$. Throughout this paper, we take graph theoretic notations from [12].

For notations and results about commutative rings, we use [9, 10, 13] as basic references. A ring R is assumed to be a finite commutative ring with or without identity. A ring R is said to be local, if it has a unique maximal ideal. The finite field with n elements is denoted by $\mathbb{F}_n$ and $\frac{m\mathbb{Z}}{n\mathbb{Z}}$ is a ring without unity whose all elements are zero-divisors. The ring $\mathbb{Z}_n$ denotes the ring of integers modulo n. The cardinality of a ring R is denoted by $|R|$. The direct product of two rings R_1 and R_2, denoted by $R_1 \times R_2$, consists of all ordered pairs (a, b) with $a \in R_1$ and $b \in R_2$. The addition rule for such pairs is $(a, b) + (c, d) = (a + c, b + d)$ and multiplication rule for such pairs is $(a, b).(c, d) = (ac, bd)$. The structure theorem for Artinian rings states that any Artinian ring R is isomorphic to the direct product of local rings, i.e., $R \cong R_1 \times R_2 \times \cdots \times R_n$ for some positive integer n, where $R_i, i = 1, 2, \ldots, n$, are local rings.

Let R be a commutative ring and $Z(R)$ be its set of all zero-divisors and $Z^*(R) = Z(R) \setminus \{0\}$ be the set of non-zero zero-divisors. The concept of zero-divisor graph $\Gamma(R)$ associated to a ring R was first defined and introduced by Beck [6] and later modified by Anderson and Livingston [4]. Some work on zero-divisor graphs can be seen in [1, 2, 5]. Several extensions on zero-divisor graphs were defined by modifying the basic definition of zero-divisor graph [7, 8]. Anderson and Badawi [3], introduced the total graph $T(\Gamma(R))$ of a commutative ring R with all elements of R as vertices, and for distinct $x, y \in R$, the vertices x and y are adjacent if and only if $x + y \in Z(R)$. Cherrabi et al. [8] introduced a new extension of zero-divisor graph, denoted by $\tilde{\Gamma}(R)$, whose vertices are the non-zero zero-divisors of a commutative ring R and for distinct elements x and y in the set $Z^*(R)$ (the set of non zero zero-divisors of R) are adjacent if and only if $xy = 0$ *or* $x + y \in Z(R)$. Liu et al. [11] investigated the commutative rings whose zero-divisor graphs have clique number one, two or three. Further, if $R \cong R_1 \times R_2 \times \cdots \times R_n$, for some positive integer n, where $R_i, i = 1, 2, \ldots, n$, are local rings, they also give the algebraic characterizations of rings R, when clique number of $\Gamma(R)$ is four. We define a new extension of the zero-divisor graph, denoted by $\Gamma_e(R)$, by taking all non-zero zero-divisors of a ring R as the vertices of $\Gamma_e(R)$ and two distinct vertices x and y are adjacent if and only if $x.y = 0$ or $x + y \in Z^*(R)$.

The following example illustrates the difference between the three graphs $\Gamma(R)$, $\tilde{\Gamma}(R)$ and $\Gamma_e(R)$. This can be seen in the figures corresponding to $R = \mathbb{Z}_2 \times \mathbb{Z}_4$.

The rest of the paper is organised as follows. In section 2, we characterize the structure of all finite commutative rings associated to the graph $\tilde{\Gamma}(R)$ to have clique number 1 or 2. In section 3, we characterize the rings of the form $R \cong R_1 \times R_2$ (each R_i is local, $i = 1, 2$) to have clique number 3, 4 or 5. In addition, we determine the rings of the form $R \cong R_1 \times R_2 \times R_3$, where each R_i, $i = 1, 2, 3$ are local rings, to have clique number equal to six. If $R \cong \frac{m\mathbb{Z}}{n\mathbb{Z}}$, then we observe that $\Gamma_e(R)$ is a complete graph.

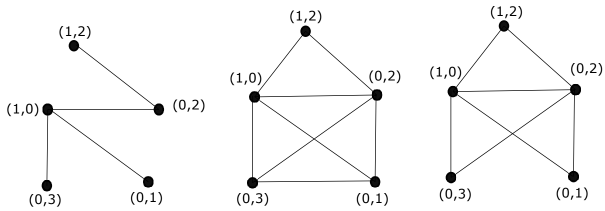

Figure 1. $\Gamma(R)$, $\tilde{\Gamma}(R)$ **and** $\Gamma_e(R)$

2. Finite commutative rings whose extended zero-divisor graphs have clique number at most 2

We start with the following facts.

Observation 1. For a finite integral domain R, we observe that $\omega(\Gamma_e(R) = 0$.

Observation 2. Let R be a finite commutative ring. We observe that $\omega(\Gamma_e(R) = 1$ if and only if R is isomorphic to either $\mathbb{Z}_4$ or $\frac{\mathbb{Z}_2[x]}{<x^2>}$. For if $R \cong \mathbb{Z}_4$, then $|Z(\mathbb{Z}_4)| = 1$, So $\Gamma_e(R)$ is a single vertex graph and thus $\omega(\Gamma_e(R) = 1$. Similarly, when $R \cong \frac{\mathbb{Z}_2[x]}{<x^2>}$, then $|Z(\frac{\mathbb{Z}_2[x]}{<x^2>})| = 1$, implies that $\omega(\Gamma_e(R) = 1$. Conversely, let R be a ring other than $\mathbb{Z}_4$ *or* $\frac{\mathbb{Z}_2[x]}{<x^2>}$. Then either $|Z^*(R)| = 0$ or $|Z^*(R)| \geq 2$. If $|Z^*(R)| = 0$, then $\omega(\Gamma_e(R) = 0$. For $|Z^*(R)| \geq 2$, clearly $\Gamma_e(R)$ contains K_2, implies that $\omega(\Gamma_e(R) \geq 2$.

The following result characterizes finite commutative rings whose extended zero-divisor graph has clique number 2.

Theorem 3. *For a finite commutative ring R, $\omega(\Gamma_e(R) = 2$ if and only if R is one of the following rings*

$$\mathbb{Z}_8, \quad \mathbb{Z}_9, \quad \frac{\mathbb{Z}_3[x]}{<x^2>}, \quad \frac{3\mathbb{Z}}{9\mathbb{Z}}, \quad \mathbb{Z}_6, \quad \mathbb{Z}_2 \times \mathbb{Z}_2, \quad \mathbb{Z}_2 \times \mathbb{Z}_3, \quad \mathbb{Z}_3 \times \mathbb{Z}_3.$$

Proof. As R is finite commutative, so R is an Artinian ring. Therefore, R can be decomposed as $R \cong R_1 \times R_2 \times \cdots \times R_n$, where each $R_i, 1 \leq i \leq n$, is a local ring. Consider the following cases.

Case 1. Let $n \geq 3$ and $|R_i| \geq 2$, for all $1 \leq i \leq n$.
Then any vertex of $\Gamma_e(R)$ is of the form $(x_1, x_2, \ldots, x_n)$, each $x_i \in R_i$, $1 \leq i \leq n$. Clearly $V_1 = \{(x_1, 0, 0, \ldots), (0, x_2, 0, \ldots), (0, 0, x_3, \ldots)\}$, is a vertex subset of $V(\Gamma_e(R))$, where each $x_i \neq 0, 1 \leq i \leq n$. The graph induced by V_1 is obviously K_3, as the vertex $(x_1, 0, 0, \ldots)$ is adjacent to vertices $(0, x_2, 0, \ldots), (0, 0, x_3, \ldots)$ and the vertex $(0, x_2, 0, \ldots)$ is adjacent to the vertex $(0, 0, x_3, \ldots)$. Therefore, $\omega(\Gamma_e(R)) = 3$.

Case 2. Let $n = 2$.
We have $R \cong R_1 \times R_2$. The following cases arise.

Subcase 2.1. Let $|R_1| \geq 4$ and $|R_2| \geq 4$. Then their exist elements, say $y_1, y_2 \in R_2$, such that $y_1 + y_2 < |R_2|$, because if $y_1, y_2 \in \mathbb{Z}_n$ and $y_1 + y_2 = n$, then the sum of two vertices of the type $(0, y_1)$ and $(0, y_2)$ is $(0, 0) \notin Z^*(R)$. Therefore, we choose a vertex subset say $V_2 = \{(0, y_1), (0, y_2), (x, 0)\}$, where $x \in R_1$, in which $(0, y_1) \sim (0, y_2)$, $(0, y_1) \sim (x, 0)$ and $(0, y_2) \sim (x, 0)$. Thus, the induced subgraph by V_2 is K_3. This implies that $\omega(\Gamma_e(R) \geq 3$.

Subcase 2.2. Now, let $|R_1| = 2$ and $|R_2| \geq 4$. Consider the vertex subset say $V_3 = \{(0, 1), (0, y), (1, 0)\}$, $y \in R_2$. As $(0, 1) \sim (0, y)$, $(0, 1) \sim (1, 0)$ and $(0, y) \sim (1, 0)$, so the induced subgraph by V_3 is K_3. This implies that $\omega(\Gamma_e(R)) \geq 3$.

Subcase 2.3. For $|R_1| = 3$ and $|R_2| \geq 4$, taking the similar vertex set as in Subcase 2.2, we have $\omega(\Gamma_e(R)) \geq 3$.

Subcase 2.4. If $|R_1| = 2$ and $|R_2| = 2$, then $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ and $\Gamma_e(R) \cong K_2$. This implies that $\omega(\Gamma_e(R)) = 2$.

Subcase2.5. For $|R_1| = 2$ and $|R_2| = 3$, clearly $R \cong \mathbb{Z}_2 \times \mathbb{Z}_3$ or $R \cong \mathbb{Z}_2 \times \frac{3\mathbb{Z}}{9\mathbb{Z}}$. If $R \cong \mathbb{Z}_2 \times \mathbb{Z}_3$, then $\Gamma_e(R) \cong P_2$, which implies that $\omega(\Gamma_e(R)) = 2$. For $R \cong \mathbb{Z}_2 \times \frac{3\mathbb{Z}}{9\mathbb{Z}}$, we choose a vertex subset $V_4 = \{(1, 0), (0, 3z), (0, 6z)\}$ such that $(1, 0) \times (0, 3z) = (0, 0)$, implies that $(1, 0) \sim (0, 3z)$. Similarly $(1, 0) \times (0, 6z) = (0, 0)$ and $(0, 3z) \times (0, 6z) = (0, 18z) = (0, 0)$, which implies that $(1, 0) \sim (0, 6z)$ and $(0, 3z) \sim (0, 6z)$. Thus, the induced subgraph by V_4 is K_3. This implies that $\omega(\Gamma_e(R)) \geq 3$.

Subcase 2.6. If $|R_1| = 3$ and $|R_2| = 3$, then the possibilities of R are $\mathbb{Z}_3 \times \mathbb{Z}_3$, $\mathbb{Z}_3 \times \frac{3\mathbb{Z}}{9\mathbb{Z}}$ and $\frac{3\mathbb{Z}}{9\mathbb{Z}} \times \frac{3\mathbb{Z}}{9\mathbb{Z}}$. In case $R \cong \mathbb{Z}_3 \times \mathbb{Z}_3$, then the vertex set of $\Gamma_e(R)$ say $V_5 = \{(0, 1), (0, 2), (1, 0)\}$ and the graph induced by V_5 is P_2, which implies that $\omega(\Gamma_e(R)) = 2$.

For $R \cong \mathbb{Z}_3 \times \frac{3\mathbb{Z}}{9\mathbb{Z}}$, we can choose a vertex subset say V_6=$\{(0, 3z)$,$(0, 6z)$,$(1, 0)$, $(2, 6z)\}$ and obviously the subgraph induced by V_6 is K_4. As K_3 is contained in K_4, therefore, we have $\omega(\Gamma_e(R)) \geq 3$.

Finally, if $R \cong \frac{3\mathbb{Z}}{9\mathbb{Z}} \times \frac{3\mathbb{Z}}{9\mathbb{Z}}$, then choose a vertex subset say $V_7 = \{(3z, 3z), (3z, 6z), (6z, 6z), (3z, 0)\}$. Clearly $(3z, 3z) \sim (3z, 6z), (6z, 6z), (3z, 0)$, as $(3z, 3z) \times (3z, 6z) = (0, 0)$, $(3z, 3z) \times (3z, 0) = (0, 0)$ and $(3z, 3z) \times (6z, 6z) = (0, 0)$. Also $(3z, 6z) \sim (6z, 6z), (3z, 0)$, as $(3z, 6z) \times (6z, 6z) = (0, 0)$ and $(3z, 6z) \times (3z, 0) = (0, 0)$ and $(6z, 6z) \times (3z, 0) = (0, 0)$, implies that $(6z, 6z) \sim (3z, 0)$. Thus, the induced subgraph by V_7 is K_4. As K_3 is contained in K_4, so $\omega(\Gamma_e(R)) \geq 3$.

Case 3. Let $n = 1$.
The following subcases arise.

Subcase 3.1. If R is an integral domain, then by observation 1, $\omega(\Gamma_e(R)) = 0$.

Subcase 3.2. Let R be not an integral domain and let $|R| \geq 10$. Then $|Z^*(R)| \geq 3$. We choose a vertex subset $V_8 = \{x_1, x_2, x_3\} \subseteq Z^*(R)$ in such a way that either $x_1 + x_2, x_1 + x_3, x_2 + x_3 \in Z^*(R)$ or $x_1x_2 = x_1x_3 = x_2x_3 = 0$. So in both the cases, the subgraph induced by V_8 is K_3, So $\omega(\Gamma_e(R)) \geq 3$.

Subcase 3.3. For $|R| \leq 9$, the following cases arise.

Subcase 3.3.1. Let $|R| = 2, 3, 4, 5, 6, 7$. If $|R| = 2$, then $\omega(\Gamma_e(R)) = 0$, when R is an integral domain or $\omega(\Gamma_e(R)) = 1$, if $R \cong \frac{2\mathbb{Z}}{4\mathbb{Z}}$. In a similar manner, if $|R| = 3$, then

$\omega(\Gamma_e(R)) = 0$ when R is an integral domain or $\omega(\Gamma_e(R)) = 2$, when $R \cong \frac{3\mathbb{Z}}{9\mathbb{Z}}$. If $|R| = 4$, then $R \cong \mathbb{Z}_4, \frac{\mathbb{Z}_2[x]}{(x^2)}, \frac{4\mathbb{Z}}{16\mathbb{Z}}$. By observation 1, $\omega(\Gamma_e(R)) = 1$, when $R \cong \mathbb{Z}_4$ *or* $\frac{\mathbb{Z}_2[x]}{(x^2)}$. For $R \cong \frac{4\mathbb{Z}}{16\mathbb{Z}}$, $\Gamma_e(R) \cong K_3$, so that $\omega(\Gamma_e(R)) = 3$. For $|R| = 5$, $R \cong \mathbb{Z}_5$ or $\frac{5\mathbb{Z}}{25\mathbb{Z}}$. Therefore, for $R \cong \mathbb{Z}_5$, $\omega(\Gamma_e(R)) = 0$ and for $R \cong \frac{5\mathbb{Z}}{25\mathbb{Z}}$, $\Gamma_e(R) \cong K_4$. Therefore, $\omega(\Gamma_e(R)) = 4$. Again, for $|R| = 6$, we have $R \cong \mathbb{Z}_6$ *or* $\frac{6\mathbb{Z}}{36\mathbb{Z}}$. So, for $R \cong \mathbb{Z}_6$, we have $\Gamma_e(R) \cong K_2$ and thus $\omega(\Gamma_e(R)) = 4$. For $R \cong \frac{6\mathbb{Z}}{36\mathbb{Z}}$, we have $\Gamma_e(R) \cong K_5$, so that $\omega(\Gamma_e(R)) = 5$. Finally, for $|R| = 7$, we have $R \cong \mathbb{Z}_7$ *or* $\frac{7\mathbb{Z}}{49\mathbb{Z}}$. Therefore, for $R \cong \mathbb{Z}_7$, we have $\omega(\Gamma_e(R)) = 0$ and for $R \cong \frac{7\mathbb{Z}}{49\mathbb{Z}}$, we have $\Gamma_e(R) \cong K_6$. Thus $\omega(\Gamma_e(R)) = 6$.

Subcase 3.3.2. If $|R| = 8$, then R is one of the following rings [13], $\mathbb{Z}_8$, $\mathbb{Z}_2 \times \mathbb{Z}_4$, $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$, $\frac{\mathbb{Z}_2[x]}{<x^3>}$, $\frac{\mathbb{Z}_4[x]}{<2x,x^2-2>}$, $\frac{\mathbb{Z}_2[x,y]}{<x,y>^2}$, $\frac{\mathbb{Z}_4[x]}{<2,x>^2}$, $\mathbb{Z}_2 \times \frac{\mathbb{Z}_2[x]}{<x^2>}$, $\frac{8\mathbb{Z}}{64\mathbb{Z}}$, $\mathbb{F}_8$. As seen in [13], the rings $\frac{\mathbb{Z}_2[x,y]}{<x,y>^2}$, $\frac{\mathbb{Z}_4[x]}{<2,x>^2}$, $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$, $\Gamma_e(R)$ contain K_3 as a subgraph and so $\omega(\Gamma_e(R)) \geq 3$. Now, for $R \cong \frac{\mathbb{Z}_2[x]}{<x^3>}$, we have $\Gamma(R) \cong K_{1,2}$. Further, the zero-divisors $x + x^2$ is adjacent to x in $\Gamma_e(R)$, which implies that $\Gamma_e(R) \cong K_3$ and thus $\omega(\Gamma_e(R)) = 3$. Similarly for $R \cong \frac{\mathbb{Z}_4[x]}{<2x,x^2-2>}$, $\Gamma_e(R) \cong K_3$ and thus $\omega(\Gamma_e(R) = 3$. For $R \cong \mathbb{Z}_2 \times \mathbb{Z}_4$ or $\mathbb{Z}_2 \times \frac{\mathbb{Z}_2[x]}{<x^2>}$, there exists a vertex subset $V_9 = \{(0,1),(0,x),(1,0)\}$ and the induced subgraph by V_9 is clearly $K_3 \subseteq \Gamma_e(R)$ and thus $\omega(\Gamma_e(R)) \geq 3$. Also, for $R \cong \frac{8\mathbb{Z}}{64\mathbb{Z}}$, we have $\Gamma_e(R) \cong K_7$, which implies that $\omega(\Gamma_e(R)) = 7$. For $R \cong \mathbb{F}_8$, we have $\omega(\Gamma_e(R)) = 0$. Finally the remaining ring is $\mathbb{Z}_8$, for which $\Gamma_e(R) \cong K_{1,2}$ and thus $\omega(\Gamma_e(R)) = 2$.

Subcase 3.3.3. If $|R| = 9$, then $R \cong \mathbb{Z}_9$, $\frac{\mathbb{Z}_3[x]}{<x^2>}$, $\mathbb{Z}_3 \times \mathbb{Z}_3$, $\frac{9\mathbb{Z}}{81\mathbb{Z}}$, $\mathbb{F}_9$. For $\mathbb{Z}_3 \times \mathbb{Z}_3$ and $\mathbb{F}_9$, the cases are discussed above. If $R \cong \mathbb{Z}_9$ *or* $\frac{\mathbb{Z}_3[x]}{<x^2>}$, then $|Z^*(R)| = 2$ and $\Gamma_e(R) \cong K_2$, so that $\omega(\Gamma_e(R)) = 2$. For $R \cong \frac{9\mathbb{Z}}{81\mathbb{Z}}$, we have $\Gamma_e(R) \cong K_8$, so that $\omega(\Gamma_e(R)) = 8$. □

3. Finite commutative rings whose extended zero-divisor graph has clique number 3,4 or 5.

We begin with the following theorem.

Theorem 4. *For $R \cong R_1 \times R_2$, where R_1 and R_2 are finite local rings, then $\omega(\Gamma_e(R) = 3$ if and only if R is one of the following rings*

$$\mathbb{Z}_2 \times \mathbb{Z}_4,\ \mathbb{Z}_2 \times \tfrac{2\mathbb{Z}}{4\mathbb{Z}},\ \mathbb{Z}_3 \times \mathbb{Z}_4,\ \tfrac{2\mathbb{Z}}{4\mathbb{Z}} \times \tfrac{2\mathbb{Z}}{4\mathbb{Z}},\ \mathbb{Z}_2 \times \tfrac{3\mathbb{Z}}{9\mathbb{Z}},\ \mathbb{Z}_3 \times \tfrac{2\mathbb{Z}}{4\mathbb{Z}},\ \mathbb{Z}_2 \times \mathbb{Z}_5,\ \tfrac{2\mathbb{Z}}{4\mathbb{Z}} \times \mathbb{Z}_5,\ \mathbb{Z}_3 \times \mathbb{Z}_5$$

Proof. As R is finite commutative, so R is an Artinian ring. Therefore, R can be decomposed as $R \cong R_1 \times R_2 \times \cdots \times R_n$, where $R_i, 1 \leq i \leq n$, is a local ring. Consider the following cases.

Case 1. Let $n \geq 3$ and $|R_i| \geq 2$, for all $1 \leq i \leq n$.
Then any vertex of $\Gamma_e(R)$ is of the form $(x_1, x_2, \ldots, x_n)$, where each $x_i \in R_i$, $1 \leq i \leq n$. Clearly $V_{10} = \{(x_1, 0, 0, \ldots), (0, x_2, 0, \ldots), (0, 0, x_3, \ldots), (0, 0, 0, x_4, 0, \ldots)\}$, is a vertex subset of $V(\Gamma_e(R))$, where each $x_i \neq 0$ and $1 \leq i \leq n$. The graph induced by V_{10} is obviously K_4, as the vertex $(x_1, 0, 0, \ldots)$ is adjacent to the vertices

$(0, x_2, 0, \dots), (0, 0, x_3, \dots)$ and $(0, 0, 0, x_4, \dots)$. Also the vertex $(0, x_2, 0, \dots)$ is adjacent to the vertices $(0, 0, x_3, \dots)$ and $(0, 0, 0, x_4, \dots)$ and $(0, 0, x_3, \dots)$ is adjacent to the vertex $(0, 0, 0, x_4, \dots)$. Therefore, $\omega(\Gamma_e(R)) \geq 4$, in this case.

Case 2. Let $n = 2$.
We have $R = R_1 \times R_2$. The following subcases arise.

Subcase 2.1. Let $|R_1| \geq 4$ and $|R_2| \geq 6$. Then their exists a vertex subset say $V_{11} = \{(x_1, 0), (x_2, 0), (0, y_1), (0, y_2)\}$, where $x_1, x_2 \in R_1$ and $y_1, y_2 \in R_2$. If $R_1 \cong \mathbb{Z}_n,\ n \geq 4$, choose $x_1, x_2 \in R_1$ such that $x_1 + x_2 \neq n$. If $R_2 \cong \mathbb{Z}_n,\ n \geq 4$ choose $y_1, y_2 \in R_2$ such that $y_1 + y_2 \neq n$. Therefore, the induced subgraph by V_{11} is K_4. This implies that $\omega(\Gamma_e(R)) \geq 4$.

Subcase 2.2. Let $|R_1| \leq 3$ and $|R_2| \geq 6$. Then we can choose a vetex subset of $\Gamma_e(R)$ as $V_{12} = \{(0, y_1), (0, y_2), (0, y_3), (x, 0)\}$, where $x \in R_1$ and $y_1, y_2, y_3 \in R_2$. Thus, the induced subgraph by V_{12} is K_4. This implies that $\omega(\Gamma_e(R)) \geq 4$.

Case 3. Let $|R_1| \leq 3$ and $|R_2| \leq 5$.
The following cases arise.

Subcase 3.1. Let $|R_1| = 2$ and $|R_2| = 5$. First let $|R_1| = 2$ and $|R_2| = 5$. So we have $R_1 \cong \mathbb{Z}_2, \frac{2\mathbb{Z}}{4\mathbb{Z}}$ and $R_2 \cong \mathbb{Z}_5$ or $\frac{5\mathbb{Z}}{25\mathbb{Z}}$. Now, if $R_1 \cong \mathbb{Z}_2$ and $R_2 \cong \mathbb{Z}_5$, then the graph shown in Figure 3(iii), contains K_3 as the maximal complete subgraph and so $\omega(\Gamma_e(R)) = 3$. Similarly, if $R_1 \cong \frac{2\mathbb{Z}}{4\mathbb{Z}}$ and $R_2 \cong \mathbb{Z}_5$, then $\omega(\Gamma_e(R)) = 3$. Again, for $R_1 \cong \mathbb{Z}_2$ and $R_2 \cong \frac{5\mathbb{Z}}{25\mathbb{Z}}$, we can construct a vertex subset, say, $V_{13} = \{(0, 5z), (0, 10z), (0, 15z), (0, 20z)\}$ and the induced subgraph by V_{13} is K_4. So, $\omega(\Gamma_e(R)) \geq 4$. Similarly, if $R \cong \frac{2\mathbb{Z}}{4\mathbb{Z}} \times \frac{5\mathbb{Z}}{25\mathbb{Z}}$, we have $\omega(\Gamma_e(R)) \geq 4$.
Now, let $|R_1| = 3$ and $|R_2| = 5$. We have $R_1 \cong \mathbb{Z}_3$ *or* $\frac{3\mathbb{Z}}{9\mathbb{Z}}$ and $R_2 \cong \mathbb{Z}_5$ or $\frac{5\mathbb{Z}}{25\mathbb{Z}}$. For the ring $\mathbb{Z}_3 \times \mathbb{Z}_5$, we have $\omega(\Gamma_e(R)) = 3$ as shown in the graph of Figure 2(iv). The graph $\Gamma_e(R)$ associated to rings $\mathbb{Z}_3 \times \frac{5\mathbb{Z}}{25\mathbb{Z}}$, $\frac{3\mathbb{Z}}{9\mathbb{Z}} \times \mathbb{Z}_5$ and $\frac{3\mathbb{Z}}{9\mathbb{Z}} \times \frac{5\mathbb{Z}}{25\mathbb{Z}}$ contains K_4 as a subgraph on the vertex subsets $\{(0, 5z), (0, 10z), (0, 15z), (0, 20z)\}$, $\{(0, 1), (0, 2), (3z, 1), (3z, 2)\}$ and $\{(0, 5z), (0, 10z), (0, 15z), (0, 20z)\}$, respectively. So $\omega(\Gamma_e(R)) \geq 4$. **Subcase 3.2.** Let $|R_1| = 2$ *or* 3 and $|R_2| = 4$. Then, we have

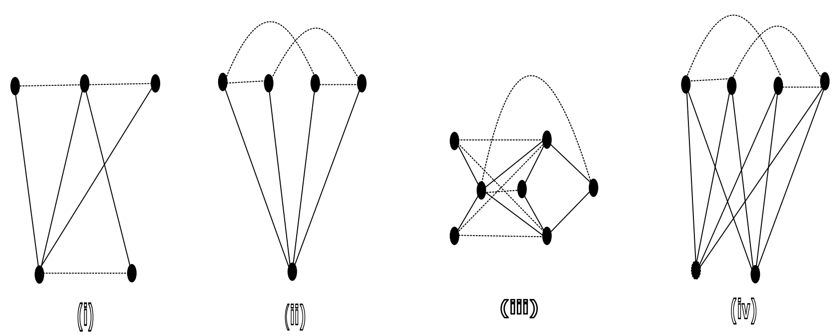

Figure 2.

$R \cong \mathbb{Z}_2 \times \mathbb{Z}_4, \mathbb{Z}_2 \times \frac{\mathbb{Z}_2[x]}{(x^2)}, \mathbb{Z}_2 \times \mathbb{F}_4, \mathbb{Z}_2 \times \frac{4\mathbb{Z}}{16\mathbb{Z}}$ and $\mathbb{Z}_3 \times \mathbb{Z}_4, \mathbb{Z}_3 \times \frac{\mathbb{Z}_2[x]}{(x^2)}, \mathbb{Z}_3 \times \mathbb{F}_4, \mathbb{Z}_3 \times \frac{4\mathbb{Z}}{16\mathbb{Z}}$.

Now for the rings $\mathbb{Z}_2 \times \frac{\mathbb{Z}_2[x]}{(x^2)}, \mathbb{Z}_2 \times \mathbb{F}_4, \mathbb{Z}_2 \times \frac{4\mathbb{Z}}{16\mathbb{Z}}$, $\mathbb{Z}_3 \times \frac{\mathbb{Z}_2[x]}{(x^2)}, \mathbb{Z}_3 \times \mathbb{F}_4, \mathbb{Z}_3 \times \frac{4\mathbb{Z}}{16\mathbb{Z}}$, $\Gamma_e(R)$ contains K_4 as a subgraph. So $\omega(\Gamma_e(R)) \geq 4$. The graphs of $\Gamma_e(R)$ corresponding to $R = \mathbb{Z}_2 \times \mathbb{Z}_4$ and $\mathbb{Z}_3 \times \mathbb{Z}_4$ are respectively shown in Figure 2(i) and 2(ii) . Clearly K_3 is a maximal complete subgraph and thus $\omega(\Gamma_e(R)) = 3$.

Subcase 3.3. Let $|R_1| = 2$ *or* 3 and $|R_2| = 2$ *or* 3. Then R is one of the following rings $\mathbb{Z}_2 \times \mathbb{Z}_2$, $\mathbb{Z}_2 \times \mathbb{Z}_3$, $\mathbb{Z}_2 \times \frac{2\mathbb{Z}}{4\mathbb{Z}}$, $\mathbb{Z}_2 \times \frac{3\mathbb{Z}}{9\mathbb{Z}}$, $\frac{2\mathbb{Z}}{4\mathbb{Z}} \times \frac{2\mathbb{Z}}{4\mathbb{Z}}$, $\mathbb{Z}_3 \times \frac{2\mathbb{Z}}{4\mathbb{Z}}$, $\frac{2\mathbb{Z}}{4\mathbb{Z}} \times \frac{3\mathbb{Z}}{9\mathbb{Z}}$, $\frac{3\mathbb{Z}}{9\mathbb{Z}} \times \frac{3\mathbb{Z}}{9\mathbb{Z}}$. For $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ *or* $\mathbb{Z}_2 \times \mathbb{Z}_3$ *or* $\mathbb{Z}_3 \times \mathbb{Z}_3$, we have $\omega(\Gamma_e(R)) = 2$, as proved in Theorem 3. Now, when $R \cong \mathbb{Z}_2 \times \frac{2\mathbb{Z}}{4\mathbb{Z}}$ *or* $\mathbb{Z}_2 \times \frac{3\mathbb{Z}}{9\mathbb{Z}}$ *or* $\frac{2\mathbb{Z}}{4\mathbb{Z}} \times \frac{2\mathbb{Z}}{4\mathbb{Z}}$ *or* $\mathbb{Z}_3 \times \frac{2\mathbb{Z}}{4\mathbb{Z}}$, the vertex set of $V(\Gamma_e(R))$ becomes $\{(0, 2z), (1, 0), (1, 2z)\}$, $\{(0, 3z), (0, 6z), (1, 0), (1, 3z), (1, 6z)\}$, $\{(0, 2z), (2z, 0), (2z, 2z)\}$, $\{(1, 0), (1, 2z), (2, 0), (2, 2z), (0, 2z)\}$ respectively. Thus each of the graphs $\Gamma_e(\mathbb{Z}_2 \times \frac{2\mathbb{Z}}{4\mathbb{Z}})$, $\Gamma_e(\frac{2\mathbb{Z}}{4\mathbb{Z}} \times \frac{2\mathbb{Z}}{4\mathbb{Z}}) \cong K_3$, $\Gamma_e(\mathbb{Z}_2 \times \frac{3\mathbb{Z}}{9\mathbb{Z}})$ and $\Gamma_e(\mathbb{Z}_3 \times \frac{2\mathbb{Z}}{4\mathbb{Z}})$ contains K_3 as a maximal complete subgraph. So $\omega(\Gamma_e(R)) = 3$. The remaining rings are discussed in Theorem 3, in which $\omega(\Gamma_e(R)) \geq 4$. □

Corollary 1. *If R is isomorphic to one of the following rings*

$$\frac{4\mathbb{Z}}{16\mathbb{Z}}, \frac{\mathbb{Z}_2[x]}{< x^3 >}, \frac{\mathbb{Z}_4[x]}{< 2x, x^2 - 2 >}, \frac{\mathbb{Z}_2[x, y]}{(x, y)^2}, \frac{\mathbb{Z}_4[x]}{(2, x)^2}, \frac{\mathbb{F}_4[x]}{< x^2 >}, \frac{\mathbb{Z}_4[x]}{(x^2 + x + 1)}$$

then $\omega(\Gamma_e(R)) = 3$.

Proof. As seen in [13], for the rings $R \cong \frac{\mathbb{Z}_2[x]}{<x^3>}, \frac{\mathbb{Z}_4[x]}{<2x,x^2-2>}, \frac{\mathbb{Z}_2[x,y]}{(x,y)^2}, \frac{\mathbb{Z}_4[x]}{(2,x)^2}, \frac{\mathbb{F}_4[x]}{<x^2>}$, $\frac{\mathbb{Z}_4[x]}{(x^2+x+1)}$, we have $|Z^*(R)| = 3$ and their zero-divisor graphs are either $K_{1,2}$ *or* K_3. Clearly, $\Gamma(\frac{\mathbb{Z}_2[x]}{<x^3>})$ and $\Gamma(\frac{\mathbb{Z}_4[x]}{<2x,x^2-2>})$ are isomorphic to $K_{1,2}$. If $R \cong \frac{\mathbb{Z}_2[x]}{<x^3>}$, we have $Z^*(R) = \{x, x^2, x + x^2\}$ and in $\Gamma(R)$ adjacency relations exists as $x \sim x^2 \sim x + x^2$. But $x + x + x^2 = 2x + x^2 = x^2 \in Z^*(R)$. So $\Gamma_e(\frac{\mathbb{Z}_2[x]}{<x^3>})) = K_3$, which implies that $\omega(\Gamma_e(R) = 3$. Similarly, $\Gamma_e(\frac{\mathbb{Z}_4[x]}{<2x,x^2-2>})) = K_3$, which implies that $\omega(\Gamma_e(R) = 3$. Further, for $R \cong \frac{\mathbb{Z}_2[x,y]}{(x,y)^2}$ *or* $\frac{\mathbb{Z}_4[x]}{(2,x)^2}$ *or* $\frac{\mathbb{F}_4[x]}{<x^2>}$ *or* $\frac{\mathbb{Z}_4[x]}{(x^2+x+1)}$, clearly $\Gamma(R) = K_3$ and $|Z^*(R)| = 3$, implies that $\Gamma_e(R) \cong K_3$, so that $\omega(\Gamma_e(R) = 3$. □

Corollary 2. *Let R be a Boolian ring of order n, i.e., $R = \prod_{i=1}^{n} \mathbb{Z}_2$. Then $\omega(\Gamma_e(R)) = |Z^*(R)|$.*

Proof. As every vertex is of the form $(x_1, x_2, x_3, \ldots, x_n)$, where $x_i \in \mathbb{Z}_2$, so in every n-tuple at least one x_i is 0. Then every vertex is adjacent to every other vertex. This implies that $\Gamma_e(R)$ is a complete graph. As $|V(\Gamma_e(R))| = |Z^*(R)|$, it follows that $\omega(\Gamma_e(R)) = |Z^*(R)|$. □

Now, we have the following results.

Theorem 5. *For $R \cong R_1 \times R_2$, where R_1 and R_2 are finite local rings, $\omega(\Gamma_e(R)) = 4$ if and only if R is one of the following rings*

$$\mathbb{Z}_2 \times \frac{\mathbb{Z}_2[x]}{< x^2 >},\ \mathbb{Z}_2 \times \mathbb{F}_4\ , \mathbb{Z}_2 \times \mathbb{Z}_7,\ \mathbb{Z}_3 \times \frac{\mathbb{Z}_2[x]}{< x^2 >},\ \mathbb{Z}_3 \times \mathbb{F}_4\ , \mathbb{Z}_3 \times \mathbb{Z}_7,\ \mathbb{Z}_4 \times \mathbb{Z}_5,\ \frac{\mathbb{Z}_2[x]}{< x^2 >} \times \mathbb{Z}_5,\ \mathbb{Z}_5 \times \mathbb{Z}_5$$

Proof. As R is finite commutative, so R is an Artinian ring and therefore, R can be decomposed as $R \cong R_1 \times R_2 \times \cdots \times R_n$, where $R_i, 1 \le i \le n$, is a local ring. Consider the following cases.

Case 1. Let $n \ge 3$ and $|R_i| \ge 2$, for all $1 \le i \le n$.
Then any vertex of $\Gamma_e(R)$ is of the form $(x_1, x_2, \ldots, x_n)$, where each $x_i \in R_i$, $1 \le i \le n$. Clearly $V_{14} = \{(x_1, 0, 0, \ldots),\ (0, x_2, 0, \ldots),\ (0, 0, x_3, \ldots),\ (0, 0, 0, x_4, 0, \ldots),\ (0, 0, 0, 0, x_5, \ldots)\}$, is a vertex subset of $V(\Gamma_e(R))$, where each $x_i \neq 0$ and $1 \le i \le n$. The graph induced by V_{14} is obviously K_5, as the vertex $(x_1, 0, 0, \ldots)$ is adjacent to the vertices $(0, x_2, 0, \ldots), (0, 0, x_3, \ldots)$, $(0, 0, 0, x_4, \ldots)$ and $(0, 0, 0, 0, x_5, \ldots)$. Also the vertex $(0, x_2, 0, \ldots)$ is adjacent to the vertices $(0, 0, x_3, \ldots)$, $(0, 0, 0, x_4, \ldots)$, $(0, 0, x_3, \ldots)$ and $(0, 0, 0, 0, x_5, \ldots)$ is adjacent to the vertex $(0, 0, 0, x_4, \ldots)$ and so on. Therefore, $\omega(\Gamma_e(R)) \ge 5$, in this case.

Case 2. Let $n = 2$.
We have $R = R_1 \times R_2$. The following subcases arise.

Subcase 2.1. Let $|R_1| \ge 4$ and $|R_2| \ge 8$. Then their exists a vertex subset say $V_{15} = \{(x_1, 0), (x_2, 0), (0, y_1), (0, y_2), (x, y)\}$, where $x, x_1, x_2 \in R_1$ and $y, y_1, y_2 \in R_2$. If $R_1 \cong \mathbb{Z}_n,\ n \ge 4$, choose $x_1, x_2 \in R_1$ such that $x_1 + x_2 \neq n$. If $R_2 \cong \mathbb{Z}_n,\ n \ge 4$, choose $y_1, y_2 \in R_2$ such that $y_1 + y_2 \neq n$. Therefore, the induced subgraph by V_{15} is K_5. This implies that $\omega(\Gamma_e(R)) \ge 5$.

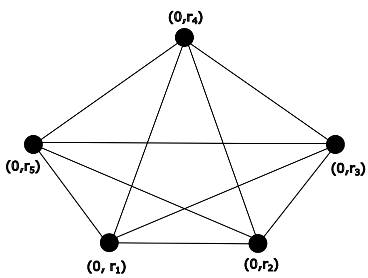

Figure 3. $\Gamma_e(R_1 \times R_2)$

Subcase 2.2. Let $|R_1| \le 3$ and $|R_2| \ge 8$. Then we can choose a vertex subset of $\Gamma_e(R)$ as $V_{16} = \{(0, y_1), (0, y_2), (0, y_3), (x_1, 0), (x_2, 0)\}$, where $x_1, x_2 \in R_1$ and $y_1, y_2, y_3 \in R_2$. Thus, the induced subgraph by V_{16} is K_4. This implies that $\omega(\Gamma_e(R)) \ge 5$.

Case 3. Let $|R_1| \le 3$ and $|R_2| \le 5$.
We have the following subcases.

Subcase 3.1. Let $|R_1| = 2$ and $|R_2| = 5$. First let $|R_1| = 2$ and $|R_2| = 5$. So we have $R_1 \cong \mathbb{Z}_2, \frac{2\mathbb{Z}}{4\mathbb{Z}}$ and $R_2 \cong \mathbb{Z}_5$ or $\frac{5\mathbb{Z}}{25\mathbb{Z}}$. Now, if $R_1 \cong \mathbb{Z}_2$ and $R_2 \cong \mathbb{Z}_5$, then the graph contains K_3 as the maximal complete subgraph and so $\omega(\Gamma_e(R)) = 3$. Similarly, if $R_1 \cong \frac{2\mathbb{Z}}{4\mathbb{Z}}$ and $R_2 \cong \mathbb{Z}_5$, then $\omega(\Gamma_e(R)) = 3$. Again, for $R_1 \cong \mathbb{Z}_2$ and $R_2 \cong \frac{5\mathbb{Z}}{25\mathbb{Z}}$, we can construct a vertex subset, say, $V_{17} = \{(0, 5z), (0, 10z), (0, 15z), (0, 20z)\}$ and the induced subgraph by V_{17} is K_4. So, $\omega(\Gamma_e(R)) = 4$. Similarly, if $R \cong \frac{2\mathbb{Z}}{4\mathbb{Z}} \times \frac{5\mathbb{Z}}{25\mathbb{Z}}$, we have $\omega(\Gamma_e(R)) = 4$.

Subcase 3.2. Now, let $|R_1| = 3$ and $|R_2| = 7$. We have $R_1 \cong \mathbb{Z}_3$ *or* $\frac{3\mathbb{Z}}{9\mathbb{Z}}$ and $R_2 \cong \mathbb{Z}_7$ or $\frac{7\mathbb{Z}}{49\mathbb{Z}}$. For the ring $\mathbb{Z}_3 \times \mathbb{Z}_7$, we have $\omega(\Gamma_e(R)) = 3$. The graph $\Gamma_e(R)$ associated to rings $\mathbb{Z}_3 \times \frac{7\mathbb{Z}}{49\mathbb{Z}}$, $\frac{3\mathbb{Z}}{9\mathbb{Z}} \times \mathbb{Z}_7$ and $\frac{3\mathbb{Z}}{9\mathbb{Z}} \times \frac{7\mathbb{Z}}{49\mathbb{Z}}$ contains K_5 as a subgraph on vertex subsets $\{(0, 7z), (0, 14z), (0, 21z), (0, 28z), (0, 35z)\}$, $\{(0, 1), (0, 2), (0, 3), (3z, 1), (3z, 2)\}$ and $\{(0, 7z), (0, 14z), (0, 21z), (0, 28z)\}$, respectively. So $\omega(\Gamma_e(R)) \geq 5$.

Subcase 3.3. Now we will investigate the rings for which $\Gamma_e(R)$ is 4. When $R \cong \mathbb{Z}_2 \times \frac{\mathbb{Z}_2[x]}{<x^2>}$, the vertex subset $\{(1, 0), (0, 1), (0, 1 + x), (0, x)\}$ forms a complete graph K_4 and hence $\Gamma_e(R)$ is 4. In a similar fashion, the graphs for the following rings $\mathbb{Z}_2 \times \mathbb{F}_4$, $\mathbb{Z}_2 \times \mathbb{Z}_7$, $\mathbb{Z}_3 \times \frac{\mathbb{Z}_2[x]}{<x^2>}$, $\mathbb{Z}_3 \times \mathbb{F}_4$, $\mathbb{Z}_3 \times \mathbb{Z}_7$, $\mathbb{Z}_4 \times \mathbb{Z}_5$, $\frac{\mathbb{Z}_2[x]}{<x^2>} \times \mathbb{Z}_5$, $\mathbb{Z}_5 \times \mathbb{Z}_5$ contains K_4 as maximal complete subgraph and hence $\Gamma_e(R)$ is 4. □

Theorem 6. *Let $R \cong R_1 \times R_2$, where R_1 and R_2 are finite local rings. Then $\omega(\Gamma_e(R) = 5$ if and only if R is one of the following rings*

$$\mathbb{Z}_2 \times \mathbb{Z}_8,\ \frac{2\mathbb{Z}}{4\mathbb{Z}} \times \frac{3\mathbb{Z}}{9\mathbb{Z}},\ \mathbb{Z}_2 \times \mathbb{Z}_9,\ \mathbb{Z}_2 \times \frac{\mathbb{Z}_3[x]}{< x^2 >},\ \mathbb{Z}_3 \times \mathbb{Z}_8,\ \mathbb{Z}_3 \times \mathbb{Z}_9,\ \mathbb{Z}_3 \times \frac{\mathbb{Z}_3[x]}{< x^2 >},\ \mathbb{Z}_4 \times \mathbb{Z}_4,$$
$$\mathbb{Z}_4 \times \mathbb{F}_4,\ \mathbb{F}_4 \times \mathbb{Z}_5,\ \mathbb{Z}_4 \times \mathbb{Z}_7,\ \frac{\mathbb{Z}_2[x]}{< x^2 >} \times \mathbb{Z}_7,\ \mathbb{Z}_5 \times \mathbb{Z}_7$$

Proof. As R is finite commutative, so R is an Artinian ring. Therefore, R can be decomposed as $R \cong R_1 \times R_2 \times \cdots \times R_n$, where $R_i, 1 \leq i \leq n$, is a local ring. In Theorems 2.3, 3.1 and 3.4. we have already proved that the rings of order less than the order of followings rings $\mathbb{Z}_2 \times \mathbb{Z}_8$, $\frac{2\mathbb{Z}}{4\mathbb{Z}} \times \frac{3\mathbb{Z}}{9\mathbb{Z}}$, $\mathbb{Z}_2 \times \mathbb{Z}_9$, $\mathbb{Z}_2 \times \frac{\mathbb{Z}_3[x]}{<x^2>}$, $\mathbb{Z}_3 \times \mathbb{Z}_8$, $\mathbb{Z}_3 \times \mathbb{Z}_9$, $\mathbb{Z}_3 \times \frac{\mathbb{Z}_3[x]}{<x^2>}$, $\mathbb{Z}_4 \times \mathbb{Z}_4$, $\mathbb{Z}_4 \times \mathbb{F}_4$, $\mathbb{F}_4 \times \mathbb{Z}_5$, $\mathbb{Z}_4 \times \mathbb{Z}_7$, $\frac{\mathbb{Z}_2[x]}{<x^2>} \times \mathbb{Z}_7$, $\mathbb{Z}_5 \times \mathbb{Z}_7$ has either clique 2, 3 or 4. Whereas, the rings of order greater than the order of above rings have clique either 6 or greater than 6 as proved in Theorems 2.3, 3.1 and 3.4. Now $\omega(\Gamma_e(R))$ corresponding to the rings $\mathbb{Z}_2 \times \mathbb{Z}_8$, $\frac{2\mathbb{Z}}{4\mathbb{Z}} \times \frac{3\mathbb{Z}}{9\mathbb{Z}}$, $\mathbb{Z}_2 \times \mathbb{Z}_9$, $\mathbb{Z}_2 \times \frac{\mathbb{Z}_3[x]}{<x^2>}$, $\mathbb{Z}_3 \times \mathbb{Z}_8$, $\mathbb{Z}_3 \times \mathbb{Z}_9$, $\mathbb{Z}_3 \times \frac{\mathbb{Z}_3[x]}{<x^2>}$, $\mathbb{Z}_4 \times \mathbb{Z}_4, \mathbb{Z}_4 \times \mathbb{F}_4$, $\mathbb{F}_4 \times \mathbb{Z}_5$, $\mathbb{Z}_4 \times \mathbb{Z}_7$, $\frac{\mathbb{Z}_2[x]}{<x^2>} \times \mathbb{Z}_7$, $\mathbb{Z}_5 \times \mathbb{Z}_7$ contains K_5 as maximal complete subgraph and hence $\omega(\Gamma_e(R)) = 5$. □

Remark 1. There does not exist any ring R that can be expressed as a product of three local rings for which $\omega(\Gamma_e(R) = 3$.

Remark 2. For any ring R, such that $R \cong R_1 \times R_2 \times R_3$, where R_i are local rings for $i = 1, 2, 3$, we have $\omega(ZT(R)) \geq 6$. So it is not possible to find any ring R, for which

$R \cong R_1 \times R_2 \times R_3$ and $\omega(\Gamma_e(R)) \leq 5$, as the smallest possibility for R_1, R_2 *and* R_3 is to be $\mathbb{Z}_2$ that is, when $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ and $\omega(\Gamma_e(R)) = 6 = |Z^*(R)|$, by Corollary 2.

Lemma 1. *If $R \cong R_1 \times R_2 \times R_3$, where each R_i, $i = 1, 2, 3$ are finite local rings, then $\omega(\Gamma_e(R) \geq 6$.*

Proof. Let the vertex subset of $\Gamma_e(R)$ be $V_{14} = \{(v_1, 0, 0), (0, v_2, 0), (0, 0, v_3), (0, v_2, v_3), (v_1, v_2, 0), (v_1, 0, v_3)\}$, then it is easy to see that either $x + y \in Z^*(R)$ or $x.y = 0$. It follows that V_{14} forms K_6 and $K_6 \subseteq \Gamma_e(R)$. Thus $\omega(\Gamma_e(R) \geq 6$. □

Now, we give necessary and sufficient conditions for $R \cong R_1 \times R_2 \times R_3$, where each R_i, $i = 1, 2, 3$, are local rings, to have clique number exactly equal to 6.

Theorem 7. *Let $R \cong R_1 \times R_2 \times R_3$, where each R_i, $i = 1, 2, 3$, are finite local rings. Then $\omega(\Gamma_e(R) = 6$ if and only if R is isomorphic to one of the following rings $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ or $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3$.*

Proof. For the rings, other than the two given rings in the statement, first we prove that $\omega(\Gamma_e(R) \geq 7$. For this, let $|R_i| \geq 3$, where $i = 1, 2, 3$. So we need to find a vertex subset based on at least seven vertices for which it contains $K_7 \subseteq \Gamma_e(R)$. We choose the vertex subset as $\{(1, 0, 0), (0, 1, 0), (0, 0, 1), (1, r_2, 0), (1, 1, 0), (r_1, 1, 0), (r_1, r_2, 0)\}$, where $r_1 \in R_1$ *and* $r_2 \in R_2$. Then its graph forms K_7, as shown in Figure 3. So, if $|R_i| \geq 3$, where $i = 1, 2, 3$, then $\omega(\Gamma_e(R) \geq 7$. The next pos-

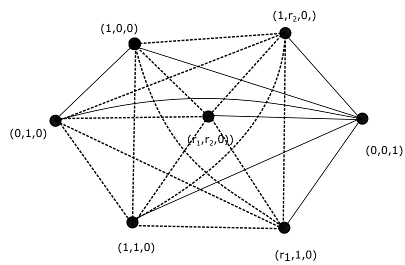

Figure 4. $\Gamma_e(R_1 \times R_2 \times R_3)$

sibility is when $|R_1| = 2$ and both $|R_2|$ and $|R_3| \geq 3$. Then choose a vertex subset as $\{(1, 0, 0), (0, 1, 0), (0, 0, 1), (0, 1, 1), (0, r_2, 1), (0, r_2, r_3), (0, 1, r_3)\}$, where $r_2 \in R_2$ *and* $r_3 \in R_3$. This forms a complete graph as shown in Figure 4. So, if $|R_1| = 2$ and both $|R_2|$ *and* $|R_3| \geq 3$, then $\omega(\Gamma_e(R) \geq 7$. Thus the remaining cases are when $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ or $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3$. If $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$, then by Corollary 2, $\omega(\Gamma_e(R) = |Z^*(R)| = 6$. For the ring $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3$, its associated graph $\Gamma_e(R)$ is based on nine vertices, shown in Figure 5, which contains K_6 as the maximal complete subgraph on a vertex subset $\{(1, 0, 0), (0, 1, 0), (1, 0, 1), (1, 0, 2), (0, 0, 1), (1, 1, 0)\}$. So $\omega(\Gamma_e(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3)) = 6$ □

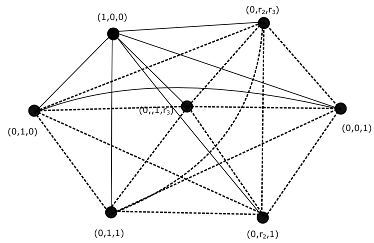

Figure 5. $\Gamma_e(R_1 \times R_2 \times R_3)$

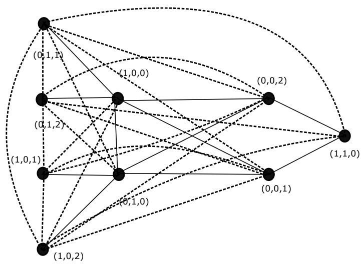

Figure 6. $\Gamma_e(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3)$

Conclusion. In this paper, we considered cliques of order up to 6 and determined the rings associated to them. Although it would be quite difficult and challenging to consider the cliques of higher orders and to characterize the finite commutative rings associated to them in the extended zero-divisor graph $\Gamma_e(R)$. But it would be theoretically interesting. Also, many others graph theoretic parameters like girth, diameter, independence number, chromatic number etc can be considered for this extended zero-divisor graph $\Gamma_e(R)$.

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Conflict of interest. The authors declare that they have no conflict of interest.
Data Availability. Data sharing is not applicable to this article as no datasets were generated or analyzed during the current study.

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