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Research Article

NP-completeness of some generalized hop and step domination parameters in graphs

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Abstract: Let $r \geq 2$. A subset S of vertices of a graph G is a r-hop independent dominating set if every vertex outside S is at distance r from a vertex of S, and for any pair $v, w \in S$, $d(v, w) \neq r$. A r-hop Roman dominating function (rHRDF) is a function f on $V(G)$ with values $0, 1$ and 2 having the property that for every vertex $v \in V$ with $f(v) = 0$ there is a vertex u with $f(u) = 2$ and $d(u, v) = r$. A r-step Roman dominating function (rSRDF) is a function f on $V(G)$ with values $0, 1$ and 2 having the property that for every vertex v with $f(v) = 0$ or 2, there is a vertex u with $f(u) = 2$ and $d(u, v) = r$. A rHRDF f is a r-hop Roman independent dominating function if for any pair v, w with non-zero labels under f, $d(v, w) \neq r$. We show that the decision problem associated with each of r-hop independent domination, r-hop Roman domination, r-hop Roman independent domination and r-step Roman domination is NP-complete even when restricted to planar bipartite graphs or planar chordal graphs.

Keywords: Dominating set, Hop dominating set, Step dominating set, Hop Independent set, Hop Roman dominating function, Hop Roman independent dominating function, Complexity.

AMS Subject classification: 05C69

1. Introduction

For a graph $G = (V, E)$ with vertex set $V = V(G)$ and edge set $E = E(G)$, the *order* of G is $n(G) = n_G = |V(G)|$ and the *size* of G is $m(G) = m_G = |E(G)|$. The *open neighborhood* of a vertex v is $N_G(v) = \{u \in V(G) \mid uv \in E(G)\}$. The *degree*

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of v, denoted by $\deg(v)$, is $|N_G(v)|$, and the *open neighborhood* of a subset $S \subseteq V$, is $N_G(S) = \bigcup_{v \in S} N_G(v)$. The *distance* between two vertices u and v in G, denoted by $d(u, v)$, is the minimum length of a (u, v)-path in G. A *bipartite graph* is a graph whose vertices can *chordal graph* is a graph that does not contain an induced cycle of length greater than 3. A *planar graph* is a graph which can be drawn in the plane without any edges crossing. A *vertex cover* of a graph is a set of vertices such that each edge of the graph is incident with at least one vertex of the set. A subset S of vertices of a graph G is a *dominating set* of G if every vertex in $V(G) - S$ has a neighbor in S. For notation and graph theory terminology not given here, we refer to [12].

Chartrand, Harary, Hossain, and Schultz [5] introduced the concept of r-step domination in graphs. For an integer $r \geq 1$, two vertices in a graph G are said to *r-step dominate* each other if they are at distance exactly r apart in G. A set S of vertices in G is a *r-step dominating set* of G if every vertex in $V(G)$ is r-step dominated by some vertex of S. The *r-step domination number*, $\gamma_{rstep}(G)$ of G, is the minimum cardinality of a r-step dominating set of G. The concept of r-step was further studied, for example in [4, 11, 14, 25]. Ayyaswamy et al. [3, 20] introduced the a similar concept, namely, hop domination in graphs. A subset S of vertices of a graph G is a *hop dominating set* (HDS) if every vertex outside S is at distance two from a vertex of S. The *hop domination number*, $\gamma_h(G)$ of G, is the minimum cardinality of an HDS of G. A subset S of vertices of a graph G is a *hop independent dominating set* (HIDS) if S is a HDS and for any pair $v, w \in S$, $d(v, w) \neq 2$. The *hop independent domination number* of G is the minimum cardinality of an HIDS of G. The concept of hop domination was further studied, for example, in [2, 13, 17]. A generalized version of hop domination, namely r-hop domination, (for any $r \geq 2$) is studied in [17]. For $r \geq 2$, a subset S of vertices of G is a *r-hop dominating set* (rHDS) if every vertex outside S is at distance r from a vertex of S. The *r-hop domination number* of G, is the minimum cardinality of a rHDS of G. For a subset $S \subseteq V(G)$ and a vertex $v \in V(G)$, we say that v is r-hop dominated by S (or S r-hop dominates v) if either $v \in S$ or $v \notin S$ and $d(u, v) = r$ for some vertex $u \in S$. Likewise, a subset S of vertices of G is a *r-hop independent dominating set* (rHIDS) if every vertex outside S is at distance r from a vertex of S, and for any pair $v, w \in S$, $d(v, w) \neq r$.

A function $f : V \longrightarrow \{0, 1, 2\}$ having the property that for every vertex $v \in V$ with $f(v) = 0$, there exists a vertex $u \in N(v)$ with $f(u) = 2$, is called a *Roman dominating function* or just an RDF. The *weight* of an RDF f is the sum $f(V) = \sum_{v \in V} f(v)$. The minimum weight of an RDF on G is called the *Roman domination number* of G and is denoted by $\gamma_R(G)$. For an RDF f in a graph G, we denote by V_i (or V_i^f to refer to f) the set of all vertices of G with label i under f. Thus an RDF f can be represented by a triple (V_0, V_1, V_2), and we can use the notation $f = (V_0, V_1, V_2)$. The mathematical concept of Roman domination, was defined and discussed by Stewart [24], and ReVelle and Rosing [21], and was subsequently developed by Cockayne et al. [10]. Many variations, generalizations and applications of Roman dominations parameters have been studied, and to see the latest progress until 2020 see [6–9].

Shabani at al. [23] introduced the concept of hop Roman dominating functions. A *hop Roman dominating function* (HRDF) is a function $f : V \longrightarrow \{0,1,2\}$ having the property that for every vertex $v \in V$ with $f(v) = 0$ there is a vertex u with $f(u) = 2$ and $d(u,v) = 2$. The weight of an HRDF f is the sum $f(V) = \sum_{v \in V} f(v)$. The minimum weight of an HRDF on G is called the *hop Roman domination number* of G and is denoted $\gamma_{hR}(G)$. For an HRDF f in a graph G, we denote by V_i (or V_i^f to refer to f) the set of all vertices of G with label i under f. Thus an HRDF f can be represented by a triple (V_0, V_1, V_2), and we can use the notation $f = (V_0, V_1, V_2)$. For a function $f = (V_0, V_1, V_2)$ and a vertex $v \in V(G)$, we say that v is hop Roman dominated by f (or f hop Roman dominates v), if either $v \in V_1 \cup V_2$ or there exist $u \in V_2$, such that $d(v,u) = 2$. An HRDF $f = (V_0, V_1, V_2)$ is a *hop Roman independent dominating function*(HRIDF) if for any pair $v, w \in V_1 \cup V_2$, $d(v,w) \neq 2$. The minimum weight of an HRIDF on G is called the *hop Roman independent domination number* of G. The concept of hop Roman domination was further studied, for example in [1, 15, 22].

We consider a generalized version of hop Roman domination. For $r \geq 2$, a *r-hop Roman dominating function* (rHRDF) is a function $f : V \longrightarrow \{0,1,2\}$ having the property that for every vertex $v \in V$ with $f(v) = 0$ there is a vertex u with $f(u) = 2$ and $d(u,v) = r$. The weight of a rHRDF f is the sum $f(V) = \sum_{v \in V} f(v)$. The minimum weight of a rHRDF on G is called the *r-hop Roman domination number* of G and is denoted $\gamma_{rhR}(G)$. For a function $f = (V_0, V_1, V_2)$ and a vertex $v \in V(G)$, we say that v is r-hop Roman dominated by f (or f r-hop Roman dominates v), if either $v \in V_1 \cup V_2$ or there exist $u \in V_2$, such that $d(v,u) = r$. A rHRDF $f = (V_0, V_1, V_2)$ is a *r-hop Roman independent dominating function*(rHRIDF) if for any pair $v, w \in V_1 \cup V_2$, $d(v,w) \neq r$. The minimum weight of a rHRIDF on G is called the *r-hop Roman independent domination number* of G. Likewise, a *r-step Roman dominating function* (rSRDF) is a function $f : V \longrightarrow \{0,1,2\}$ having the property that for every vertex $v \in V_0 \cup V_2$ there is a vertex $u \in V_2$ such that $d(u,v) = r$. The weight of a rSRDF f is the sum $f(V) = \sum_{v \in V} f(v)$. The minimum weight of a rSRDF on G is called the *r-step Roman domination number* of G.

Farhadi et al. [17] proved that for $r \geq 2$, the decision problems associated with both r-step domination and r-hop domination are NP-complete for planar bipartite graphs and planar chordal graphs. Jafari Rad et al. [16] proved that the decision problems associated with hop independent domination, r-hop Roman domination and the hop Roman independent domination are NP-complete even when restricted to planar bipartite graphs or planar chordal graphs.

In this paper we study the complexity of decision problems associated with the r-hop independent domination, r-hop Roman domination, r-hop Roman independent domination and r-step Roman domination. We show that the decision problem associated to each of these problems is NP-complete even when restricted to planar bipartite graphs or planar chordal graphs. We use a transformation of the Vertex Cover Problem which was one of Karp's 21 NP-complete problems [19] (see also [18]). The Vertex Cover Problem is the following decision problem.

Vertex Cover Problem (VCP).

Instance: A non-empty graph G, and a positive integer k.

Question: Does G have a vertex cover of size at most k?

2. r-Hop Independent Domination

Consider the following decision problem:

r-Hop Independent Dominating Problem (rHIDP).
Instance: A non-empty graph G and two positive integers $r \geq 2$ and $k \geq 1$.
Question: Does G have a r-hop independent dominating set of size at most k?
We show that the decision problem for rHIDP is NP-complete even when restricted to planar bipartite graphs or planar chordal graphs.

Theorem 1. *r-HIDP is NP-complete for planar bipartite graphs.*

Proof. Clearly, the rHIDP is NP, since it is easy to verify a "yes" instance of the rHIDP in polynomial time. Now we transform the vertex cover problem to the rHIDP so that one of them has a solution if and only if the other has a solution. Let G be a connected planar bipartite graph of order n_G and size $m_G \geq 2$. Let H be the graph obtained from G as follows. For each edge $e = uv \in E(G)$, we subdivide the edge e, $2r-1$ times. Let $x_e^1, x_e^2, \ldots, x_e^{2r-1}$ be the subdivided vertices that are produced by subdividing e, where x_e^i is adjacent to x_e^{i+1}, for $i = 1, 2, \ldots, 2r-2$, u is adjacent to x_e^1, and v is adjacent to x_e^{2r-1}. For every vertex $v \in V(G) \cup \{x_e^1, x_e^2, \ldots, x_e^{2r-1}\}$, we add a P_{2r+1}-path $P_{2r+1}^v : v_1 v_2 \ldots v_{2r+1}$, and join v_{r+1} to v, and then subdivide the edge $v_{r+1}v$ $2r-2$ times. Let y_v^1, y_v^2, $\ldots$, y_v^{2r-2} be the subdivided vertices that were produced by subdividing the edge $v_{r+1}v$, where y_v^1 is adjacent v_{r+1} and y_v^{2r-2} is adjacent to v. For every vertex $v \in \{x_e^r \mid e \in E(G)\}$ we subdivide the edge vy_v^{2r-2}, and let z_v be the subdivided vertex, where z_v is adjacent to both v and y_v^{2r-2}. Finally, for every vertex $v \in \{x_e^r \mid e \in E(G)\}$, add a vertex v' and join v' to both x_e^1 and x_e^{2r-1} and then subdivide each edge $v'x_e^1$ and $v'x_e^{2r-1}$, $r-2$ times. The resulting graph H has order $n_H = 4rn_G + (8r^2-2r-2)m_G$ and size $m_H = (4r-1)n_G + (8r^2-2r-1)m_G$. Figure 1 illustrates the graph H if G is a path P_3 and $r = 2$.

We show that G has a vertex cover of size at most k if and only if H has an rHIDS of size at most $k + rn_G + rm_G(2r-1)$. Assume S_G is a vertex cover of size at most k. Let

$$\begin{aligned} S_H &= S_G \cup \{v_{r+1}, v_{r+2}, \ldots, v_{2r} \mid v \in S_G\} \\ &\cup \ \big\{v_{r+1}, y_v^1, y_v^2, \ldots, y_v^{r-1} \mid v \in \big((V(G) - S_G) \cup \{x_e^1, x_e^2, \ldots, x_e^{2r-1} \mid e \in E(G)\}\big)\big\}. \end{aligned}$$

Clearly $d(a,b) \neq r$ for any pair $a, b \in S_H$. We show S_H is a rHIDS of size at most $k+rn_G+rm_G(2r-1)$. For each $e \in E(G)$, the vertices x_e^r and $x_e^{r}{'}$ are r-hop dominated by S_G, any vertex on the path from $x_e^{r}{'}$ to x_e^1 is r-hop dominated by $\{{x_e^1}_{r+1}, y_{x_e^1}^1, y_{x_e^1}^2, \ldots, y_{x_e^1}^{r-1}\}$, and any vertex on the path from $x_e^{r}{'}$ to x_e^{2r-1} is r-hop dominated by $\{{x_e^{2r-1}}_{r+1}, y_{x_e^{2r-1}}^1, y_{x_e^{2r-1}}^2, \ldots, y_{x_e^{2r-1}}^{r-1}\}$. For any vertex $v \in S_G$, any vertex in $\{v_1, v_2, \ldots, v_{2r+1}\} \cup \{y_v^1, y_v^2, \ldots, y_v^{2r-2}\}$ is hop dominated by $\{v_{r+1}, v_{r+2}, \ldots, v_{2r}\}$. For any vertex $v \in V(G) - S_G$, any vertex in $\{v_1, v_2, \ldots, v_{2r+1}\} \cup \{y_v^1, y_v^2, \ldots, y_v^{2r-2}\}$ is hop dominated

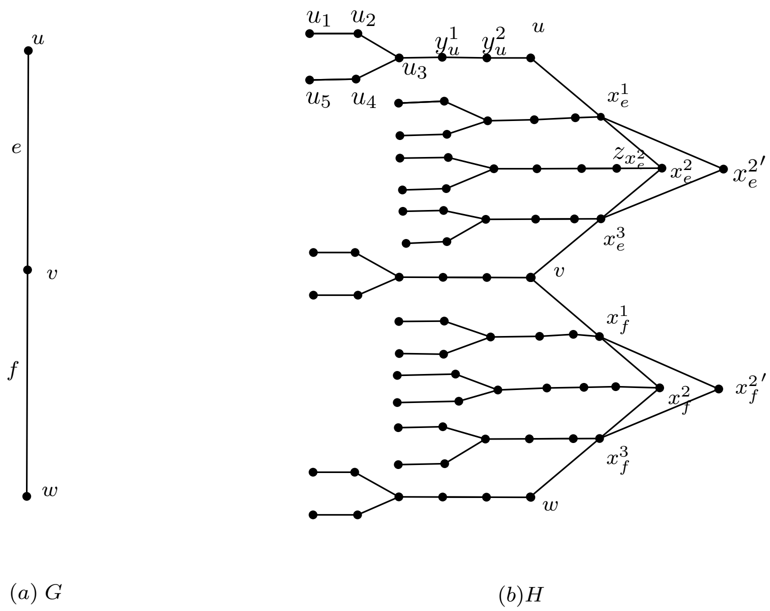


Figure 1. **The graphs G and H in the proof of Theorem 1**

by $\{v_{r+1}, y_v^1, y_v^2, \dots, y_v^{r-1}\}$. For any edge $e \in E(G)$, any vertex in

$$\{x^r_{e1}, x^r_{e2}, \dots, x^r_{e2r+1}\} \cup \{y^1_{x^r_e}, y^2_{x^r_e}, \dots, y^{2r-2}_{x^r_e}\}$$

is r-hop dominated by $\{x^r_{er+1}, y^1_{x^r_e}, y^2_{x^r_e}, \dots, y^{r-1}_{x^r_e}\}$. Similarly, for any edge $e \in E(G)$, any vertex in $\{x^i_e, x^i_{e1}, x^i_{e2}, \dots, x^i_{e2r+1}\} \cup \{y^1_{x^i_e}, y^2_{x^i_e}, \dots, y^{2r-2}_{x^i_e}\}$, where $i \neq r$, is r-hop dominated by $\{x^i_{er+1}, y^1_{x^i_e}, y^2_{x^i_e}, \dots, y^{r-1}_{x^i_e}\}$. Consequently, S_H is a rHIDS of size at most $k + rn_G + rm_G(2r-1)$.

Assume next that H has a rHIDS, S_H, of size at most $k + rn_G + rm_G(2r-1)$. It is evident that for any vertex $v \in V(G) \cup \{x^1_e, x^2_e, \dots, x^{2r-1}_e \mid e \in E(G)\}$,

$$|S_H \cap \{v_1, v_2, \dots, v_{2r+1}, y^1_v, y^2_v, \dots, y^{2r-2}_v\}| \geq r.$$

Let

$$A = S_H \cap \bigcup_{v \in V(G) \cup \{x^1_e, x^2_e, \dots, x^{2r-1}_e | e \in E(G)\}} (\{v_1, v_2, \dots, v_{2r+1}, y^1_v, y^2_v, \dots, y^{2r-2}_v\}).$$

Then $|A| \geq rn_G + rm_G(2r-1)$, and so $|S_H - A| \leq k$. For any edge $e = uv$, since $x^{r\prime}_e$ is r-hop dominated by S_H, either $x^{r\prime}_e \in S_H$ or $S_H \cap \{u, v\} \neq \emptyset$. If for an edge $e = uv$, $S_H \cap \{u, v\} = \emptyset$, then $x^{r\prime}_e \in S_H$, and we replace S_H by $(S_H - \{x^{r\prime}_e\}) \cup \{u\}$. Thus we assume that for any edge $e = uv$, $S_H \cap \{u, v\} \neq \emptyset$. Thus $S_H \cap V(G)$ is a vertex cover for G of size at most k. Therefore G has a vertex cover of size at most k, as desired. □

We next prove the NP-completeness of rHIDP for planar chordal graphs.

Theorem 2. *rHIDP is NP-complete for planar chordal graphs.*

Proof. Let G be a planar chordal graph of order n_G and size $m_G \geq 2$, and let H be the graph presented in the proof of Theorem 1. For any edge $e \in E(G)$, let $x_e^1, x_e^{2'}, \ldots, x_e^{r-1'}, x_e^{r'}$ be vertices on the path from x_e^1 to $x_e^{r'}$, and $x_e^{r'}, x_e^{r+1'}, \ldots, x_e^{2r-1}$ be the vertices on the path from $x_e^{r'}$ to x_e^{2r-1}. We join x_e^i to both $x_e^{i'}$ and $x_e^{i+1'}$ for each $i = 2, 3, \ldots, 2r-3$, and join x_e^{2r-2} to $x_e^{2r-2'}$. Let H' be the constructed graph. Clearly H' is a planar chordal graph. Now with the same argument given in the proof of Theorem 1, we can see that G has a vertex cover of size at most k if and only if H' has an rHIDS of size at most $k + rn_G + rm_G(2r-1)$. □

3. r-Hop Roman Domination

Consider the following decision problem:
r-Hop Roman Dominating Function Problem (rHRDFP).
Instance: A non-empty graph G, and two positive integers $r \geq 2$ and $k \geq 1$.
Question: Does G have a r-hop Roman dominating function of weight at most k?

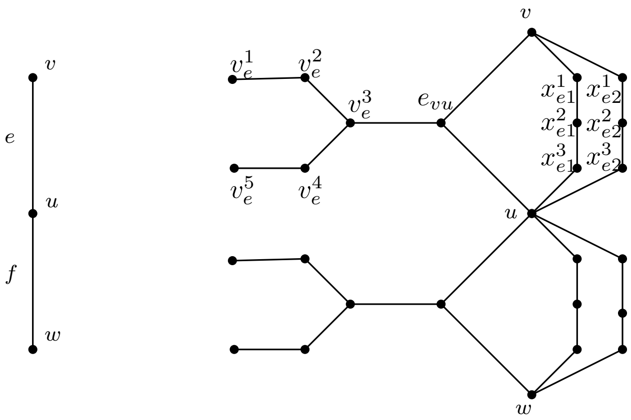
We show that the decision problem for the rHRDFP is NP-complete even when restricted to planar bipartite graphs or planar chordal graphs.

Theorem 3. *For $r \geq 2$, rHRDFP is NP-complete for planar bipartite graphs.*

Proof. Clearly, the rHRDFP is in NP. We transform the vertex cover problem to the rHRDFP so that one of them has a solution if and only if the other one has a solution. Let G be a connected planar bipartite graph of order n_G and size $m_G \geq 2$, and let H be the graph obtained from G as follows: We convert each edge $e = vu \in E(G)$ into a double edge $e_1 = vu$, and $e_2 = vu$, and then subdivide each of edges e_1 and e_2, $2r-1$ times. Let the vertices $x_{e_i}^1, x_{e_i}^2, \ldots, x_{e_i}^{2r-2}$ be the vertices that were produced from subdividing the edge e_i, for $i = 1, 2$, where the vertex $x_{e_i}^1$ is adjacent to v, for $i = 1, 2$. For each edge $e = vu \in E(G)$, we add a new vertex e_{vu} and a P_{2r+1}-path $v_e^1 v_e^2 \ldots v_e^{2r+1}$, join the vertex e_{vu} to u, v and v_e^{r+1}. Finally, we subdivide the edge $e_{vu}v_e^{r+1}$, $r-2$ times. Let $y_v^1, \ldots, y_v^{r-2}$ be the subdivided vertices produced by subdivision of $e_{vu}v_e^{r+1}$, where y_v^1 is adjacent to v_e^{r+1} and y_v^{r-2} is adjacent to e_{uv}. The resulting graph H has order $n_H = n_G + (7r-2)m_G$ and size $m_H = (7r+1)m_G$. Figure 2 illustrates the graph H if G is a path P_3 and $r = 2$. We note that since G is connected and planar, so H is connected and planar. Further, by construction, H is bipartite. Thus, H is a connected planar bipartite graph.

We show that G has a vertex cover of size at most k if and only if H has a rHRDF of weight $2k + 2rm_G$. Assume that G has a vertex cover, S_G, of size at most k. Let

$$S_H = S_G \cup \bigcup_{e=uv \in E(G)} \{v_e^{r+1}, y_v^1, \ldots, y_v^{r-2}, e_{vu}\}.$$



$(a)G$ $(b)H$

Figure 2. **The graph G and H in the proof of Theorem 3**

We show that $f = (V(H) - S_H, \emptyset, S_H)$ is an rHRDF for H of weight at most $2k + 2rm_G$. For every edge $e = vu \in E(G)$, the vertex v_e^{r+1} r-hop Roman dominates the vertices v_e^1, v_e^{2r+1}, u and v in H, while the vertex y_v^i $(i = 1, 2, \ldots, r-2)$ r-hop dominates the vertices v_e^{i+1}, v_e^{2r+1-i}, $x_{e_1}^i$, $x_{e_2}^i$, $x_{e_1}^{2r-i}$ and $x_{e_2}^{2r-i}$. Furthermore, e_{vu} r-hop Roman dominates the vertices $x_{e_1}^{r+1}$ and $x_{e_2}^{r+1}$, since S_G is a vertex cover in G. Therefore, the function f is a rHRDF for H of weight at most $2k + 2rm_G$.

Assume next that $f = (V_0^f, V_1^f, V_2^f)$ is a rHRDF for H of weight $2k + 2rm_G$. Without loss of generality we assume that f has minimum weight. If for an edge $e \in E(G)$, $f(v_e^1) + \cdots + f(v_e^{2r+1}) + f(y_v^1) + \cdots + f(y_v^{r-2}) + f(e_{vu}) < 2r$, then there is a vertex in $\{v_e^1, \ldots, v_e^{2r+1}\}$ such that it is not r-hop Roman dominated by f, a contradiction. Therefore, $f(v_e^1) + \cdots + f(v_e^{2r+1}) + f(y_v^1) + \cdots + f(y_v^{r-2}) + f(e_{vu}) \geq 2r$ for every edge $e \in E(G)$. If for an edge $e \in E(G)$, $f(v_2^e) + f(v_4^e) + f(e_{vu}) \leq 1$, then v_2^e or v_4^e is not hop Roman dominated by f, a contradiction. Therefore, $f(v_2^e) + f(v_4^e) + f(e_{vu}) \geq 2$ for every edge $e \in E(G)$. Suppose that there exists an edge $e = uv \in E(G)$ such that $f(x_{e_i}^r) > 0$ for each $i = 1, 2$. Assume that $f(u) \geq f(v)$. Then the function g defined by $g(x_{e_1}^r) = g(x_{e_2}^r) = 0$, $g(u) = \max\{f(u), 2\}$ and $g(z) = f(z)$ otherwise, is an rHRDF. If $f(u) \neq 0$ then $g(V) < f(V)$, a contradiction by the choice of f. Thus, assume that $f(u) = 0$, and so g is a minimum rHRDF. Thus we may assume that $f(x_{e_1}^r) = f(x_{e_2}^r) = 0$ for any edge $e = uv \in E(G)$. Then either $f(u) = 2$ or $f(v) = 2$. Hence, $S_G = V_2^f \cap V(G)$ is a vertex cover of G of size at most $\frac{1}{2}\big(w(f) - 2rm_G\big)$. Thus, G has a vertex cover of size at most k. $\square$

4. r-Hop Roman Independent Domination

We next study the complexity issue of the r-hop Roman independent domination. Consider the following decision problem:

r-Hop Roman Independent Dominating Function Problem (HRIDFP).
Instance: A non-empty graph G, and two positive integers $r \geq 2$ and $k \geq 1$.
Question: Does G have a r-hop Roman independent dominating function of weight at most k?

We show that the decision problem for rHRIDFP is NP-complete even when restricted to planar bipartite graphs or planar chordal graphs.

Theorem 4. *For $r \geq 2$, rHRIDFP is NP-complete for planar bipartite graphs.*

Proof. Let G be a graph of order n_G and size m_G, and let H be the connected planar bipartite graph constructed in the proof of Theorem 1. Note that H has order $n_H = 4rn_G + (8r^2 - 2r - 2)m_G$ and size $m_H = (4r-1)n_G + (8r^2 - 2r - 1)m_G$. We show that G has a vertex cover of size at most k if and only if H has an rHRIDF of weight at most $2k + 2rn_G + 2rm_G(2r-1)$. Assume first that G has a vertex cover, S_G, of size at most k. Let

$$\begin{aligned} S_H &= S_G \cup \{v_{r+1}, v_{r+2}, \ldots, v_{2r} \mid v \in S_G\} \\ &\cup \ \{v_{r+1}, y_v^1, y_v^2, \ldots, y_v^{r-1} \mid v \in \big((V(G) - S_G) \cup \{x_e^1, x_e^2, \ldots, x_e^{2r-1} \mid e \in E(G)\}\big)\}. \end{aligned}$$

Clearly $d(a,b) \neq r$ for any pair $a, b \in S_H$. We set $f = (V(H) - S_H, \emptyset, S_H)$. As it is proved in the proof of Theorem 1, that S_H is a rHIDS for H, we conclude that any vertex v with $f(v) = 0$ is r-hop dominated by a vertex u with $f(u) = 2$. Hence H has a rHRIDF of weight at most $2k + 2rn_G + 2rm_G(2r-1)$.

Assume now that H has a rHRIDF f, of weight at most $2k + 2rn_G + 2rm_G(2r-1)$. It is evident that for any vertex $v \in V(G) \cup \{x_e^1, x_e^2, \ldots, x_e^{2r-1} \mid e \in E(G)\}$,

$$\sum_{v \in \{v_1, v_2, \ldots, v_{2r+1}, y_v^1, y_v^2, \ldots, y_v^{2r-2}\}} f(v) \geq 2r.$$

Let

$$A = S_H \cap \bigcup_{v \in V(G) \cup \{x_e^1, x_e^2, \ldots, x_e^{2r-1} \mid e \in E(G)\}} (\{v_1, v_2, \ldots, v_{2r+1}, y_v^1, y_v^2, \ldots, y_v^{2r-2}\}).$$

Then $\sum_{v \in A} f(v) \geq 2rn_G + 2rm_G(2r-1)$. For any edge $e = uv$, since both x_e^r and $x_e^{r\prime}$ are r-hop dominated by f, either $f(x_e^r) \geq 1$ and $f(x_e^{r\prime}) \geq 1$, or $2 \in \{f(u), f(v)\}$. If

$2 \notin \{f(u), f(v)\}$, then we replace $f(u)$ by 2 and both $f(x_e^r)$ and $f(x_e^{r\prime})$ by 0. Thus we my assume that for any edge $e = uv$, $2 \in \{f(u), f(v)\}$. Then $\{v \in V(G) : f(v) = 2\}$ is a vertex cover for G of size at most $2k$. Therefore G has a vertex cover of size at most $2k$. □

Theorem 5. *For $r \geq 2$, rHRIDFP is NP-complete for planar chordal graphs.*

Proof. Let G be a graph of order n_G and size m_G, and let H' be the connected planar chordal graph constructed in the proof of Theorem 2. With a similar argument as it is given in proof of Theorem 4, we can see that G has a vertex cover of size at most k if and only if H' has an rHRIDS of weight at most $2k + 2rn_G + 2rm_G(2r - 1)$. □

5. r-Step Roman domination

Consider the following decision problem:

r-Step Roman Dominating Function Problem (rSRDFP).
Instance: A non-empty graph G, and two positive integers $r \geq 2$ and $k \geq 1$.
Question: Does G have a r-step Roman dominating function of weight at most k?

We show that the decision problem for rSRDFP is NP-complete even when restricted to planar bipartite graphs or planar chordal graphs.

Theorem 6. *For $r \geq 2$, rSRDFP is NP-complete for planar bipartite graphs.*

Proof. Clearly, the rSRDFP is in NP, since it is easy to verify a "yes" instance of rSRDFP in polynomial time. Now we transform the vertex cover problem to the rSRDFP so that one of them has a solution if and only if the other has a solution. Let G be a connected planar bipartite graph of order n_G and size $m_G \geq 2$. Let H be the graph obtained from G as follows. For each edge $e = uv \in E(G)$ we subdivide the edge e, $2r - 1$ times, and add a path $v_1^e v_2^e \ldots v_{2r}^e$, and join v_1^e to both u and v. For any edge $e = uv \in E(G)$, let e_{uv} be the subdivided vertex at distance r from both u and v in H that resulted from subdividing the edge e, $2r - 1$ times. Then add a vertex $e_{uv}{}'$ and join it to both neighbors of e_{uv}. Let H be the resulted graph. Then H has order $n_H = n_G + 4rm_G$ and size $m_H = (4r + 3)m_G$. The transformation can clearly be performed in polynomial time. We note that since G is connected and planar, so H is connected and planar. Further, by construction, H is bipartite. Thus, H is a connected planar bipartite graph. Figure 3 depicts the graph H if $r = 2$ and $G = P_3$.

We show that G has a vertex cover of size at most k if and only if H has a r-step Roman dominating function of weight at most $2k + 2rm_G$. Assume that G has a

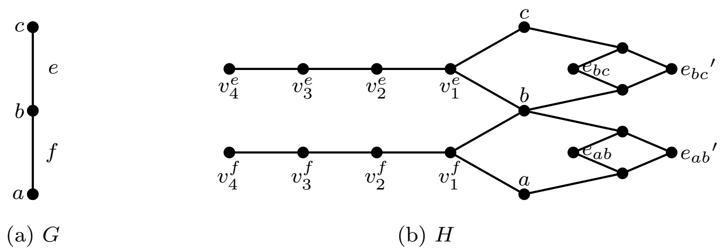


Figure 3. **The graphs G and H in the proof of Theorem 6 for $r = 2$**

vertex cover, namely S_G, of size at most k. Let

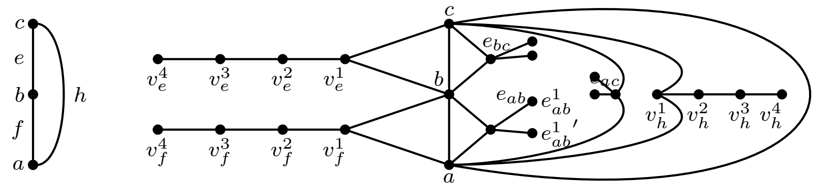
$$S_H = S_G \cup \bigcup_{e \in E(G)} \{v_e^1, v_e^2, \ldots, v_e^r\}.$$

We show that $f = (V(H) - S_H, \emptyset, S_H)$ is a r-step Roman dominating function. Clearly $S_G \neq \emptyset$, since $m_G \geq 2$. For every edge $e = uv \in E(G)$, the vertex v_r^e r-step dominates the vertices v_e^{2r}, u and v in H, while the vertex v_e^i $(i = 1, 2, \ldots, r-1)r$-step dominates the vertex v_e^{i+r} and the r-neighbors of u and v in H that belong to the (u, v)-path in H that resulted from subdividing the edge $e = uv$ of G. Since S_G is a vertex cover in G, every subdivided vertex that is not a neighbor of a vertex in $V(G)$ is r-step dominated by the set S_G in H. Further, the set S_G r-step dominates the vertex v_e^r for every edge $e \in E(G)$. Since G is connected and $m_G \geq 2$, for every two adjacent edges e and f in G the vertices v_e^i and v_f^j r-step dominate each other for $1 \leq i, j < r$, where $i + j = r$. Therefore, S_H is a r-step dominating set for H, and thus $f = (V(H) - S_H, \emptyset, S_H)$ is a r-step Roman dominating function for H of weight at most $2k + 2rm_G$ in H.

Suppose next that H has a r-step Roman dominating function f of weight at most $2k + 2rm_G$. Without loss of generality we assume that f has minimum weight. Let $e = uv \in E(G)$. For $i = r+1, \ldots, 2r$, in order to r-step Roman dominate v_e^i in H, it is required that $\sum_{i=1}^{2r} f(v_e^i) \geq 2r$. If $2 \notin \{f(u), f(v)\}$, then $f(e_{uv}) \neq 0$ and $f({e_{uv}}') \neq 0$. Let g be a function obtained by changing both $f(e_{uv})$ and $f({e_{uv}}')$ to 0 and $f(u)$ to 2. Since f has minimum weight, we find that $w(g) = w(f)$. Thus we may assume that $2 \in \{f(u), f(v)\}$. Hence, $\{v \in V(G) : f(v) = 2\}$ is a vertex cover of G. Further, $|\{v \in V(G) : f(v) = 2\}| \leq k$, since $\sum_{i=1}^{2r} f(v_e^i) \geq 2r$ for every edge $e \in E(G)$. Thus, G has a vertex cover of size at most k. □

Theorem 7. *For $r \geq 2$, rSRDFP is NP-complete for planar chordal graphs.*

Proof. Let G be a connected planar chordal graph of order n_G and size $m_G \geq 2$. Let H be the graph obtained from G as follows. For each edge $e = uv \in E(G)$ we add a new vertex e_{uv} adjacent to both u and v in H and we add a P_{r-1}-path ${e^1}_{uv}{e^2}_{uv} \ldots {e^{r-1}}_{uv}$ and join e_{uv} to e_{uv}^1. Further, we add a P_{2r}-path $v_e^1 v_e^2 \ldots v_e^{2r}$, and join v_e^1 to u and v. Finally for each edge $e = uv \in E(G)$ add a new vertex ${e^{r-1}}_{uv}{}'$ and join it to the neighbor of ${e^{r-1}}_{uv}$. The resulting graph H has order $n_H = n_G + (3r+1)m_G$ and size $m_H = (3r+4)m_G$. The transformation can clearly be performed in polynomial time. We note that since H is a connected planar chordal graph.



(a) G (b) H

Figure 4. **The graphs G and H in the proof of Theorem 7 for $r = 2$**

We show that G has a vertex cover of size at most k if and only if H has a r-step Roman dominating function of weight at most $2k + 2rm_G$. Let S_G be a vertex cover of size at most k, and let

$$S_H = S_G \cup \bigcup_{e \in E(G)} \{v_e^1, v_e^2, \dots, v_e^r\}.$$

Let $f = (V(H) - S_H, \emptyset, S_H)$. Note that $S_G \neq \emptyset$. For every edge $e = uv \in E(G)$, the vertex v_e^r r-step dominates the vertices v_e^{2r}, u and v in H, while the vertex v_e^i $(1 \leq i < r)$ r-step dominates the vertices v_e^{i+r} and ${e_{uv}}^{r-i-1}$, where ${e_{uv}}^0 =: e_{uv}$. Since S_G is a vertex cover in G, every vertex ${e_{uv}}^{r-1}$ is r-step dominated by S_G in H. Further, S_G r-step dominates v_e^r for every edge $e \in E(G)$. Since G is connected and $m_G \geq 2$, for every two adjacent edges e and f in G the vertices v_e^i and v_f^j r-step dominate each other for $1 \leq i, j < r$, where $i + j = r$. Therefore, f is a r-step Roman dominating function of weight at most $2k + 2rm_G$.
Suppose next that H has a r-step Roman dominating function f of weight at most $2k + 2rm_G$. Let $e = uv \in E(G)$. For $i = r+1, \dots, 2r$, in order to r-step Roman dominate v_e^i in H, it is required that $\sum_{i=1}^{2r} f(v_e^i) \geq 2r$. If $2 \notin \{f(u), f(v)\}$, then $f({e^{r-1}}_{uv}{}') \neq 0$ and $f({e^{r-1}}_{uv}) \neq 0$. Let g be a function obtained by changing both $f({e^{r-1}}_{uv})$ and $f({e^{r-1}}_{uv}{}')$ to 0 and $f(u)$ to 2. Since f has minimum weight, we find that $w(g) = w(f)$. Thus we may assume that $2 \in \{f(u), f(v)\}$. Hence, $\{v \in V(G) : f(v) = 2\}$ is a vertex cover of G. Further, $|\{v \in V(G) : f(v) = 2\}| \leq k$, since $\sum_{i=1}^{2r} f(v_e^i) \geq 2r$ for every edge $e \in E(G)$. Thus, G has a vertex cover of size at most k. □

Conflict of interest. The authors declare that they have no conflict of interest.

Data Availability. Data sharing is not applicable to this article as no datasets were generated or analyzed during the current study.

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