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***Research Article***

# The zero-divisor associate graph over a finite commutative ring

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**Abstract:** In this paper, we introduce the zero-divisor associate graph $\Gamma_D(R)$ over a finite commutative ring $R$. It is a simple undirected graph whose vertex set consists of all non-zero elements of $R$, and two vertices $a, b$ are adjacent if and only if there exist non-zero zero-divisors $z_1, z_2$ in $R$ such that $az_1 = bz_2$. We determine the necessary and sufficient conditions for connectedness and completeness of $\Gamma_D(R)$ for a unitary commutative ring $R$. The chromatic number of $\Gamma_D(R)$ is also studied. Next, we characterize the rings $R$ for which $\Gamma_D(R)$ becomes a line graph of some graph. Finally, we give the complete list of graphs with at most 15 vertices which are realizable as $\Gamma_D(R)$, characterizing the associated ring $R$ in each case.

**Keywords:** Zero-divisor, Commutative ring, Chromatic number, Complete graph, Line graph.

**AMS Subject classification:** 05C25, 05E40, 05C76, 13A70

## 1. Introduction

The study of algebraic structures using graph-theoretical tools has been an exciting research area since the previous two decades. Several graphs have been defined over

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various algebraic structures, which provided valuable insights into the interplay between the algebraic properties and graph-theoretical properties. Among such graphs, the zero-divisor graph deserves special mention as it is significant from various perspectives. The concept of a zero-divisor graph was first introduced by Beck (cf. [1, 3]), and the zero-divisor graph of a commutative ring was defined in its present form by Anderson and Livingston [2]. Then, Redmond [13] generalized the zero-divisor graphs for non-commutative rings. The richness of results obtained from the zero-divisor graphs motivated the definition of several such graphs ([5–7, 16], for example) over rings and other structures.

With the motivation of further exploring the set of zero-divisors of a ring, we here introduce a simple undirected graph in the following way:

**Definition 1.** Let $R$ be a finite commutative ring. The zero-divisor associate graph $\Gamma_D(R)$ over the ring $R$ is a simple undirected graph $(V, E)$ where $V = R - \{0\}$, and two distinct vertices $a, b$ are adjacent if and only if $az_1 = bz_2$ for some non-zero zero-divisors $z_1, z_2$ (not necessarily distinct) of $R$.

Clearly, $\Gamma_D(R)$ is an empty graph when $R$ is an integral domain.

The term 'zero-divisor associate' alludes to the concept of associate elements of a ring. In this paper, we first characterize the rings $R$ for which the graph $\Gamma_D(R)$ is connected, and show that $\Gamma_D(R)$ is connected if and only if $(Z(R))^2 \neq \{0\}$. This, in turn, shows that $\Gamma_D(R)$ is connected for all commutative unital rings except local rings of characteristic $p$ or $p^2$ for some prime $p$. It is also shown that $\Gamma_D(R)$ is complete if and only if $Ann(Z(R)) = \{0\}$. The chromatic number of $\Gamma_D(R)$ is found to be $|R| - |Ann(Z(R))|$. Then, we characterize the rings $R$ for which $\Gamma_D(R)$ is realized as the line graph of some graph. In the appendix, we completely characterize the graphs having at most 15 vertices which are realizable as $\Gamma_D(R)$.

Throughout this paper, $S^*$ denotes the non-zero elements of a set $S$. For any ring $R$, $U(R)$ denotes the set of all units of $R$, $Z(R)$ denotes the set of all zero-divisors of $R$, and $Ann(Z(R)) = \{z \in Z(R) \mid zz_1 = 0 \text{ for each } z_1 \in Z(R)\}$. For any $S \subseteq R$, $S^2$ is the set $\{xy \mid x, y \in S\}$. $Char(R)$ denotes the characteristic of the ring $R$. One may see [4, 9] for general algebraic notations. The complement of the graph $G$ is denoted by $\overline{G}$. Also, $a \leftrightarrow b$ denotes that the vertices $a$ and $b$ are adjacent. For other graph-theoretical properties, we refer to [17].

## 2. Connectedness of the graph $\Gamma_D(R)$

In this section we mainly consider necessary and sufficient conditions for $\Gamma_D(R)$ to be connected. We begin with several lemmas, which lead us to the main result.

**Lemma 1.** *Let $R$ be a finite commutative unital ring, such that $Z(R) \neq \{0\}$. Then the set of vertices corresponding to $U(R)$ (i.e., the set of all units of $R$) induces a clique in $\Gamma_D(R)$. Also, the set of vertices corresponding to $Z(R)^*$ induces a clique in $\Gamma_D(R)$.*

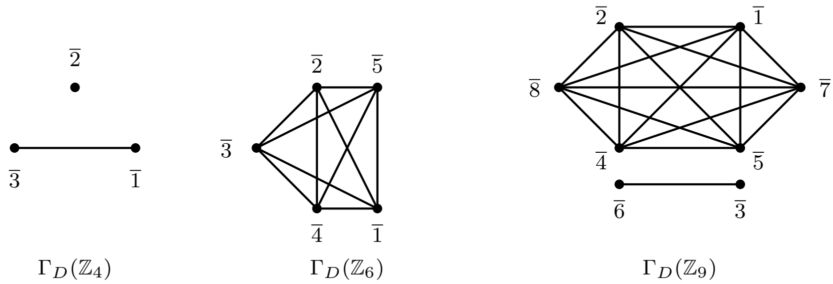


**Figure 1.** $\Gamma_D(R)$ **for** $R \in \{\mathbb{Z}_4, \mathbb{Z}_6, \mathbb{Z}_9\}$

*Proof.* Let $zy = 0$ for all $y \in Z(R)$. If possible, let $\exists\ u \in U(R)$ such that $u \leftrightarrow z$ in $\Gamma_D(R)$. Then $uz_1 = zz_2$ for some $z_1, z_2 \in Z(R)^*$. This shows that $uz_1 = 0$, contradicting that $u$ is a unit. So $z \not\leftrightarrow u$ for any unit $u$ of $R$. Next, let $\exists\ z_1 \in Z(R)^*$ with $zz_1 \neq 0$. Then for any unit $u$, we have that $u(u^{-1}zz_1) = zz_1$. So $z$ is adjacent to every unit $u$ of $R$. □

**Lemma 2.** *Let $R$ be a finite communtative unital ring, and $z \in Z(R)^*$. If $zy = 0$ for all $y \in Z(R)$, then $z$ is not adjacent to any unit $u \in U(R)$ in $\Gamma_D(R)$. Else, $z$ is adjacent to every unit of $R$ in $\Gamma_D(R)$ .*

*Proof.* Let $zy = 0$ for all $y \in Z(R)$. If possible, let $\exists\ u \in U(R)$ such that $u \leftrightarrow z$ in $\Gamma_D(R)$. Then $uz_1 = zz_2$ for some $z_1, z_2 \in Z(R)^*$. This shows that $uz_1 = 0$, contradicting that $u$ is a unit. So $z \not\leftrightarrow u$ for any unit $u$ of $R$. Next, let $\exists\ z_1 \in Z(R)^*$ with $zz_1 \neq 0$. Then for any unit $u$, we have that $u(u^{-1}zz_1) = zz_1$. So $z$ is adjacent to every unit $u$ of $R$. □

**Lemma 3.** *Let $R$ be a finite commutative unital ring, and let $u \in U(R)$ and $z \in Z(R)^*$. Then in $\Gamma_D(R)$, $u \leftrightarrow z$ if and only if $1 \leftrightarrow z$.*

*Proof.* $u \leftrightarrow z$ in $\Gamma_D(R)$ if and only if $uz_1 = zz_2$ for some $z_1, z_2 \in Z(R)^*$, i.e., $1(uz_1) = z(z_2)$, which happens if and only if $1 \leftrightarrow z$. Hence the result. □

**Example 1.** Consider the finite commutative rings $\mathbb{Z}_4, \mathbb{Z}_6$ and $\mathbb{Z}_9$. It is seen that $\Gamma_D(\mathbb{Z}_4)$ and $\Gamma_D(\mathbb{Z}_9)$ are not connected, but $\Gamma_D(\mathbb{Z}_6)$ is connected (cf. Figure 1). It is interesting to note that $(Z(\mathbb{Z}_4))^2 = (Z(\mathbb{Z}_9))^2 = \{0\}$, but $(Z(\mathbb{Z}_6))^2 \neq \{0\}$. We next show that $(Z(R))^2 \neq \{0\}$ is, in fact, a necessary and sufficient condition for $\Gamma_D(R)$ to be connected.

**Theorem 1.** *Let $R$ be a finite commutative unital ring. Then $\Gamma_D(R)$ is connected if and only if $(Z(R))^2 \neq \{0\}$.*

*Proof.* Suppose that $(Z(R))^2 \neq \{0\}$. By Lemma 1, we already have that the sets $U(R)$ and $Z(R)^*$ both induce cliques in $\Gamma_D(R)$. Since $R^* = U(R) \cup Z(R)^*$, it then suffices to show that $u \leftrightarrow z$ for some $u \in U(R)$ and $z \in Z(R)^*$ to prove that $\Gamma_D(R)$ is connected. As $(Z(R))^2 \neq \{0\}$, $\exists\ a, b \in Z(R)^*$ such that $ab \neq 0$. Noting that $1(ab) = ab$, we have that the unit 1 is adjacent to the zero-divisor $a$. So the graph $\Gamma_D(R)$ is connected.

Conversely, let $\Gamma_D(R)$ be connected. So we necessarily have $z \leftrightarrow u$ in $\Gamma_D(R)$ for some $z \in Z(R)^*$ and $u \in U(R)$. Thus, by Lemma 2, $zy \neq 0$ for some $y \in Z(R)$, which implies that $(Z(R))^2 \neq \{0\}$. This completes the proof. □

**Corollary 1.** *The graph $\Gamma_D(\mathbb{Z}_n)$ is connected if and only if $n \neq 4$ and $n \neq p, p^2$ for any odd prime $p$.*

*Proof.* Let $\Gamma_D(\mathbb{Z}_n)$ be connected. Figure 1 shows that $n \neq 4$. If $n = p$ for any odd prime $p$, then $\mathbb{Z}_n$ is a field and we arrive at a contradiction as $\Gamma_D(\mathbb{Z}_p)$ is a disjoint union of more than one isolated vertices. If possible, let $n = p^2$ for some odd prime $p$. Then $(Z(\mathbb{Z}_n))^2 = \{\overline{0}\}$, which contradicts the connectedness of $\Gamma_D(\mathbb{Z}_n)$ (by Theorem 1). Thus, $n \neq p, p^2$ for any odd prime $p$. Conversely, let $n \neq 4$ and $n \neq p, p^2$ for any odd prime $p$. Then either $n = p^m$ for some prime $p$ (and $m > 2$), or $n$ has at least two distinct prime factors. In either case, $(Z(\mathbb{Z}_n))^2 \neq \{\overline{0}\}$. So, by Theorem 1, the graph $\Gamma_D(\mathbb{Z}_n)$ is connected. □

**Theorem 2.** *Let $R$ be a finite commutative unital ring. If $R$ is either a local ring with $Char(R) \neq p, p^2$ for any prime $p$, or a non-local ring, then $\Gamma_D(R)$ is connected.*

*Proof.* We first observe that $\Gamma_D(\mathbb{Z}_2 \times \mathbb{Z}_2)$ is connected. Next, let $R \ncong \mathbb{Z}_2 \times \mathbb{Z}_2$. So $R$ is neither (isomorphic to) $\mathbb{Z}_2 \times \mathbb{Z}_2$ nor local with $Char(R) = p$ or $p^2$ for any prime $p$. Thus, it follows from [Theorem 2.10, [2]] that the zero-divisor graph $\Gamma(R)$ is not complete. This implies that $\exists\ z, w \in Z(R)$ such that $zw \neq 0$, and hence, we have that $(Z(R))^2 \neq \{0\}$. Thus, by Theorem 1, $\Gamma_D(R)$ is connected. □

Next, we consider the completeness of $\Gamma_D(R)$. It is seen that $\Gamma_D(\mathbb{Z}_6)$ is complete but $\Gamma_D(\mathbb{Z}_8)$ is not (cf. Figure 1 and Figure 2). It is interesting to observe that $Ann(Z(\mathbb{Z}_6)) = \{\overline{0}\}$ and $Ann(Z(\mathbb{Z}_8)) \neq \{\overline{0}\}$. We show in the next result that $Ann(Z(R) = \{0\}$ is indeed a necessary and sufficient condition for $\Gamma_D(R)$ to be complete.

**Theorem 3.** *Let $R$ be a finite commutative unital ring $R$. Then $\Gamma_D(R)$ is complete if and only if $Ann(Z(R)) = \{0\}$.*

*Proof.* Let $Ann(Z(R)) = \{0\}$. Then if $z \in Z(R)^*$, $\exists\ z_1 \in Z(R)^*$ such that $zz_1 \neq 0$. Since $1(zz_1) = z(z_1)$, we have that $1 \leftrightarrow z$ in $\Gamma_D(R)$. It then follows by Lemma 3 that $u \leftrightarrow z$ for every $u \in U(R)$. As $z$ was arbitrary, it follows that $u_2 \leftrightarrow z_2$ for any

$u_2 \in U(R), z_2 \in Z(R)^*$. Since $R^* = U(R) \cup Z(R)^*$, this together with Lemma 1 implies that $\Gamma_D(R)$ is complete.

Conversely, let $\Gamma_D(R)$ be complete. Let $z \in Z(R)^*$. Clearly, $z \leftrightarrow u$ for all $u \in U(R)$. By Lemma 2, it then follows that $zy \neq 0$ for some $y \in Z(R)$. This shows that $z \notin Ann(Z(R))$ and hence, we have that $Ann(Z(R)) = \{0\}$. □

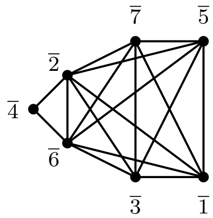


**Figure 2.** $\Gamma_D(\mathbb{Z}_8)$

**Corollary 2.** *Let $n \in \mathbb{N} - \{1, 2\}$. Then $\Gamma_D(\mathbb{Z}_n)$ is complete if and only if $n \neq p^m$ for any prime $p$ and $m \in \mathbb{N}$.*

*Proof.* Let $\Gamma_D(\mathbb{Z}_n)$ be complete. If possible, let $n = p^m$ for some prime $p$ and $m \in \mathbb{N}$. By Corollary 1 and Figure 1, we then have that $m > 2$. So $Ann(Z(\mathbb{Z}_n)) \neq \{\overline{0}\}$, which contradicts the completeness of $\Gamma_D(\mathbb{Z}_n)$ (cf. Theorem 3). Hence, $n \neq p^m$ for any prime $p$ and $m \in \mathbb{N}$. Conversely, let $n \neq p^m$ for any prime $p$ and $m \in \mathbb{N}$. Then $n = p_1^{r_1} p_2^{r_2} \cdots p_k^{r_k}$ for $k(\geq 2)$ distinct primes $p_i$'s. Consider any non-zero zero-divisor $z$ in $\mathbb{Z}_n$. Clearly, $z = \overline{p_1^{r_1-s_1} p_2^{r_2-s_2} \cdots p_k^{r_k-s_k}}$, where $s_i \geq 0$ for $i = 1, 2, \ldots, k$, and $s_j > 0$ for some $j \in \{1, 2, \ldots, k\}$. Since $z\overline{p_t} \neq \overline{0}$ for any $t \neq j$, we have that $z \notin Ann(Z(\mathbb{Z}_n))$. This shows that $Ann(Z(\mathbb{Z}_n)) = \{\overline{0}\}$. Hence, by Theorem 3, $\Gamma_D(\mathbb{Z}_n)$ is complete. □

**Theorem 4.** *Let $R$ be a finite commutative unital ring.*
*(i) If $R$ is not a local ring, then $\Gamma_D(R)$ is complete.*
*(ii) If all the zero-divisors of $R$ are nilpotent of order 2, then $\Gamma_D(R)$ is complete if and only if $R$ is not local.*

*Proof.* (i) Let $R$ be a non-local ring. First, let $R \cong \mathbb{Z}_2 \times F$ for some finite field $F$. Then any element $z$ of $Z(R)^*$ can be expressed either as $(\overline{1}, 0_F)$ or in the form $(\overline{0}, a)$ for some $a \in F^*$. As $z^2 \neq 0$, it follows that $Ann(Z(R)) = \{0\}$. Hence, $\Gamma_D(R)$ is complete by Theorem 3. Next, let $R \ncong \mathbb{Z}_2 \times F$, for any finite field $F$. Since $R$ is not local, we have from [Corollary 2.7, [2]] that there is no vertex in the zero-divisor graph $\Gamma(R)$ which is adjacent to every other vertex. This shows that $Ann(Z(R)) = \{0\}$, and so, $\Gamma_D(R)$ is complete by Theorem 3.
(ii) If $R$ is not local, then $\Gamma_D(R)$ is complete by part (i). Conversely, let $\Gamma_D(R)$ be complete. So $Ann(Z(R)) = \{0\}$ by Theorem 3. Since all elements of $Z(R)^*$ are

nilpotent of order 2, we have that $Ann(Z(R)) = \{0\}$ if and only if there is no vertex in the zero-divisor graph $\Gamma(R)$ which is adjacent to all other vertices. It then follows from [Corollary 2.7, [2]], that $R$ is not local. This completes the proof. □

We now show that apart from $Z(R)^*$ and $U(R)$, there is another important subset of $R$ which also induces a clique in $\Gamma_D(R)$.

**Lemma 4.** *Let $R$ be a finite commutative unital ring. Then $R - Ann(Z(R))$ induces a clique in $\Gamma_D(R)$.*

*Proof.* By Lemma 1, both $U(R)$ and $Z(R)^*$ induce cliques in $\Gamma_D(R)$. If $a \in Z(R) - Ann(Z(R))$, then $az \neq 0$ for some $z \in Z(R)^*$. So, by Lemma 2, $a \leftrightarrow u$ for any $u \in U(R)$. Since $U(R) \cup \big(Z(R) - Ann(Z(R))\big) = R - Ann(Z(R))$, this shows that $R - Ann(Z(R))$ induces a clique in $\Gamma_D(R)$. □

The above result helps us in determining the chromatic number of $\Gamma_D(R)$.

**Theorem 5.** *Let $R$ be a finite commutative unital ring. Then $\chi(\Gamma_D(R)) = \omega(\Gamma_D(R)) = |R - Ann(Z(R))|$.*

*Proof.* Since $R$ is a finite commutative unital ring, it can be expressed in the form $R_1 \times R_2 \times \cdots \times R_k$, where the $R_i$'s local rings and $k \in \mathbb{N}$. If $k = 1$, then $R$ is local and hence, $|U(R)| > |Z(R)| \geq |Ann(Z(R))|$. Again, if $k \geq 2$, then $R$ is not local and $Ann(Z(R)) = \{(0, 0, 0, \cdots, 0)\}$. So $|U(R)| = |U(R_1) \times U(R_2) \times \cdots \times U(R_k)| \geq |Ann(Z(R))|$ in this case as well. By Lemma 1, $Z(R)^*$ induces a clique in $\Gamma_D(R)$. Let $(Ann(Z(R)))^* = \{z_1, z_2, \ldots, z_m\}$ and $Z(R)^* = \{z_1, z_2, \ldots, z_m, z_{m+1}, \ldots, z_k\}$. We assign the colours $c_1, c_2, \ldots, c_k$ to the vertices $z_1, z_2, \ldots, z_k$, respectively. Let $U(R) = \{u_1, u_2, \ldots, u_m, u_{m+1}, \ldots, u_r\}$. By Lemma 2, $z_j \not\leftrightarrow u_s$ for any unit $u_s$ and for $j = 1, 2, \ldots, m$. So one can assign the colours $c_1, c_2, \ldots, c_m$ to the vertices $u_, u_2, \ldots, u_m$, respectively. Again, by Lemma 2, $z_j \leftrightarrow u_s$ for any unit $u_s$ and for $j = m + 1, m + 2, \ldots, k$. So one can assign the colours $c_{k+1}, c_{k+2}, \ldots, c_{k+r-m}$ to the remaining vertices $u_{m+1}, u_{m+2}, \ldots, u_r$. This shows that $\chi(\Gamma_D(R)) \leq k + r - m = |R - Ann(Z(R))|$. By Lemma 4, $\omega(\Gamma_D(R)) \geq |R - Ann(Z(R))|$. So $\chi(\Gamma_D(R)) \geq \omega(\Gamma_D(R)) \geq |R - Ann(Z(R))|$. Combining these results, we have that $\chi(\Gamma_D(R)) = \omega(\Gamma_D(R)) = |R - Ann(Z(R))|$. □

**Corollary 3.**

$$\chi(\Gamma_D(\mathbb{Z}_n)) = \begin{cases} n - p & \text{if } n = p^m \text{ for some prime } p \text{ and } m > 2 \\ n - 1 & \text{otherwise .} \end{cases}$$

*Proof.* If $n = p^m$ for some prime $p$ and $m > 2$, then $|Ann(Z(\mathbb{Z}_n))| = p$. Next, let $n = p_1^{r_1} p_2^{r_2} \cdots p_k^{r_k}$, where $k \geq 2$ and the $p_i$'s are distinct primes. Then $\mathbb{Z}_n \cong \mathbb{Z}_{p_1^{r_1}} \times \mathbb{Z}_{p_2^{r_2}} \times \cdots \times \mathbb{Z}_{p_k^{r_k}}$ and hence, $Ann(Z(\mathbb{Z}_n)) = \{\overline{0}\}$. So the result follows from Theorem 5. □

## 3. $\Gamma_D(R)$ as a line graph of some graph $G$

In this section, we characterize those commutative unital rings $R$ for which the graph $\Gamma_D(R)$ is realizable as the line graph of some graph $G$.

**Definition 2.** [12] Let $G$ be a graph. Then the line graph of $G$ is the graph $L(G)$ whose vertex set consists of the edges of $G$, and two vertices in $L(G)$ are adjacent if the corresponding edges in $G$ have a common end-point.

**Theorem 6.** *The graph $\Gamma_D(R)$ is a line graph of some graph $G$ if and only if one of the following holds.*
*(i) $Ann(Z(R)) = \{0\}$,*
*(ii) $|Ann(Z(R))| = 2$ and $|Z(R) - Ann(Z(R))| = 2$,*
*(iii) $|Ann(Z(R))| \geq 2$ and $|Z(R) - Ann(Z(R))| \leq 1$.*

*Proof.* If one of the said conditions holds, then $\Gamma_D(R)$ is a line graph as shown below.

**Case 1.** Assume $Ann(Z(R)) = \{0\}$.
Then, by Theorem 3, $\Gamma_D(R) \cong K_{|R|-1} = L(K_{1,|R|-1})$.

**Case 2.** Suppose that $|Ann(Z(R))| = |Z(R) - Ann(Z(R))| = 2$.
Clearly, $|Z(R)^*| = 3$ and the zero-divisor graph $\Gamma(R)$ is not complete. It then follows from a list given in [14] that $R$ is isomorphic to one of the rings $\mathbb{Z}_8, \mathbb{Z}_2[x]/(x^3), \mathbb{Z}_4[x]/(2x, x^2-2)$. For each of the said rings, $\Gamma_D(R) = G_1^* = L(G_2^*)$, where $G_1^*$ and $G_2^*$ are shown in Figure 3, respectively.

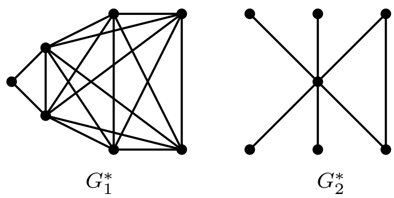


**Figure 3.** **The graphs $G_1^*$ and $G_2^*$**

**Case 3.** Assume that $|Ann(Z(R))| \geq 2$ and $|Z(R) - Ann(Z(R))| \leq 1$.
Let $|U(R)| = m$ and $|Z(R)^*| = n$. First, let $Z(R) = Ann(Z(R))$. Then $(Z(R))^2 = \{0\}$. In this case, $\Gamma_D(R) \cong K_m + K_n \cong L(K_{1,n} + K_{1,m})$. Next, let $|Z(R) - Ann(Z(R))| = 1$. Then $\Gamma_D(R)$ is isomorphic to $\overline{K_1 + K_{n-1,m}}$, which is the line graph of the graph $G_3^*$ shown in the Figure 4.

Conversely, let $\Gamma_D(R)$ be the line graph of some graph $G$. If possible, let the ring $R$ satisfy none of the conditions (i)-(iii). Then we have two possibilities as shown below.

**Case I.** Suppose that $|Z(R) - Ann(Z(R))| \geq 3$ and $|Ann(Z(R))| \geq 2$.
We consider a vertex subset $V_1 = \{z_1, z_2, z_3, z, 1\}$, where $z \in Ann(Z(R))^*$, and

$z_1, z_2, z_3 \in Z(R) - Ann(Z(R))$. The subgraph of $\Gamma_D(R)$ induced by $V_1$ is isomorphic to $K_5 - e$ (cf. Figure 5), which is a forbidden subgraph for a line graph [Theorem 1,[15]]. So we arrive at a contradiction.

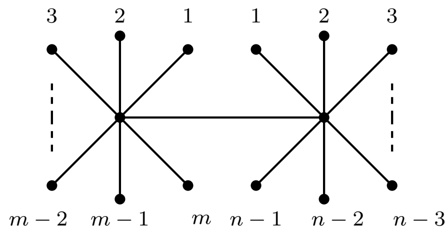


**Figure 4.** **The graph $G_3^*$**

**Case II.** Let $|Ann(Z(R))| \geq 3$ and $|Z(R) - Ann(Z(R))| = 2$.
Here we consider a vertex subset $V_2 = \{z_1, z_2, z_3, z_4, u_1, u_2\}$, where $z_1, z_2 \in Ann(Z(R))^*, z_3, z_4 \in Z(R) - Ann(Z(R))$ and $u_1, u_2 \in U(R)$. In $\Gamma_D(R)$, $V_2$ induces a subgraph isomorphic to $G_4$ (cf. Figure 5), which is a forbidden subgraph for a line graph (cf. [Theorem 1, [15]]). So we again reach a contradiction.

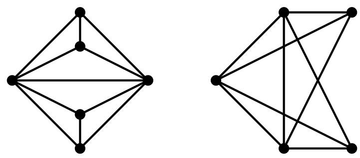

**Figure 5.** **Graphs $G_4$ and $K_5 - e$**

Thus, $\Gamma_D(R)$ is a line graph if and only if one of the conditions (i)-(iii) is satisfied. □

**Corollary 4.** *$\Gamma_D(\mathbb{Z}_n)$ is a line graph of some graph $G$ if and only if $n = 8$ or $n = p_1^{r_1} p_2^{r_2} \cdots p_k^{r_k}$, where $k \geq 2$, $p_i$'s are distinct primes and $r_i \geq 1$ $\forall i = 1, 2, \ldots, k$.*

*Proof.* Since $|Ann(\mathbb{Z}_8)| = 2$ and $|Z(\mathbb{Z}_8) - Ann(\mathbb{Z}_8)| = 2$, it follows by Theorem 6 that $\Gamma_D(\mathbb{Z}_8)$ is a line graph. Next, let $n = p_1^{r_1} p_2^{r_2} \cdots p_k^{r_k}$ for $k (\geq 2)$ distinct primes $p_i$'s, where $r_t \geq 1$ for $t = 1, 2, \ldots, k$. Then $Ann(Z(\mathbb{Z}_n)) = \{0\}$ and so, $\Gamma_D(\mathbb{Z}_n)$ is a line graph. For all other values of $n$, it is easy to show that $\mathbb{Z}_n$ satisfies none of the three conditions mentioned in Theorem 6 and hence $\Gamma_D(R)$ is not a line graph for those values of $n$. This completes the proof. □

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**Conflict of interest.** The authors declare that they have no conflict of interest.

**Data Availability.** Data sharing is not applicable to this article as no datasets were generated or analyzed during the current study.

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## Appendix: All zero-divisor associate graphs of at most 15 vertices

We here give the complete list of graphs realizable as $\Gamma_D(R)$ with number of vertices at most 15. For this, we need to consider the local (i.e., those having a unique maximal ideal) as well as non-local rings of order at most 16 (since $\Gamma_D(R)$ has $|R|-1$ vertices). Several lists given in [8, 10, 11, 14] help us list all such commutative unital rings, following some routine calculations. In this regard, the following result particularly helps in the characterization of the aforementioned local rings.

**Theorem 7.** *Let $(R, M)$ be a finite commutative local ring with 1 and with unique maximal ideal $M$. Then $|R| = p^{nr}$ and $|J(R)| = |M| = |Z(R)| = p^{(n-1)r}$ for some prime $p$ and $n, r \in \mathbb{Z}$.*

*Proof.* Clearly, $M = Z(R) = J(R)$. This shows that $Z(R)$ forms an additive group. So, by [Proposition 1, [10]], $|R| = p^{nr}$ and $|J(R)| = |M| = |Z(R)| = p^{(n-1)r}$ for some prime $p$ and $n, r \in \mathbb{Z}$. □

The corresponding zero-divisor associate graphs having at most 16 vertices are given in the following tables given in this appendix.

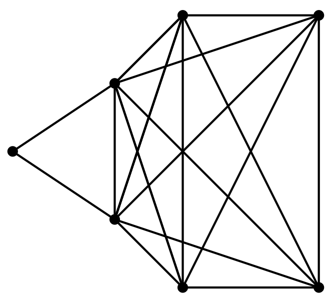

**Figure 6.** **The graph** $G_5$

| No. of Vertices | The Ring $R$ | $\|(Z(R))^*\|$ | The Graph $\Gamma_D(R)$ | Planarity |
|---|---|---|---|---|
| 1 | $\mathbb{Z}_2$ | 0 | $K_1$ | planar |
| 2 | $\mathbb{Z}_3$ | 0 | $2K_1$ | planar |
| 3 | $\mathbb{Z}_4$, $\mathbb{Z}_2[x]/(x^2)$ | 1 | $K_1 + K_2$ | planar |
| 3 | $\mathbb{F}_4$ | 0 | $3K_1$ | planar |
| 3 | $\mathbb{Z}_2 \times \mathbb{Z}_2$ | 2 | $K_3$ | planar |
| 4 | $\mathbb{Z}_5$ | 0 | $4K_1$ | planar |
| 5 | $\mathbb{Z}_6$ | 3 | $K_5$ | not planar |
| 7 | $\mathbb{Z}_7$ | 0 | $7K_1$ | planar |
| 7 | $\mathbb{Z}_8$, $\mathbb{Z}_2[x]/(x^3)$, $\mathbb{Z}_4[x]/(2x, x^2-2)$ | 3 | $G_5$ (cf. Figure 6) | not planar |
| 7 | $\mathbb{F}_8$ | 0 | $7K_1$ | planar |
| 7 | $\mathbb{Z}_2[x,y]/(x,y)^2$, $\mathbb{Z}_4[x]/(2,x)^2$ | 3 | $K_3 + K_4$ | planar |
| 7 | $\mathbb{Z}_2 \times \mathbb{F}_4$ | 4 | $K_7$ | not planar |
| 7 | $\mathbb{Z}_2 \times \mathbb{Z}_4$, $\mathbb{Z}_2 \times \mathbb{Z}_2[x]/(x^2)$ | 5 | $K_7$ | not planar |
| 7 | $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ | 6 | $K_7$ | not planar |
| 8 | $\mathbb{Z}_3 \times \mathbb{Z}_3$ | 4 | $K_8$ | not planar |
| 8 | $\mathbb{Z}_3[x]/(x^2)$, $\mathbb{Z}_9$ | 2 | $K_6 + K_2$ | not planar |
| 8 | $\mathbb{F}_9$ | 0 | $8K_1$ | planar |
| 9 | $\mathbb{Z}_2 \times \mathbb{Z}_5$ | 5 | $K_9$ | not planar |
| 10 | $\mathbb{Z}_{11}$ | 0 | $10K_1$ | planar |
| 11 | $\mathbb{Z}_3 \times \mathbb{F}_4$ | 5 | $K_{11}$ | not planar |
| 11 | $\mathbb{Z}_3 \times \mathbb{Z}_4$, $\mathbb{Z}_3 \times \mathbb{Z}_2[x]/(x^2)$ | 7 | $K_{11}$ | not planar |
| 11 | $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3$ | 9 | $K_{11}$ | not planar |
| 12 | $\mathbb{Z}_{13}$ | 0 | $12K_1$ | planar |
| 13 | $\mathbb{Z}_2 \times \mathbb{Z}_7$ | 7 | $K_{13}$ | not planar |
| 14 | $\mathbb{Z}_3 \times \mathbb{Z}_5$ | 6 | $K_{14}$ | not planar |
| 15 | $\mathbb{F}_{16}$ | 0 | $15K_1$ | planar |
| 15 | $\mathbb{Z}_2[x]/(x^4), \mathbb{Z}_{16}, \mathbb{Z}_4[x]/(2+x^2), \mathbb{Z}_4[x]/(x^3-2, 2x^2, 2x)$, $\mathbb{Z}_4[x]/(x^2+2x)$ | 7 | $\overline{K_{1,8}+6K_1}$ | not planar |
| 15 | $\mathbb{Z}_8[x]/(2x, x^2+4), \mathbb{Z}_4[x]/(x^2), \mathbb{Z}_4[x]/(x^3-x^2-2, 2x^2, 2x)$, $\mathbb{Z}_2[x,y]/(x^2, y^2-xy)$ | 7 | $\overline{K_{1,8}+6K_1}$ | not planar |
| 15 | $\mathbb{Z}_4[x,y]/(x^2, y^2-xy, xy-2, 2x, 2y)$, $\mathbb{Z}_4[x,y]/(x^2, y^2, xy-2, 2x, 2y)$, $\mathbb{Z}_2[x,y]/(x^2, y^2)$ | 7 | $\overline{K_{1,8}+6K_1}$ | not planar |
| 15 | $\mathbb{Z}_4[x]/(x^2+3x)$ | 7 | $K_{15}$ | not planar |
| 15 | $\mathbb{Z}_8[x]/(2x, x^2)\mathbb{Z}_4[x]/(x^3, 2x^2, 2x)$, $\mathbb{Z}_2[x,y]/(x^3, xy, y^2)$ | 7 | $\overline{G_5}$ (cf. Figure 6) | not planar |
| 15 | $\mathbb{Z}_2[x,y,z]/(x,y,z)^2$, $\mathbb{Z}_4[x,y]/(x^2, y^2, xy, 2x, 2y)$ | 7 | $K_7 + K_8$ | not planar |
| 15 | $\mathbb{F}_4[x]/(x^2)$, $\mathbb{Z}_4[x]/(x^2+x+1)$ | 3 | $K_3 + K_{12}$ | not planar |
| 15 | $\mathbb{F}_4 \times \mathbb{F}_4$ | 6 | $K_{15}$ | not planar |
| 15 | $\mathbb{Z}_2 \times \mathbb{F}_8$ | 8 | $K_{15}$ | not planar |
| 15 | $\mathbb{Z}_4 \times \mathbb{F}_4$, $\mathbb{Z}_2[x]/(x^2) \times \mathbb{F}_4$ | 9 | $K_{15}$ | not planar |
| 15 | $\mathbb{Z}_2 \times \mathbb{Z}_8$, $\mathbb{Z}_2 \times \mathbb{Z}_2[x]/(x^2)$, $\mathbb{Z}_2 \times \mathbb{Z}_4[x]/(2x, x^2-2)$ | 11 | $K_{15}$ | not planar |
| 15 | $\mathbb{Z}_2 \times \mathbb{Z}_2[x,y]/(x,y)^2$, $\mathbb{Z}_2 \times \mathbb{Z}_4[x]/(2,x)^2$ | 11 | $K_{15}$ | not planar |
| 15 | $\mathbb{Z}_4 \times \mathbb{Z}_4$, $\mathbb{Z}_4 \times \mathbb{Z}_2[x]/(x^2)$, $\mathbb{Z}_2[x]/(x^2) \times \mathbb{Z}_2[x]/(x^2)$ | 11 | $\overline{K_{1,4}+10K_1}$ | not planar |
| 15 | $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{F}_4$ | 12 | $K_{15}$ | not planar |
| 15 | $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_4$, $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2[x]/(x^2)$ | 13 | $K_{15}$ | not planar |
| 15 | $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ | 14 | $K_{15}$ | not planar |