Communications in Combinatorics and Optimization
Vol. xx, No. xx (xxxx), pp. 1-11
https://doi.org/10.22049/cco.2023.28704.1679

CCO
Commun. Comb. Optim.

Research Article

On the distance-transitivity of the folded hypercube

Seyed Morteza Mirafzal

Department of Mathematics, Faculty of Basic Science, Lorestan University,
Khorramabad, Iran
mirafzal.m@lu.ac.ir

Received: 1 June 2023; Accepted: 27 October 2023
Published Online: 3 November 2023

Abstract: The folded hypercube FQ_n is the Cayley graph $\mathrm{Cay}(\mathbb{Z}_2^n, S)$, where $S = \{e_1, e_2, \ldots, e_n\} \cup \{u = e_1 + e_2 + \cdots + e_n\}$, and $e_i = (0, \ldots, 0, 1, 0, \ \ldots, 0)$, with 1 at the ith position, $1 \leq i \leq n$. In this paper, we show that the folded hypercube FQ_n is a distance-transitive graph. Then, we study some properties of this graph. In particular, we show that if $n \geq 4$ is an even integer, then the folded hypercube FQ_n is an *automorphic* graph, that is, FQ_n is a distance-transitive primitive graph which is not a complete or a line graph.

Keywords: distance-transitive graph, folded hypercube, distance regular graph, primitive graph, automorphic graph.

AMS Subject classification: 05C25, 94C15

1. Introduction

In this paper, a graph $\Gamma = (V, E)$ is considered as an undirected simple graph where $V = V(\Gamma)$ is the vertex-set and $E = E(\Gamma)$ is the edge-set. For all the terminology and notation not defined here, we follow [1, 3, 6].

Let $n \geq 3$ be an integer. The hypercube Q_n of dimension n is the graph with the vertex-set $\{(x_1, x_2, \ldots, x_n) \mid x_i \in \{0, 1\}\}$, two vertices $(x_1, x_2, \ldots, x_n)$ and $(y_1, y_2, \ldots, y_n)$ are adjacent if and only if $x_i = y_i$ for all but one i. As a topology for an interconnection network of a multiprocessor system, the hypercube is a widely used and well-known model. The hypercube Q_n possesses many interesting properties, for example, its regularity, diameter and connectivity all are n. Also, it is bipartite and thus Q_n is 2-colorable. Moreover it is highly semmetric, that is, Q_n is vertex and edge-transitive [1, 6, 22]. There are many invariants of Q_n, for instance, generalized hypercube, folded hypercube, twisted hypercube, augmented hypercube and enhanced hypercube [2, 8, 22].

As a variant of the hypercube, the n-dimensional folded hypercube proposed first in [4]. The folded hypercube FQ_n of dimension n, is the graph obtained from the hypercube Q_n by adding edges, called complementary edges, between any two vertices $x = (x_1, x_2, \dots, x_n)$, $y = (\bar{x_1}, \bar{x_2}, \dots, \bar{x_n})$, where $\bar{1} = 0$ and $\bar{0} = 1$. The folded hypercube FQ_n has some interesting properties, for example although it is regular of degree $n+1$ (while the hypercube Q_n is regular of degree n), its diameter is almost half of the hypercube Q_n, that is, $\lceil \frac{n}{2} \rceil$ [4]. It can be shown that the hypercube Q_n is the Cayley graph $\mathrm{Cay}(\mathbb{Z}_2^n, B)$, where $B = \{e_1, e_2, \dots, e_n\}$, e_i is the element of $\mathbb{Z}_2^n$ with 1 in the ith position and 0 in the other positions for, $1 \le i \le n$. Also, the folded hypercube FQ_n is the Cayley graph $\mathrm{Cay}(\mathbb{Z}_2^n, S)$, where $S = B \cup \{u = e_1 + e_2 + \dots + e_n\}$. Hence the hypercube Q_n and the folded hypercube FQ_n are vertex-transitive graphs. Since Q_n is Hamiltonian [9, 23] and it is a spanning subgraph of FQ_n, so FQ_n is Hamiltonian. Some properties of the folded hypercube FQ_n are discussed in [5, 9, 11, 21, 24].
The graphs shown in Figure 1. are the folded hypercubes FQ_3 and FQ_4

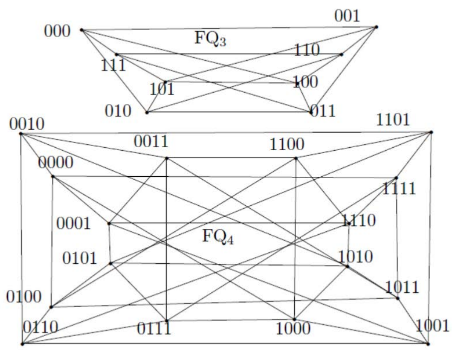

Figure 1. **The folded hypercubes FQ_3 and FQ_4**

We say that the graph Γ is *distance-transitive* if for all vertices u, v, x, y of Γ such that $d(u, v) = d(x, y)$, where $d(u, v)$ denotes the distance between the vertices u and v in Γ, there is an automorphism π in $\mathrm{Aut}(\Gamma)$ such that $\pi(u) = x$ and $\pi(v) = y$. The class of distance-transitive graphs contains many of interesting and important graphs. It is easy to see that the complete graphs K_n and the complete bipartite graph $K_{n,n}$ are distance-transitive. Also, it is not hard to check that the cycle C_n is distance-transitive. A more interesting example is the Petersen graph [6]. Another interesting example is the crown graph [12, 13, 17]. The class of Johnson graphs is one the important subclass of distance-transitive graphs [3, 13, 14, 18]. Another family of examples is the hypercube Q_n [1, 3, 6]. Distance-transitive graphs have been extensively studied from various aspects, by various authors and some of the works include [7, 10, 14, 16].
The fact that the folded hypercube is an edge-transitive graph, is one of the main results that has been shown in [9]. The result has been generalized in [11] by showing that the folded hypercube is in fact an arc-transitive graph.

In this paper we show, by an elementary and self-contained method, that the folded hypercube is in fact distance-transitive and hence distance-regular. Then, we study some properties of this graph. In particular, we show that if $n \geq 4$ is an even integer, then the hypercube FQ_n is an *automorphic* graph, that is, FQ_n is a distance-transitive primitive graph which is not a complete or a line graph.

2. Preliminaries

The graphs $\Gamma_1 = (V_1, E_1)$ and $\Gamma_2 = (V_2, E_2)$ are called *isomorphic*, if there is a bijection $\alpha : V_1 \longrightarrow V_2$ such that $\{a, b\} \in E_1$ if and only if $\{\alpha(a), \alpha(b)\} \in E_2$ for all $a, b \in V_1$. In such a case the bijection α is called an *isomorphism*. An *automorphism* of a graph Γ is an isomorphism of Γ with itself. The set of automorphisms of Γ with the operation of composition of functions is a group called the *automorphism group* of Γ and denoted by $\mathrm{Aut}(\Gamma)$.

The group of all permutations of a set V is denoted by $\mathrm{Sym}(V)$ or just $\mathrm{Sym}(n)$ when $|V| = n$. A *permutation group* G on V is a subgroup of $\mathrm{Sym}(V)$. In this case we say that G *acts* on V. If G acts on V we say that G is *transitive* on V (or G acts *transitively* on V) if given any two elements u and v of V, there is an element β of G such that $\beta(u) = v$. If Γ is a graph with vertex-set V then we can view each automorphism of Γ as a permutation on V and so $\mathrm{Aut}(\Gamma) = G$ is a permutation group on V.

A graph Γ is called *vertex-transitive* if $\mathrm{Aut}(\Gamma)$ acts transitively on $V(\Gamma)$. We say that Γ is *edge-transitive* if the group $\mathrm{Aut}(\Gamma)$ acts transitively on the edge-set E, namely, for any $\{x, y\}, \{v, w\} \in E(\Gamma)$, there is some π in $\mathrm{Aut}(\Gamma)$, such that $\pi(\{x, y\}) = \{v, w\}$. We say that Γ is *symmetric* (or *arc-transitive*) if for all vertices u, v, x, y of Γ such that u and v are adjacent, and also, x and y are adjacent, there is an automorphism π in $\mathrm{Aut}(\Gamma)$ such that $\pi(u) = x$ and $\pi(v) = y$. Note that if Γ is arc-transitive, then it is edge-transitive. Also, it is not hard to see that every distance-transitive graph is an arc-transitive graph. The automorphism group of a graph and its action on the vertex and edge or arc sets of a graph have crucial roles in finding some topological properties of the graph. Some recent works in this field include [11, 14, 15, 17, 19].

Let G be any abstract finite group with identity 1 and suppose Ω is a subset of G with the properties:
(i) $x \in \Omega \Longrightarrow x^{-1} \in \Omega$, (ii) $1 \notin \Omega$.
The *Cayley graph* Γ=Cay(G, Ω) is the (simple) graph whose vertex-set and edge-set are defined as follows: $V(\Gamma) = G$, $E(\Gamma) = \{\{g, h\} \mid g^{-1}h \in \Omega\}$.
It can be shown that the Cayley graph Γ=Cay(G, Ω) is connected if and only if the set Ω is a generating set in the group G [1].

The group G is called a semidirect product of N by Q, denoted by $G = N \rtimes Q$, if G contains subgroups N and Q such that: (i)$N \trianglelefteq G$ (N is a normal subgroup of G); (ii) $NQ = G$; and (iii) $N \cap Q = 1$ [20].
It has been shown in [11] that if $n > 3$, then $\mathrm{Aut}(FQ_n)$ is a semidirect product of N

by M, where N is isomorphic to the Abelian group $\mathbb{Z}_2^n$ and M is isomorphic to the group $Sym(n+1)$.

3. Main results

Let $\Gamma = (V, E)$ be a graph with diameter D. For each vertex v of Γ we let $\Gamma_i(v) = \{x \in V \mid d(x, v) = i\}$, $0 \leq i \leq D$. In other words $\Gamma_i(v)$ is the set of vertices of Γ which are at distance i from the vertex v. The stabilizer subgroup of v in A=Aut(Γ) denoted by A_v is defined to be the subgroup of automorphisms g of Γ such that $g(v) = v$. We have the following result.

Proposition 1. *[1, 6] Let $\Gamma = (V, E)$ be a vertex-transitive graph with diameter D and v be an arbitrary vertex of Γ. Then Γ is a distance-transitive graph if and only if there is a subgroup H of $Aut(\Gamma)_v$=A_v such that H acts transitively on every Γ_i, $0 \leq i \leq D$.*

One of the interesting properties in the folded hypercube, concerning the distances between vertices, is shown in the following result.

Proposition 2. *Let $\Gamma = FQ_n$. If $1 \leq i \leq \lceil \frac{n}{2} \rceil$, then $\Gamma_i(0)$=$\{v \mid w(v) = i\} \cup \{x \mid w(x) = n - i + 1\}$=$\{v \mid w(v) = i\} \cup \{v + u \mid w(v) = i - 1\}$,where $w(v)$ is the number of 1s in the n-tuple v ($u = e_1 + \cdots + e_n$).*

Proof. Let v be a vertex in the hypercube Q_n. Let $w(v)$ denote the weight of v, that is, the number of 1s in the n-tuple v. Let $0 = (0, 0, \ldots, 0)$ be the zero n-tuple in Q_n. It is easy to see that $d_{Q_n}(0, v) = w(v)$. Thus in the hypercube Q_n we have $Q_{n_i}(0) = \{y \in V(Q_n) \mid w(y) = i\}$. We know that the diameter of the folded hypercube FQ_n is $\lceil \frac{n}{2} \rceil$. Now it is easy to check that if $1 \leq i \leq \lceil \frac{n}{2} \rceil$, and $w(v) = i$ or $w(v) = n - i + 1$, then the distance between the zero vertex and v in FQ_n is i. In fact we can check that if $\Gamma = FQ_n$, then $\Gamma_i(0)$=$\{v \mid w(v) = i\}$ $\cup\{v + u \mid w(v) = i - 1\}$, where $u = e_1 + e_2 + \cdots + e_n$, e_j is the element of $\mathbb{Z}_2^n$ with 1 in the jth position and 0 in the other positions for $1 \leq j \leq n$. Note that if $w(x) = j - 1$, $1 \leq j \leq \lceil \frac{n}{2} \rceil$, then $w(u + x) = n - (j - 1) = n - j + 1$, but $d_{FQ_n}(0, u + x) = j$. □

We now are ready to prove the following important theorem.

Theorem 1. *Let $n \geq 4$ be an integer. Then the folded hypercube FQ_n is a distance-transitive graph.*

Proof. Let $\Gamma = FQ_n$ and A=Aut(Γ). Let $v = 0$. In the rest of the proof we need some information about A_0, the stabilizer subgroup of the vertex 0 in the group A, and its action on the vertex-set of Γ explicitly. Note that the Abelian group Z_2^n is also a vector space over the field $F = \{0, 1\}$ and $B = \{e_1, e_2, \ldots, e_n\}$ is a basis of this vector space. It is easy to check that any n-subset of the set $S = B \cup \{u = e_1 + e_2 + \cdots + e_n\}$

is linearly independent over F and hence it is a basis of the vector space $\mathbb{Z}_2^n$. Let T be a subset of S with n elements and $f: B \longrightarrow T$ be a one to one function. We can extend f over $\mathbb{Z}_2^n$ linearly to a mapping $e(f)$, that is, if $v = a_1e_1 + a_2e_2 + \cdots + a_ne_n$, then $e(f)(v) = a_1f(e_1) + a_2f(e_2) + \cdots + a_nf(e_n)$. Thus $e(f)$ is a non-singular linear mapping of the vector space $\mathbb{Z}_2^n$ into itself such that $e(f)|_B = f$. Since B and T are bases of the vector space $\mathbb{Z}_2^n$, hence $e(f)$ is a permutation of $\mathbb{Z}_2^n$. Since $e(f)$ is an automorphism of the group $\mathbb{Z}_2^n$ which fixes the generating set S of the Cayley graph FQ_n, hence it is an automorphism of the folded hypercube FQ_n. Now it is easy to check that, $H = \{e(f) \mid f: B \longrightarrow T,\ T \subset S, |T| = n,\ f\ \textit{is a one to one mapping}\}$, is a subgroup of the stabilizer group of the vertex $v = 0$. (In fact, it is not hard to show that H=A_0.) The graph FQ_n is a Cayley graph, thus it is a vertex-transitive graph, hence by Proposition 1, it is sufficient to show that the action of H on the set $\Gamma_i(0) = \Gamma_i$ is transitive, where $\Gamma_i(0)$ is the set of vertices at distance i from the vertex $v = 0$. Let x and y be two vertices in Γ_i. Then either $w(x) = w(y)$ or $w(x) \neq w(y)$.
First suppose that $w(x) = w(y)$. Let $x = e_{k_1} + \cdots + e_{k_i}$ and $y = e_{j_1} + \cdots + e_{j_i}$. There are vertices $e_{x_1}, \ldots, e_{x_{n-i}}$ and $e_{y_1}, \ldots, e_{y_{n-i}}$ in FQ_n such that $\{e_{k_1}, \ldots, e_{k_i}, e_{x_1}, \ldots, e_{x_{n-i}}\}$=$B$=$\{e_1, e_2, \ldots, e_n\}$=$\{e_{j_1}, \ldots, e_{j_i}, e_{y_1}, \ldots, e_{y_{n-i}}\}$. Let f be the permutation on the set B which is defined by the rule, $f(e_{k_r}) = e_{j_r}, 1 \leq r \leq i$, and $f(e_{x_l}) = e_{y_l}, 1 \leq l \leq n - i$. We now can see that $e(f)(x) = y$, where $e(f)$ is the linear extension of f to $\mathbb{Z}_2^n$. Note that $e(f) \in H$.
Now suppose that $w(x) \neq w(y)$. Without loss of generality we can assume that $w(x) = i$ and $w(y) = n - i + 1$. By Proposition 2, there is a vertex y_1 in Γ_{i-1} such that $w(y_1) = i - 1$ and $y = u + y_1$ (in fact $y_1 = y + u$).
Let $x = e_{k_1} + \cdots + e_{k_i}$ and $y_1 = e_{j_2} + \cdots + e_{j_i}$. There are vertices $e_{x_1}, \ldots, e_{x_{n-i}}$ and $e_{y_1}, \ldots, e_{y_{n-i}}$ in FQ_n such that $\{e_{k_1}, \ldots, e_{k_i}, e_{x_1}, \ldots, e_{x_{n-i}}\}$=$B$=$\{e_1, e_2, \ldots, e_n\}$ and $\{u, e_{j_2}, \ldots, e_{j_i}, e_{y_1}, \ldots, e_{y_{n-i}}\} = T$, $|T| = n$, $T \subset S$.
Let $f: B \longrightarrow T$ be a one to one function such that$f(e_{k_1}) = u$, $f(e_{k_r}) = e_{y_r}$, $2 \leq r \leq i$, $f(e_{x_r}) = e_{y_r}$, $1 \leq r \leq n - i$.
Now it is clear that for the automorphism $e(f)$ we have $e(f)(x) = y$. Now, since $e(f) \in H$, the result follows.

□

A block B, in the action of a group G on a set X, is a subset of X such that $B \cap g(B) \in \{B, \emptyset\}$, for each g in G. If G is transitive on X, then we say that the permutation group (X, G) is primitive if the only blocks are the trivial blocks, that is, those with cardinality 0,1 or $|X|$. In the case of an imprimitive permutation group (X, G), the set X is partitioned into a disjoint union of non-trivial blocks, which are permuted by G. We refer to this partition as a block system. A graph Γ is said to be primitive or imprimitive according to the group $\mathrm{Aut}(\Gamma)$ acting on $V(\Gamma)$ has the corresponding property. In the sequel, we need the following definition.

Definition 1. A graph $\Gamma = (V, E)$ of diameter D is said to be *antipodal* if for any $x, v, w \in V$ such that $d(x, v) = d(x, w) = D$, then we have $d(v, w) = D$ or $v = w$.

Let $\Gamma_i(x)$ denote the set of vertices of Γ at distance i from the vertex x. Let Γ be a distance-transitive graph. From Definition 1, it follows that if $\Gamma_D(x)$ is a singleton set, then the graph Γ is antipodal. It is easy to see that the hypercube Q_n is antipodal, since every vertex u has a unique vertex at maximum distance from it. Note that this graph is at the same time bipartite. We have the following result [1].

Proposition 3. *A distance-transitive graph Γ of diameter D has a block $X = \{v\} \cup \Gamma_D(v)$ if and only if Γ is antipodal, where $\Gamma_D(v)$ is the set of vertices of Γ at distance D from the vertex v.*

Also, we have the following important result [1].

Theorem 2. *An imprimitive distance-transitive graph is either bipartite or antipodal. (Both possibilities can occur in the same graph.)*

We have the following result.

Proposition 4. *[23] The folded hypercube FQ_n is a bipartite graph if and only if n is an odd integer.*

We now can state and prove the following fact concerning the folded hypercube FQ_n.

Theorem 3. *Let $n \geq 4$ be an integer. Then, the folded hypercube FQ_n is a primitive distance-transitive graph if and only if n is an even integer.*

Proof. By Theorem 1, the folded hypercube FQ_n is a distance-transitive graph. If n is an odd integer, then by Proposition 4, the folded hypercube FQ_n is a bipartite graph, thus by Theorem 2, it is imprimitive.
Let n be an even integer. Therefore, by Proposition 4, FQ_n is not bipartite. Let $n = 2m$. Thus the diameter of the FQ_n is m. Let v be a vertex in FQ_n such that $w(v) = m$. Let $t = u + v$, where $u = e_1 + e_2 + \cdots + e_n$. Hence $w(t) = m$. This follows that $d(0, v) = d(0, t) = m$, but $d(v, t) = 1 \neq m$. Hence FQ_{2m} is not antipodal. Thus, by Theorem 2, FQ_{2m} is primitive. □

Let $\Gamma = (V, E)$ be a simple connected graph with diameter D. A *distance-regular* graph $\Gamma = (V, E)$, with diameter D, is a regular connected graph of valency k with the following property. There are positive integers

$$b_0 = k, b_1, \ldots, b_{D-1}; c_1 = 1, c_2, \ldots, c_D,$$

such that for each pair (u, v) of vertices satisfying $u \in \Gamma_i(v)$, we have
(1) the number of vertices in $\Gamma_{i-1}(v)$ adjacent to u is c_i, $1 \leq i \leq D$.

(2) the number of vertices in $\Gamma_{i+1}(v)$ adjacent to u is b_i, $0 \leq i \leq D-1$.
The intersection array of Γ is $i(\Gamma) = \{k, b_1, \ldots, b_{D-1}; 1, c_2, ..., c_D\}$.
It is easy to show that if Γ is a distance-transitive graph, then it is distance-regular [1]. Hence, the hypercube $Q_n,\ n > 2$ is a distance-regular graph. We can verify by an easy argument that the intersection array of Q_n is

$$\{n, n-1, n-2, \ldots, 1; 1, 2, 3, \ldots, n\}.$$

In other words, for hypercube Q_n, we have $b_i = n - i$, $c_i = i$, $1 \leq i \leq n-1$, and $b_0 = n$, $c_n = n$. In the following theorem, we determine the intersection array of the Folded hypercube FQ_n.

Proposition 5. *Let $n > 3$ be an integer and $\Gamma = FQ_n$ be the folded hypercube. Let D denote the diameter of FQ_n. Then for the intersection array of this graph we have $b_i{=}n+1-i$, $0 \leq i < D$. $c_i{=}i$, $1 \leq i \leq D$ (note that $D{=}\lceil \frac{n}{2} \rceil$).*

Proof. Nothing to what is stated in the proof of Proposition 2, the proof of the theorem is straightforward.

□

An *automorphic* graph is a distance-transitive graph whose automorphism group acts primitively on its vertices, and not a complete graph or a line graph [1].
Automorphic graphs are apparently very rare. For instance, there are exactly three cubic automorphic graphs [1]. It is clear that for $n \geq 3$, the graph FQ_n is not a complete graph. In the sequel, we show that if $n \geq 4$ is an even integer, then the graph FQ_n is an automorphic graph. In the first step, we show that FQ_n is not a line graph. In the rest of our paper , we need some information about the eigenvalues of FQ_n. We do not need the spectrum of FQ_n, that is, all the eigenvalues of FQ_n.
Let Γ be a graph with vertex set $V(\Gamma) = V = \{v_1, v_2, \ldots, v_n\}$ and edge set $E = E(\Gamma)$. The adjacency matrix $A = A(\Gamma) = [a_{ij}]$ of Γ is an $n \times n$ symmetric matrix of $0s$ and $1s$ with $a_{ij} = 1$ if and only if v_i and v_j are adjacent. The characteristic polynomial of Γ is the polynomial $P(G) = P(G, x) = \det(xI_n - A)$, where I_n denotes the $n \times n$ identity matrix. The spectrum of $A(\Gamma)$ is also called the spectrum of Γ. If the distinct eigenvalues are ordered by $\lambda_1 > \lambda_2 > \cdots > \lambda_r$, and their multiplicities are $m_1, m_2, \ldots, m_r$, respectively, then we write,

$$Spec(\Gamma) = \binom{\lambda_1, \lambda_2, \ldots, \lambda_r}{m_1, m_2, \ldots, m_r} \text{ or } Spec(\Gamma) = \{\lambda_1^{m_1}, \lambda_2^{m_2}, ..., \lambda_r^{m_r}\}.$$

Let Γ be a graph with vertex set $V = \{v_1, v_2, \ldots, v_n\}$ and adjacency matrix A, and the rows and columns of A are labeled by the set V. Let π be a permutation of the set V. We know that π can be represented by a permutation matrix $P_\pi = P = (p_{ij})$, where $p_{ij} = 1$ if $v_i = \pi(v_j)$, and $p_{ij} = 0$ otherwise. It is a well known fact that π is an automorphism of the graph Γ if and only if $AP = PA$ [1].

Let $\Gamma = (V, E)$ be a graph. The line graph $L(\Gamma)$ of the graph Γ is constructed by taking the edges of Γ as vertices of $L(\Gamma)$, and joining two vertices in $L(\Gamma)$ whenever the corresponding edges in Γ have a common vertex. Note that if $e = \{v, w\}$ is an edge of Γ, then its degree in the graph $L(\Gamma)$ is $\deg(v) + \deg(w) - 2$. Concerning the eigenvalues of the line graphs, we have the following fact [1].

Proposition 6. *If λ is an eigenvalue of a line graph $L(\Gamma)$, then $\lambda \geq -2$.*

Therefore, if $\lambda < -2$ is an eigenvalue of a graph Γ, then Γ is not a line graph.
In the proof of the following theorem, we need the following fact.

Proposition 7. *Let $\Gamma = FQ_n$. Then the mapping $\alpha : V(\Gamma) \to V(\Gamma)$, $\alpha(v) = v^c$, where v^c is the complement of v ($v^c = (\bar{x_1}, \bar{x_2}, ..., \bar{x_n})$, when $v = (x_1, x_2, \ldots, x_n)$, $\bar{1}$=0, $\bar{0}$=1), is an automorphism of Γ and the hypercube Q_n.*

Proof. The proof is straightforward.

□

Using this result we show that, without having the spectrum of the folded hypercube FQ_n in the hand, if $n \geq 4$, then FQ_n has an eigenvalue less than -2, hence it is not a line graph.

Theorem 4. *If $n \geq 4$, then FQ_n is not a line graph.*

Proof. If $\Gamma = FQ_n$, then by Proposition 7, the permutation $\alpha : V(\Gamma) \to V(\Gamma)$, $\alpha(v) = v^c$, where v^c is the complement of the set v, is an automorphism of the graph Γ and the hypercube Q_n. Thus, if P is the permutation matrix of α, then we have $MP = PM$ where M is the adjacency matrix of the graph FQ_n.
It is not hard to check that the adjacency matrix of FQ_n is of the form $M = A + P$, where A is the adjacency matrix of the hypercube Q_n. Since α is of order 2, then $P^2 = E$ where $E = I_h$ is the identity matrix of size h ($h = 2^n$). Hence if $p(x) = x^2 - 1$, then $p(P) = 0$. Thus, if μ is an eigenvalue of the matrix P, then $p(\mu) = 0$, namely, $\mu \in \{1, -1\}$. Since α is an automorphism of the graph Q_n, thus $AP = PA$. On the other hand, the matrices A and P are symmetric, hence the matrices A and P are diagonalizable, and therefore there is a basis $B = \{u_1, \ldots, u_h\}$ of $\mathbb{R}^h$ such that each u_i is an eigenvector of the matrices A and P [6]. Therefore, if $Au_i = \lambda_i u_i$, then $Mu_i = (A + P)u_i = \lambda_i u_i + t_i u_i = (\lambda_i + t_i)u_i$, where $t_i \in \{1, -1\}$. Every eigenvalue of the hypercube Q_n is of the form $n - 2i$, $0 \leq i \leq n$, [1]. Thus, for $i = n$, $n - 2n + 1 = -n + 1$, or $n - 2n - 1 = -n - 1$ is an eigenvalue of the folded hypercube FQ_n. Nothing that $n \geq 4$, FQ_n has an eigenvalue δ such that $\delta \leq -3$. Now, by Proposition 6, the hypercube FQ_n is not a line graph.

□

Theorem 5. *Let $n \geq 4$ be an integer. Then the folded hypercube FQ_n is an automorphic graph if and only if n is an even integer.*

Proof. By Theorem 3, the folded hypercube FQ_n is a primitive distance-transitive graph if and only if n is an even integer. By Theorem 4, FQ_n is not a line graph. It is clear that FQ_n is not a complete graph. We now conclude that the folded hypercube FQ_n is automorphic if and only if n is an even integer.

□

Acknowledgements. The author is thankful to the anonymous reviewers for their valuable comments and suggestions.

Conflict of interest. The authors declare that they have no conflict of interest.

Data Availability. Data sharing is not applicable to this article as no datasets were generated or analyzed during the current study.

References

[1] N. Biggs, *Algebraic Graph Theory. 2nd ed.*, no. 67, Cambridge university press, London, 1993.

[2] D. Boutin, S. Cockburn, L. Keough, S. Loeb, and P. Rombach, *Symmetry parameters of various hypercube families*, Art Discret. Appl. Math. **6** (2021), no. 2, Article number P2.06
https://doi.org/10.26493/2590-9770.1481.29d.

[3] A.E. Brouwer, A.M. Cohen, and A. Neumaier, *Distance-Regular Graphs*, Springer Berlin, 1989.

[4] A. El-Amawy and S. Latifi, *Properties and performance of folded hypercubes*, IEEE Trans. Parallel Distrib. Syst. **2** (1991), no. 1, 31–42
https://doi.ieeecomputersociety.org/10.1109/71.80187.

[5] M. Ghasemi, *Some results about the reliability of folded hypercubes*, Bull. Malays. Math. Sci. Soc. **44** (2021), 1093–1099
https://doi.org/10.1007/s40840-020-00999-4.

[6] C. Godsil and G.F. Royle, *Algebraic Graph Theory*, vol. 207, Springer Science & Business Media, New York, 2001.

[7] C.D. Godsil, R.A. Liebler, and C.E. Praeger, *Antipodal distance transitive covers of complete graphs*, European J. Combin. **19** (1998), no. 4, 455–478
https://doi.org/10.1006/eujc.1997.0190.

[8] L. Lu and Q. Huang, *Automorphisms and isomorphisms of enhanced hypercubes*, Filomat **34** (2020), no. 8, 2805–2812
http://dx.doi.org/10.2298/FIL2008805L.

[9] M. Ma and J.M. Xu, *Algebraic properties and panconnectivity of folded hypercubes*, Ars Combin. **95** (2010), 179–186.

[10] D. Marušič, *Bicirculants via imprimitivity block systems*, Mediterr. J. Math. **18** (2021), 1–15 https://doi.org/10.1007/s00009-021-01771-z.

[11] S.M. Mirafzal, *Some other algebraic properties of folded hypercubes*, Ars Combin. **124** (2011), 153–159.

[12] ______, *The automorphism group of the bipartite Kneser graph*, Proc. Math. Sci. **129** (2019), Article number: 34 https://doi.org/10.1007/s12044-019-0477-9.

[13] ______, *On the automorphism groups of connected bipartite irreducible graphs*, Proc. Math. Sci. **130** (2020), Article number 57 https://doi.org/10.1007/s12044-020-00589-1.

[14] ______, *Some remarks on the square graph of the hypercube*, Ars Math. Contemp. **23** (2023), no. 2, 2–6 https://doi.org/10.26493/1855-3974.2621.26f.

[15] ______, *Some algebraic properties of the subdivision graph of a graph*, Commun. Comb. Optim. (In press), https://doi.org/10.22049/cco.2023.28270.1494.

[16] S.M. Mirafzal and A. Zafari, *On the spectrum of a class of distance-transitive graphs*, Electron. J. Graph Theory Appl. **5** (2017), no. 1, 63–69 https://dx.doi.org/10.5614/ejgta.2017.5.1.7.

[17] ______, *Some algebraic properties of bipartite Kneser graphs*, Ars Combin. **153** (2020), 3–14.

[18] S.M. Mirafzal and M. Ziaee, *Some algebraic aspects of enhanced johnson graphs*, Acta Math. Univ. Comenian. **88** (2019), no. 2, 257–266.

[19] ______, *A note on the automorphism group of the Hamming graph*, Trans. Comb. **10** (2021), no. 2, 129–136 https://doi.org/10.22108/toc.2021.127225.1817.

[20] J.J. Rotman, *An Introduction to the Theory of Groups*, vol. 148, Springer Science & Business Media, New York, 2012 https://doi.org/10.1007/978-1-4612-4176-8.

[21] E. Sabir and J. Meng, *Structure fault tolerance of hypercubes and folded hypercubes*, Theoret. Comput. Sci. **711** (2018), 44–55 https://doi.org/10.1016/j.tcs.2017.10.032.

[22] J.M. Xu, *Topological Structure and Analysis of Interconnection Networks*, vol. 7, Springer Science & Business Media, New York, 2013 https://doi.org/10.1007/978-1-4757-3387-7.

[23] J.M. Xu and M. Ma, *Cycles in folded hypercubes*, Appl. Math. Lett. **19** (2006), no. 2, 140–145 https://doi.org/10.1016/j.aml.2005.04.002.

[24] M.M. Zhang and J.X. Zhou, *On g-extra connectivity of folded hypercubes*, Theoret. Comput. Sci. **593** (2015), 146–153 https://doi.org/10.1016/j.tcs.2015.06.008.