Communications in Combinatorics and Optimization
Vol. xx, No. x (xxxx), pp. 1-23
https://doi.org/10.22049/cco.2023.28619.1635

CCO
Commun. Comb. Optim.

Research Article

Global restrained Roman domination in graphs

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Received: 29 April 2023; Accepted: 6 November 2023
Published Online: 13 November 2023

Abstract: A global restrained Roman dominating function on a graph $G = (V, E)$ to be a function $f : V \to \{0, 1, 2\}$ such that f is a restrained Roman dominating function of both G and its complement $\overline{G}$. The weight of a global restrained Roman dominating function is the value $w(f) = \Sigma_{u \in V} f(u)$. The minimum weight of a global restrained Roman dominating function of G is called the global restrained Roman domination number of G and denoted by $\gamma_{grR}(G)$. In this paper we initiate the study of global restrained Roman domination number of graphs. We then prove that the problem of computing γ_{grR} is NP-hard even for bipartite and chordal graphs. The global restrained Roman domination of a given graph is studied versus to the other well known domination parameters such as restrained Roman domination number γ_{rR} and global domination number γ_g by bounding γ_{grR} from below and above involving γ_{rR} and γ_g for general graphs, respectively. We characterize graphs G for which $\gamma_{grR}(G) \in \{1, 2, 3, 4, 5\}$. It is shown that: for trees T of order n, $\gamma_{grR}(T) = n$ if and only if diameter of T is at most 5. Finally, the triangle free graphs G for which $\gamma_{grR}(G) = |V|$ are characterized.

Keywords: Roman dominating function, restrained domination, global domination, global restrained Roman domination.

AMS Subject classification: 05C69

1. Introduction

Cockayne et al. [8] defined *Roman dominating function* (RDF) on a graph $G = (V, E)$ to be a function $f : V \to \{0, 1, 2\}$ satisfying the condition that every vertex u for which $f(u) = 0$ is adjacent to at least one vertex v for which $f(v) = 2$. For a real valued

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function $f : V \to \mathbb{R}$, the *weight* of f is $w(f) = \Sigma_{v\in V} f(v)$ and for $S \subseteq V$, $f(S) = \Sigma_{v\in S} f(v)$, so $w(f) = f(V)$. The *Roman domination number*, denoted by $\gamma_R(G)$, is the minimum weight of an RDF in G; that is, $\gamma_R(G) = min\{w(f) : f \text{ is an RDF in } G\}$. An RDF of weight $\gamma_R(G)$ is called a $\gamma_R(G)$-function. Roman domination in graphs has been studied in several papers, some of them are [5–8, 14, 16, 17, 21, 23, 24, 27]. The definition of a Roman dominating function was motivated by an article in Scientific American by Ian Stewart entitled "Defend the Roman empire !" [26].

In the 4th century A.D., when the Roman Empire was under attack during the period of emperor Constantine the Great, it had the requirement that an army or a legion could be sent from its home to defend a neighboring location only if there was a second army which would stay and protect the home. Thus there are two types of armies, stationary and travelling. Each vertex with no army must have neighboring vertex with a travelling army. Stationary armies then dominate their own vertices and a vertex with two armies are dominated by its stationary army and its open neighborhood is dominated by the travelling army. The objective, of course, is to minimize the total number of legions needed.

By a graph $G = (V, E)$, we mean a simple, finite, undirected graph with $|V = V(G)| = n$ and $E = E(G)$. For graph theoretic terminology we refer to Charatrand and Lesniak [4]. A set of vertices S in a graph G is a *dominating set* of G if $N_G[S] = V$. The *domination number* $\gamma(G)$ of G is the minimum cardinality of a dominating set S in G, and a dominating set S of minimum cardinality is called a γ-set of G. The literature on domination and its variations in graphs has been surveyed and detailed in the book by Haynes et al [15]. A set $S \subseteq V$ is a *restrained dominating set* if every vertex not in S is adjacent to a vertex in S and to a vertex in $V - S$. The *restrained domination number* of G, denoted by $\gamma_r(G)$, is the smallest cardinality of a restrained dominating set of G. Restrained domination in graphs has been in studied, for example in [9, 11–13, 24]. A *restrained Roman dominating function* on a graph $G = (V, E)$ is a function $f : V \to \{0, 1, 2\}$ satisfying the condition that every vertex u for which $f(u) = 0$ is adjacent to at least one vertex v for which $f(v) = 2$ and at least one vertex w for which $f(w) = 0$. The minimum weight of a restrained Roman dominating function on a graph G is called the *restrained Roman domination number* of G and denoted by $\gamma_{rR}(G)$. A restrained Roman dominating function of weight $\gamma_{rR}(G)$ is called a $\gamma_{rR}(G)$-function, [1, 24].
A *Total restrained Roman dominating function* on a graph is a restrained Roman dominating function in which the subgraph induced by the vertices of positive weight has no isolated vertex [3].

A set $S \subseteq V$ is a *global dominating set* if S dominates both G and its complement $\overline{G}$. The *global domination number* $\gamma_g(G)$ is the minimum cardinality of a global dominating set in G. A global dominating set of minimum cardinality is called a $\gamma_g(G)$-set, [15]. The other global domination parameters are for instance, total global domination, global connected domination, global restrained domination, global outer connected domination number and outer independent global domination number of a graph and etc [2, 10, 18, 19, 22].

Pushpam and Padmapriea et al. [25] defined *global Roman dominating function*

(GRDF) on a graph $G = (V, E)$ to be a function $f : V \to \{0, 1, 2\}$ such that f is an RDF for both G and its complement $\overline{G}$.
A set $S \subseteq V$ is a *global restrained dominating set (GRDS)* if S is a restrained dominating set of both G and its complement $\overline{G}$. The *global restrained domination number* $\gamma_{gr}(G)$ is the minimum cardinality of a global restrained dominating set in G. A global restrained dominating set of minimum cardinality is called a $\gamma_{gr}(G)$-set [18].
In this paper, we extend the global restrained dominating set and the global Roman dominating function of a graph to the global restrained Roman dominating function as follows.

A *global restrained Roman dominating function* (GRRDF) on a graph $G = (V, E)$ is defined to be a function $f : V \to \{0, 1, 2\}$ such that f is a restrained Roman dominating function for both G and its complement $\overline{G}$. The minimum weight of a global restrained Roman dominating function on a graph G is called the *global restrained Roman domination number* of G and denoted by $\gamma_{grR}(G)$. A global restrained Roman dominating function of weight $\gamma_{grR}(G)$ is called a $\gamma_{grR}(G)$-function. For a global restrained Roman dominating function $f : V \to \{0, 1, 2\}$, we denote $V_i = \{v \in V : f(v) = i\}$, where $i \in \{0, 1, 2\}$ and sometimes write $f = (V_0, V_1, V_2)$.
This paper organized as follows. After introducing the notations in Section 2, we study the specific value of $\gamma_{grR}(G)$-function of the graphs in Section 3. In Section 4 the complexity of $\gamma_{grR}(G)$-function of graphs is investigated and show that GRRD is NP-complete for bipartite and chordal graphs. The GRRD versus other parameters domination are studied in Section 5 and in Section 6 we characterize the graphs G with $\gamma_{grR}(G) \in \{1, 2, 3, 4, 5\}$. Finally in Section 7, we show that, for trees T of order n, $\gamma_{grR}(T) = n$ if and only if $\mathrm{diam}(T) \leq 5$, and characterize the triangle free graphs G for which $\gamma_{grR}(G) = |V(G)|$.

2. Notations

The *degree* of a vertex v in a graph G is denoted by $d_G(v)$ or simply $d(v)$ when the graph is clear from the context. A vertex of degree zero in G is called an *isolated vertex*, while a vertex of degree one is called a *leaf* or a *pendant vertex* of G and a *support vertex* is a vertex which is adjacent to at least one leaf. We use $L(G)$ and $S(G)$ to show the set of leaves of G and the set of support vertices of G respectively. The *minimum degree* of G is the minimum degree among the vertices of G and is denoted by $\delta(G)$. The *maximum degree* of G is defined as the maximum degree among the vertices of G and is denoted by $\Delta(G)$. In a connected graph G, the *distance between two vertices* u and v is the number of edges in a shortest path joining u and v if any; and is denoted by $d(u, v)$. If $u \in V$ and $S \subseteq V$, then $d(u, S)$ denotes the minimum distance between u and any vertex of S. A set S of vertices is called *independent* if no two vertices in S are adjacent. The *eccentricity* of a vertex v is $ecc(v) = \max\{d(v, w); w \in V\}$. The radius of a graph G is $rad(G) = \min\{ecc(v) : v \in V\}$ and the *diameter* of the graph G is $\mathrm{diam}(G) = \max\{ecc(v) : v \in V\}$. The *center* of a graph is the set of all vertices of minimum eccentricity, that is, the set of all vertices u where the greatest distance

$d(u, v)$ to other vertices v is minimal. The *complement* of a graph G, denoted by $\overline{G}$, is a graph on the same vertices such that two distinct vertices of $\overline{G}$ are adjacent if and only if they are not adjacent in G.

For any set $S \subseteq V$, the *induced subgraph* S is the maximal subgraph of G with vertex set S and is denoted by $G[S]$. A *triangle free* graph is a graph with no induced cycle C_3. For any vertex $v \in V$, the *open neighborhood* of v is the set $N(v) = \{u \in V : uv \in E\}$ and the *closed neighborhood* is the set $N[v] = N(v) \cup \{v\}$. For a set $S \subseteq V$, the *open neighborhood* is $N(S) = \bigcup_{v \in S} N(v)$ and the *closed neighborhood* is $N[S] = N(S) \cup S$. Let $v \in S \subseteq V$. A vertex u is called a *private neighbor* of v with respect to S if $u \in N[v] - N[S - \{v\}]$.
A *complete graph*, *star graph*, *cycle*, and *path* with n vertices are denoted by K_n, $K_{1,n-1}$, C_n and P_n while $K_{m,n}$ and $S_{p,q}$ denote *complete bipartite graph* and *double star graph* of order $m + n$ and $p + q + 2$ respectively. We use [4, 15] as references for terminology and notation which are not explicitly defined here.

3. Specific values of global restrained Roman domination

In this section some basic properties of global restrained Roman dominating function are studied.
The function $f = (\emptyset, V(G), \emptyset)$ is a global restrained Roman dominating function of G, and any $\gamma_{grR}(G)$-function is a restrained Roman dominating function of G and $\overline{G}$. Therefore, it is routine to obtain.

Observation 1. For any graph G, $max\{\gamma_{rR}(G), \gamma_{rR}(\overline{G})\} \leq \gamma_{grR}(G) \leq n$.

One of the properties of global restrained domination number and global restrained Roman domination number of a graph is the following which is applicable for the other results.

Proposition 1. *Let G be a graph of order n and $f = (V_0, V_1, V_2)$ be a γ_{grR}-function of G. If $\gamma_{grR}(G) < n$, then $|V_2| \geq 2$ and $|V_0| \geq 4$.*

Proof. Since $\gamma_{grR}(G) < n$, so $V_1 \neq V(G)$, therefore $V_0 \neq \emptyset$ and $V_2 \neq \emptyset$. From definition of γ_{grR}-function, the induced subgraphs by V_0 in G and $\overline{G}$ have no isolated vertex. If $|V_0| \in \{2, 3\}$, then the subgraph induced by V_0 in G or in $\overline{G}$ has isolated vertex, which is a contradiction. Therefore $|V_0| \geq 4$. On the other hand, for every vertex $u \in V_0$, u has at least one neighbor in V_2 and also u is nonadjacent to at least one vertex of V_2. Therefore $|V_2| \geq 2$. □

As an immediate consequence, we have.

Corollary 1. *Let G be a graph of order n. If $n \leq 5$, then $\gamma_{grR}(G) = n$.*

In the following, we provide the γ_{grR} for specific graphs. Since $\gamma_{rR}(\overline{K_n}) = n$ and $\gamma_{rR}(K_{1,n-1}) = n$, we have.

Observation 2. For the complete graph K_n, $\gamma_{grR}(K_n) = n$ and for the star graph $K_{1,n-1}$, $\gamma_{grR}(K_{1,n-1}) = n$.

Proposition 2.

$$\gamma_{grR}(K_{m,n}) = \begin{cases} m+n & m \le 2 \text{ or } n \le 2. \\ 4 & m \ge 3 \text{ and } n \ge 3. \end{cases}$$

Proof. Let M and N be two partite sets of $K_{m,n}$ with $|M| = m$, $|N| = n$. If $m \ge 3$ and $n \ge 3$, suppose $x \in M$ and $y \in N$, then the function $g = (V(G) - \{x, y\}, \emptyset, \{x, y\})$ is a global restrained roman dominating function of $K_{m,n}$ and by Proposition 1, $\gamma_{grR}(K_{m,n}) = 4$.
Now let $m \le 2$ or $n \le 2$. Without lose of generality let $m \le 2$. If $m = 1$, then $K_{m,n}$ is a star, and $\gamma_{grR}(K_{m,n}) = |V(G)| = m + n$.
If $m = 2$ and $n \ge 2$, suppose $M = \{s, t\}$ and $f = (V_0, V_1, V_2)$ is a γ_{grR}-function. We claim that $V_2 = \emptyset$ and so $V_0 = \emptyset$, $V_1 = V(G)$ and $\gamma_{grR}(G) = m + n$. Let to the contrary $V_2 \ne \emptyset$. Then $|V_2| \ge 2$ and $|V_0| \ge 4$. Also $V_2 \cap M \ne \emptyset$, $V_2 \cap N \ne \emptyset$ and $M \nsubseteq V_2$, Since V_0 is not independent. Hence, $|V_2 \cap M| = 1$. Let $s \in V_2$ and $t \notin V_2$. If $f(t) = 0$, then t is independent in $\overline{G}$ a contradiction. Then we have $f(t) = 1$ and so $V_0 \subseteq N$ is independent, a contradiction. Hence $V_2 = 0$ and the proof is complete. □

For any $\gamma_{grR}(G)$-function $f = (V_0, V_1, V_2)$ of G, since V_0 has no isolated vertex in G and $\overline{G}$, and each vertex of V_0 is adjacent to at least one vertex of V_2 in G and also in $\overline{G}$, then it is clearly obtained.

Observation 3. Let $f = (V_0, V_1, V_2)$ be a $\gamma_{grR}(G)$-function. Then for $u \in V_0$, we have $2 \le d(u) \le n - 3$.

Proposition 3. *Let $k \ge 3$ and $n_1 \le n_2 \le \cdots \le n_k$ be integers. Then*

$$\gamma_{grR}(K_{n_1,n_2,\cdots,n_k}) = \begin{cases} \sum_{i=1}^{k} n_i & \textit{if } 1 \le n_{k-1} \le 2 \textit{ and } 1 \le n_k. \\ \sum_{i=1}^{t} n_i + 2(k-t) & \textit{if } t \le k-2, 1 \le n_t \le 2 \textit{ and } n_{t+1} \ge 3 \end{cases}$$

Proof. Let $G = K_{n_1,n_2,\cdots,n_k}$ and $W_1, W_2, \cdots, W_k$ be the partite sets of sizes $n_1, n_2, \cdots, n_k$ of G, respectively. Let $f = (V_0, V_1, V_2)$ be a γ_{grR}-function of G. If $n_i \le 2$, $i \in \{1, 2, \cdots, k\}$, then by Lemma 3, $W_i \cap V_0 = \emptyset$. Let $n_{k-1} \le 2$ and $n_k \ge 3$. If $W_k \cap V_0 \ne \emptyset$, then every vertex $u \in W_k \cap V_0$ will be an isolated vertex in $G[V_0]$, which is a contradiction. So $V_0 = \emptyset$, hence $V_2 = \emptyset$ and $V_1 = V(G)$, therefore $\gamma_{grR}(G) = \sum_{i=1}^{k} n_i$.
Let $n_i \ge 3$ for $t + 1 \le i \le k$, $(t \le k - 3)$ and for otherwise $n_i \le 2$. We can assign 2 to one vertex of ith partite set for $t + 1 \le i \le k$ and 0 to other vertices

of these partite sets and 1 otherwise. Other assignments are impossible. Therefore $\gamma_{grR}(G) = \sum_{i=1}^{t} n_i + 2(k-t)$. □

Theorem 4. ([24] Theorem 3.1) *For $n \geq 4$, $\gamma_{rR}(P_n) = \frac{2n+3+r}{3}$, where $n \equiv r \pmod 3$ and $r \in \{1,2,3\}$.*

We can simply show that $\frac{2n+3+r}{3} = \theta(n) = n - \lfloor \frac{n-4}{3} \rfloor$, where $n \equiv r \pmod 3$ and $r \in \{1,2,3\}$.

Proposition 4. *For $n \geq 4$, $\gamma_{grR}(P_n) = \theta(n)$.*

Proof. By Theorem 4, we have $\gamma_{rR}(P_n) = \theta(n)$, $n \geq 4$. If $n = 4$ or $n = 5$, then by Corollary 1, $\gamma_{grR}(P_n) = n = \theta(n)$. If $n = 6$, then $f = (\emptyset, V(P_6), \emptyset)$ is a global restrained Roman dominating function of P_6, so $\gamma_{grR}(P_6) \leq w(f) = 6 = \theta(6)$. Since $\gamma_{grR}(P_6) \geq \gamma_{rR}(P_6) = \theta(6)$, therefore $\gamma_{grR}(P_6) = \theta(6)$. Let $n \geq 7$, and $P_n = u_1u_2u_3 \ldots u_n$. If $n \equiv 0 \pmod 3$, then consider $V_0 = \{u_{3k+2} : k = 0, 1, \ldots, \frac{n-6}{3}\} \cup \{u_{3k} : k = 1, 2, \ldots, \frac{n-3}{3}\}$, $V_1 = \{u_{n-1}, u_n\}$, $V_2 = \{u_{3k+1} : k = 0, 1, \ldots, \frac{n-3}{3}\}$.
If $n \equiv 1 \pmod 3$, then put $V_0 = \{u_{3k+2} : k = 0, 1, \ldots, \frac{n-4}{3}\} \cup \{u_{3k} : k = 1, 2, \ldots, \frac{n-1}{3}\}$, $V_1 = \emptyset$, $V_2 = \{u_{3k+1} : k = 0, 1, \ldots, \frac{n-1}{3}\}$.
If $n \equiv 2 \pmod 3$, then consider $V_0 = \{u_{3k+2} : k = 0, 1, \ldots, \frac{n-5}{3}\} \cup \{u_{3k} : k = 1, 2, \ldots, \frac{n-2}{3}\}$, $V_1 = \{u_n\}$, $V_2 = \{u_{3k+1} : k = 0, 1, \ldots, \frac{n-2}{3}\}$.
In each of three above cases the function $f = (V_0, V_1, V_2)$ is a global restrained Roman dominating function of size $\theta(n)$, so $\gamma_{grR}(P_n) \leq \theta(n)$. Since $\gamma_{grR}(P_n) \geq \gamma_{rR}(P_n) = \theta(n)$ therefore $\gamma_{grR}(P_n) = \theta(n)$. □

Theorem 5. ([24] Theorem 3.2) *For cycles C_n,*

$$\gamma_{rR}(C_n) = \begin{cases} \frac{2n+3+r}{3} & n \equiv r \pmod 3, r \in \{1,2\} \\ \frac{2n}{3} & n \equiv 0 \pmod 3. \end{cases}$$

For any positive integer n, suppose that $\eta(n) = 2\lfloor \frac{n+2}{3} \rfloor + \lfloor \frac{n-3\lfloor \frac{n}{3} \rfloor}{2} \rfloor$. It can be easily seen that:

$$\eta(n) = \begin{cases} \frac{2n+3+r}{3} & n \equiv r \pmod 3, r \in \{1,2\} \\ \frac{2n}{3} & n \equiv 0 \pmod 3. \end{cases}$$

According to the above description, we have.

Proposition 5. *For cycles C_n, $\gamma_{grR}(C_n) = \eta(n)$, $n \geq 4$.*

Proof. By Theorem 5 we have $\gamma_{rR}(C_n) = \eta(n)$. If $n = 3$, then $\gamma_{rR}(C_n) = 2$ and $\gamma_{grR}(C_n) = 3$. For $n = 4$ or $n = 5$, by Corollary 1, $\gamma_{grR}(C_n) = n = \eta(n)$. Now let $n \geq 6$, and $C_n = u_1u_2 \ldots u_n$.

If $n \equiv 0 \pmod 3$, then put $V_0 = \{u_{3k+2} : k = 0, 1, \ldots, \frac{n-3}{3}\} \cup \{u_{3k} : k = 1, 2, \ldots, \frac{n}{3}\}$, $V_1 = \emptyset$, $V_2 = \{u_{3k+1} : k = 0, 1, \ldots, \frac{n-3}{3}\}$.
If $n \equiv 1 \pmod 3$, then put $V_0 = \{u_{3k+2} : k = 0, 1, \ldots, \frac{n-4}{3}\} \cup \{u_{3k} : k = 1, 2, \ldots, \frac{n-1}{3}\}$, $V_1 = \emptyset$, $V_2 = \{u_{3k+1} : k = 0, 1, \ldots, \frac{n-1}{3}\}$.
If $n \equiv 2 \pmod 3$, then put $V_0 = \{u_{3k+2} : k = 0, 1, \ldots, \frac{n-5}{3}\} \cup \{u_{3k} : k = 1, 2, \ldots, \frac{n-2}{3}\}$, $V_1 = \{u_n\}$, $V_2 = \{u_{3k+1} : k = 0, 1, \ldots, \frac{n-2}{3}\}$.
In each of three above cases show that, $f = (V_0, V_1, V_2)$ is a global restrained Roman dominating function of size $\eta(n)$, so $\gamma_{grR}(C_n) \leq \eta(n)$, $n \geq 4$. Since $\gamma_{grR}(C_n) \geq \gamma_{rR}(C_n) = \eta(n)$, therefore $\gamma_{grR}(C_n) = \eta(n)$. □

In what follows, we present a sharp upper bound for $\gamma_{grR}(G)$ in terms of diameter of G.

Proposition 6. *Let G be a graph of order n and $diam(G) = d$. Then $\gamma_{grR}(G) \leq n - \lfloor \frac{d-3}{3} \rfloor$.*
This bound is sharp.

Proof. Let $d =$diam(G). If $0 \leq d \leq 5$, then the equality is trivial since $n - \lfloor \frac{d-3}{3} \rfloor \geq n$. Let $d \geq 6$ and $P := u_1u_2u_3 \cdots u_du_{d+1}$ be a path of length d between u and v, where $u = u_1$ and $v = u_{d+1}$. We define $f : V(G) \to \{0, 1, 2\}$ with $f(u_{3i+1}) = 2$ for $0 \leq i \leq \lfloor \frac{d}{3} \rfloor$, $f(u_{3i-1}) = f(u_{3i}) = 0$ for $1 \leq i \leq \lfloor \frac{d}{3} \rfloor$ and $f(w) = 1$, otherwise. Then f is a global restrained Roman dominating function of G and $w(f) = n - (3\lfloor \frac{d}{3} \rfloor + 1) + 2(\lfloor \frac{d}{3} \rfloor + 1) = n - \lfloor \frac{d-3}{3} \rfloor$, therefore $\gamma_{grR}(G) \leq n - \lfloor \frac{d-3}{3} \rfloor$.
From Proposition 4, this upper bound is sharp for path P_n $(n \geq 4)$. □

4. Complexity and computational issues

In this section we first show that the restrained Roman domination problem is NP-complete for bipartite graphs and chordal graphs and we then show that the global restrained Roman domination problem is NP-complete for those graphs.
We consider the problem of deciding whether a graph G has an restrained Roman domination (RRD) function of weight at most a given integer. That is stated in the following decision problem. Note that a chordal graph is a graph with no induced cycle of length at least four.

RESTRAINED ROMAN DOMINATION problem (RRD problem)
INSTANCE: A graph G and an integer $j \leq \lvert V(G)\rvert$.
QUESTION: Is there an RRD function f of weight at most j?

We shall prove the NP-completeness results by reducing the following Roman domination problem, which is known to be NP-complete for bipartite graphs and chordal graphs [20].

Roman DOMINATION problem (RD problem)
INSTANCE: A graph G and an integer $k \leq |V(G)|$.
QUESTION: Does G have an RDF of weight at most k?

Theorem 6. (Liu and Chang, [20]) *The RD problem is NP-complete for bipartite graphs and chordal graphs.*

Now we discuss on the following.

Theorem 7. *The RRD problem is NP-complete even when restricted to bipartite graphs and chordal graphs.*

Proof. The problem clearly belongs to NP since checking that a given function is indeed an RRD function of weight at most j can be done in polynomial time. Set $j = 5n + k$. Let G be a graph (bipartite or chordal) with $V(G) = \{v_1, \cdots, v_n\}$. For any $1 \leq i \leq n$, we add a new vertex w_i and a tree (star) $T_i = K_{1,3}$ with $V(T_i) = \{a_i, b_i, c_i, d_i\}$ in which a_i is the support vertex, and b_i, c_i, d_i be the leaves. We then join v_i to both a_i and w_i, for all $1 \leq i \leq n$. Let H be the constructed graph, see the Figure 1.

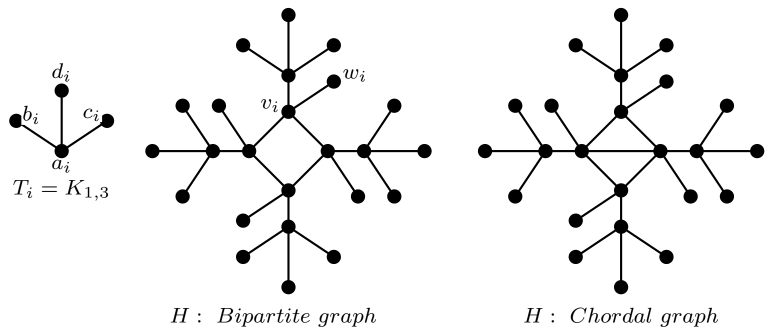

Figure 1. The graph T_i and the graph H constructed from bipartite graph and chordal graph.

Let $f = (V_0, V_1, V_2)$ be a $\gamma_{rR}(H)$-function. Clearly, $f(V(T_i) \cup \{w_i\}) \geq 5$ for all $1 \leq i \leq n$. Moreover, if there exists a vertex $v_j \in V(G) \cap V_0$ which does not have any neighbor in $V_2 \cap V(G)$, then without lose of generality, we assign label 0 to a_j, label 2 to w_i and to one of leaves b_j, c_j, d_j. It is easy to see that $f(V(T_j) \cup \{w_j\}) \geq 6$. We define X to be the set of such vertices, that is, $X = \{v_j \in V(G) \cap V_0 | N_{V(G)}(v_j) \cap V_2 = \emptyset\}$. We have

$$\begin{aligned} \gamma_{rR}(H) = \omega(f) &= \sum_{i=1}^{n} f(V(T_i) \cup \{w_i\}) + f(V(G)) \\ &= \sum_{v_i \in V(G) \setminus X} f(V(T_i) \cup \{w_i\}) + \sum_{v_i \in X} f(V(T_i) \cup \{w_i\}) + f(V(G)) \\ &\geq 5|V(G) \setminus X| + 6|X| + |V_1 \cap V(G)| + 2|V_2 \cap V(G)| \\ &= 5n + |X| + |V_1 \cap V(G)| + 2|V_2 \cap V(G)|. \end{aligned} \tag{4.1}$$

On the other hand, by assigning 1 to the vertices in X, we obtain $|X|+|V_1\cap V(G)|+2|V_2\cap V(G)|\geq \gamma_R(G)$. Therefore, by using the inequality (4.1), we deduce that $\gamma_{rR}(H)\geq 5n+\gamma_R(G)$.

Conversely, let g be a $\gamma_R(G)$-function. We define f' by $f'(w_i)=1$ and $f'(v_i)=g(v_i)$, $i\in\{1,2,\cdots,n\}$. Also $f'(a_i)=0$, $f'(d_i)=2$, $f'(b_i)=f'(c_i)=1$, if $g(v_i)=0$, and $f'(a_i)=f'(b_i)=f'(c_i)=f'(d_i)=1$ if $g(v_i)\neq 0$, $i\in\{1,2,\cdots,n\}$. It is readily checked that f' is an RRD function of H with weight $3n+\gamma_R(G)$. Therefore, $\gamma_{rR}(H)\leq 3n+\gamma_R(G)$. This shows that $\gamma_{rR}(H)=3n+\gamma_R(G)$.
Our reduction is now completed by taking into account the fact that $\gamma_{rR}(H)\leq j$ if and only if $\gamma_R(G)\leq k$. Since the RD problem is NP-complete for both bipartite graphs and chordal graphs, we have the same with the RRD problem. □

Now we show that the global restrained Roman domination is NP complete for bipartite graphs and chordal graphs. We shall prove the NP-completeness results by reducing the Roman domination problem, which is known to be NP-complete for bipartite graphs and chordal graphs [20], or Theorem 6.

We consider the problem of deciding whether a graph G has a global restrained Roman dominating (GRRD) function of weight at most a given integer $j\leq k$. That is stated in the following decision problem.

GLOBAL RISTRAINED ROMAN DOMINATION problem (GRRD problem)
INSTANCE: A graph G and an integer $j\leq |V(G)|$.
QUESTION: Is there a GRRDF of weight at most j?

Theorem 8. *The GRRD problem is NP-complete for bipartite graphs and chordal graphs.*

Proof. The problem clearly belongs to NP since checking that a given function is indeed a GRRD function of weight at most j can be done in polynomial time. Set $j=9n+k$. Let G be a graph (bipartite or chordal) with $V(G)=\{v_1,\cdots,v_n\}$. For any $1\leq i\leq n$, we add a new vertex w_i and two trees (stars) $T_{i_1}=K_{1,3}=T_{i_2}$ with $V(T_{i_j})=\{a_{i_j},b_{i_j},c_{i_j},d_{i_j}\}$ where a_{i_j} is the support vertex, and b_{i_j},c_{i_j},d_{i_j} are the corresponding leaves for $1\leq j\leq 2$ and $1\leq i\leq n$. We then join v_i to both a_{i_j} for $1\leq j\leq 2$ and to w_i, for all $1\leq i\leq n$. Let F be the constructed graph, see the Figure 2.

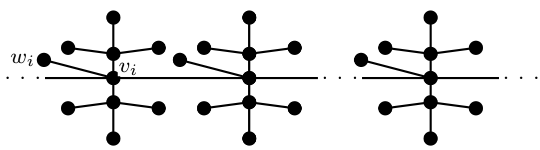

Figure 2. **The graph F constructed from bipartite graph and chordal graph.**

Let $f=(V_0,V_1,V_2)$ be a $\gamma_{grR}(F)$-function. Clearly, $f(V(T_{i_1})\cup V(T_{i_2})\cup\{w_i\})\geq 9$ for all $1\leq i\leq n$. Moreover, if there exists a vertex $v_p\in V(G)\cap V_0$ which does not have any neighbor in $V_2\cap V(G)$, then it is easy to see that $f(V(T_{p_1})\cup V(T_{p_2})\cup\{w_p\})\geq 10$. We define X to be the set of such vertices, that is, $X=\{v_p\in V(G)\cap V_0|N_{V(G)}(v_p)\cap V_2=\emptyset\}$. We have

$$\begin{aligned}\gamma_{grR}(F)=\omega(f) &= \textstyle\sum_{i=1}^{n} f(V(T_{i_1})\cup V(T_{i_2})\cup\{w_i\})+f(V(G))\\ &= \textstyle\sum_{v_i\in V(G)\setminus X} f(V(T_{i_1})\cup V(T_{i_2})\cup\{w_i\})\\ &+ \textstyle\sum_{v_i\in X} f(V(T_{i_1})\cup V(T_{i_2})\cup\{w_i\})+f(V(G))\\ &\geq 9|V(G)\setminus X|+10|X|+|(V_1\cap V(G)|+2|V_2)\cap V(G)|\\ &= 9n+|X|+|(V_1\cap V(G)|+2|V_2)\cap V(G)|.\end{aligned}\tag{4.2}$$

On the other hand, $|X|+|(V_1\cap V(G)|+2|V_2)\cap V(G)|\geq\gamma_R(G)$. Therefore, by using the inequality (4.2), we deduce that $\gamma_{grR}(F)\geq 9n+\gamma_R(G)$.
Conversely, let g be a $\gamma_R(G)$-function. If $n\geq 3$, then $\gamma_R(G)\leq n-1$ and so G has at least two vertices of weight 0. In this case we define f' by $f'(w_i)=1$ and $f'(v_i)=g(v_i)$, $i\in\{1,2,\cdots,n\}$. If $g(v_i)=0$, then put $f'(a_{i_1})=f'(a_{i_2})=0$, $f'(d_{i_1})=f'(d_{i_2})=2$, $f'(b_{i_1})=f'(b_{i_2})=f'(c_{i_1})=f'(c_{i_2})=1$. If $g(v_i)\neq 0$, then put $f'(a_{i_j})=f'(b_{i_j})=f'(c_{i_j})=f'(d_{i_j})=1$, $j\in\{1,2\}$.
Now let $n\leq 2$, so $g=(\emptyset,V(G),\emptyset)$ is a $\gamma_R(G)$-function and we can define f' by $f'(w_i)=1$ and $f'(v_i)=g(v_i)$, $i\in\{1,2,\cdots,n\}$ and $f'(a_{i_j})=f'(b_{i_j})=f'(c_{i_j})=f'(d_{i_j})=1$, $i\in\{1,2,\cdots,n\}$, $j\in\{1,2\}$. It is readily checked that in each three above cases f' is an RRD function of F with weight $9n+\gamma_R(G)$. Therefore, $\gamma_{rR}(F)\leq 9n+\gamma_R(G)$. This shows that $\gamma_{rR}(F)=9n+\gamma_R(G)$.
It is easy to see, in the graph F, for every vertex $v\in V_0$ there is a vertex $v'\in V_0$ such that v is not adjacent to v' and for every vertex $v\in V_0$ there is a vertex $v''\in V_2$ such that v is not adjacent to v''. Therefore, f' is an RRD function of $\overline{F}$, in the other hand f' is a GRRD function of F and so $\gamma_{grR}(F)\leq 9n+\gamma_R(G)$. Therefore $\gamma_{rR}(F)=9n+\gamma_R(G)$. Our reduction is now completed by taking into account the fact that $\gamma_{grR}(F)\leq j$ if and only if $\gamma_R(G)\leq k$. Since the RD problem is NP-complete for both bipartite graphs and chordal graphs, we have the same with the GRRD problem. □

5. GRRD versus GRD and RD

In this section the global restrained Roman domination is compared with other domination parameters.

Proposition 7. *Let G be a graph and $f=(V_0,V_1,V_2)$ be a $\gamma_{grR}(G)$-function. Then $V_1\cup V_2$ is a GRDS of G. In particular, if V_1 and V_2 have the minimum and maximum size between the set of vertices take value 1 and value 2 respectively under any $\gamma_{grR}(G)$-functions, then $V_1\cup V_2$ is a $\gamma_{gr}(G)$-set.*

Proof. Let $f=(V_0,V_1,V_2)$ be an arbitrary $\gamma_{grR}(G)$-function. It is obvious that $V_1\cup V_2$ is a GRDS of G. Let $\gamma_{grR}(G)=k$, $t=max\{|V_2|:f=(V_0,V_1,V_2)$ is a

$\gamma_{grR}(G)$-function} and $f' = (V'_0, V'_1, V'_2)$ is a $\gamma_{grR}(G)$-function such that $|V'_2| = t$. For any arbitrary $\gamma_{grR}(G)$-function $f = (V_0, V_1, V_2)$ we have $|V_1| + 2|V_2| = k$ and $|V'_1| + 2|V'_2| = k$ and $|V_2| \leq |V'_2|$, so $|V'_1 \cup V'_2| = |V'_1| + |V'_2| = k - |V'_2| \leq k - |V_2| = |V_1| + |V_2| = |V_1 \cup V_2|$. Therefore $|V'_1 \cup V'_2|$ is a $\gamma_{gr}(G)$-set. □

Let $\mathcal{G}_1$ be a family of graphs G in which:
1. The order of G is at most 5. Or
2. G is a graph of order $n \geq 6$ in which,
2.1. for any 6 vertices v_1, v_2, a, b, c, d of G, we cannot find two K_2 like ab, cd, such that $\{a, b, c, d\} \subseteq N(v_1) \cup N(v_2)$ and $N(v_1) \cap N(v_2) \cap \{a, b, c, d\} = \emptyset$, and
2.2. for any 6 vertices v_1, v_2, a, b, c, d of G, we cannot find a path P_4 like $abcd$, such that $\{a, b, c, d\} \subseteq N(v_1) \cup N(v_2)$ and $N(v_1) \cap N(v_2) \cap \{a, b, c, d\} = \emptyset$, and
2.3. for any 6 vertices v_1, v_2, a, b, c, d of G, we cannot find a cycle C_4 like $abcda$, such that $\{a, b, c, d\} \subseteq N(v_1) \cup N(v_2)$ and $N(v_1) \cap N(v_2) \cap \{a, b, c, d\} = \emptyset$,

Let $\mathcal{G}_2$ be a family of graphs G formed from a subgraphs C and H in which:
1. C has no isolated vertices in G and $\overline{G}$.
2. Every vertex of C is adjacent to at least one vertex of H and is adjacent to at least one vertex of $\overline{H}$.
3. Every vertex of H has a private neighbor in C, in G or $\overline{G}$.
4. H with two properties of 2 and 3, between the subgraphs of G respected to C has minimum cardinality.

Proposition 8. *Let G be a graph. Then $\gamma_{gr}(G) \leq \gamma_{grR}(G) \leq 2\gamma_{gr}(G)$. The equality of lower bound holds if and only if $G \in \mathcal{G}_1$. The equality of upper bound holds if and only if $G \in \mathcal{G}_2$.*

Proof. Let S be a γ_{gr}-set of G. Define $V_0 = V - S$; $V_1 = \emptyset$; $V_2 = S$. It is clear that $f = (V_0, V_1, V_2)$ is a global restrained Roman dominating function of G. Therefore $\gamma_{grR}(G) \leq w(f) = 2|V_2| = 2|S| = 2\gamma_{gr}(G)$.
Now, let $f = (V_0, V_1, V_2)$ be a γ_{grR}-function of G. Then $S = V_1 \cup V_2$ is a global restrained dominating set of G. Then $\gamma_{gr}(G) \leq |S| = |V_1| + |V_2| \leq |V_1| + 2|V_2| = \gamma_{grR}(G)$.
For seeing the lower bound, let $G \in \mathcal{G}_1$. If $|V(G)| \leq 5$, then by Corollary 1 $\gamma_{grR}(G) = n$. Since every vertex is assigned by 1, this shows that $\gamma_{gr}(G) = n$. Let G be a graph of order $n \geq 6$ with the given properties, suppose on the contrary $\gamma_{gr}(G) \neq \gamma_{grR}(G)$. We deduce $V_0 \neq \emptyset$, $V_2 \neq \emptyset$ with V_0 and V_2 are of orders at least 4 and 2 respectively. By the properties of the restrained Roman domination and global domination, $G(V_0)$ has at least $2K_2$, P_4 or C_4 with vertices $\{a, b, c, d\}$ and $G(V_2)$ has at least two vertices v_1, v_2 such that $N(v_1) \cup N(v_2)$ includes $\{a, b, c, d\}$ and $N(v_1) \cap N(v_2) \cap \{a, b, c, d\} = \emptyset$, which is a contradiction. Therefore every vertex is assigned by label 1 and $\gamma_{gr}(G) = \gamma_{grR}(G)$.
Conversely, assume that $\gamma_{gr}(G) = \gamma_{grR}(G)$ and the order of G is at least 6. On the contrary, let at least one of the properties 2.1, 2.2 and 2.3 not hold. Then, there

are 6 vertices v_1, v_2, a, b, c, d of G in which 4 of them induces K_2 like ab, cd, or a path P_4 like $abcd$ or a cycle C_4 like $abcda$ such that $\{a, b, c, d\} \subset N(v_1) \cup N(v_2)$ and $N(v_1) \cap N(v_2) \cap \{a, b, c, d\} = \emptyset$.
Consider $f = (V_0' = \{a, b, c, d\}, V_1' = V(G) - \{a, b, c, d, v_1, v_2\}, V_2' = \{v_1, v_2\})$ is a GRRD function of size $n-2$, so $\gamma_{grR}(G) \leq n-2$. Therefore for any $\gamma_{grR}(G)$-function like $f = (V_0, V_1, V_2)$ we have $V_2 \neq \emptyset$ and then $\gamma_{grR}(G) = 2 \mid V_2 \mid + \mid V_1 \mid$. In this case $\gamma_{gr}(G) =\mid V_2 \mid + \mid V_1 \mid < 2 \mid V_2 \mid + \mid V_1 \mid$ which is a contradiction. Thus $G \in \mathcal{G}_1$.
For seeing the upper bound, let $G \in \mathcal{G}_2$. Then we assign 0 to the vertices of C and 2 to the vertices in H, it is easy to see that $\gamma_{gr}(G) \leq |V(H)|$ and $\gamma_{grR}(G) \leq 2|V(H)|$. Since every vertex in H has a private neighbor in C then we should assign value 2 to each vertex of H. Therefore $\gamma_{grR}(G) = 2|V(H)|$ and $\gamma_{gr}(G) = |V(H)|$ and $\gamma_{grR}(G) = 2\gamma_{gr}(G)$.
Conversely, let G be a graph and $\gamma_{grR}(G) = 2\gamma_{gr}(G)$. Let S be a γ_{gr}-set. Then we take $H = G(S)$ and $C = G(V(G) \setminus S)$. It is readily seen that by the property of global restrained the subgraphs C and $\overline{C}$ have no isolated vertices, and every vertex of C has a neighbor in H and has a neighbor in H. On the other hand if a vertex in H has no private neighbor in C, neither in G nor in $\overline{G}$, then we can assign 1 to this vertex in any GRRD function f of G and then $\gamma_{grR}(G) < 2\gamma_{gr}(G)$, which is a contradiction. Therefore $G \in \mathcal{G}_2$. □

It is obvious, for nontrivial graph, we have.

Observation 9. If G is a nontrivial graph, then $\gamma_{rR}(G) \geq 2$.

It is well known that the domination parameters of the components of a disconnected graph are independent together. Therefore it can be had the following result.

Observation 10. Let G be a disconnected graph with components $W_1, W_2, \ldots, W_k$ and let $\gamma_{rR}^{(1)}, \gamma_{rR}^{(2)}, \ldots, \gamma_{rR}^{(k)}$ be the restrained Roman domination numbers of $W_1, W_2, \ldots, W_k$, respectively, then $\gamma_{rR}(G) = \Sigma_{i=1}^{K} \gamma_{rR}^{(i)}$.

Theorem 11. *Let G be a disconnected graph and $W_1, W_2, \ldots, W_k$ be the components of G with $k \geq 2$. If $f = (V_0, V_1, V_2)$ is a γ_{rR}-function of G and there exist $i, j \in \{1, 2, \ldots, k\}$ $(i \neq j)$ in which $W_i \cap V_2 \neq \emptyset$ and $W_j \cap V_2 \neq \emptyset$, then $\gamma_{grR}(G) = \gamma_{rR}(G)$.*

Proof. According to the $W_i \cap V_2 \neq \emptyset$ and $W_j \cap V_2 \neq \emptyset$, we have $W_i \cap V_0 \neq \emptyset$ and $W_j \cap V_0 \neq \emptyset$. On the other hand, every vertex in $V_0 - V(W_i)$ is adjacent to the vertices of $V_2 \cap V(W_i)$ in $\overline{G}$ and every vertex of $V_0 \cap V(W_i)$ is adjacent to vertices of $V_2 \cap V(W_j)$ in $\overline{G}$ and also every vertex in $V_0 - V(W_i)$ is adjacent to the vertices of $V_0 \cap V(W_i)$ in $\overline{G}$ and every vertex of $V_0 \cap V(W_i)$ is adjacent to the vertices of $V_0 \cap V(W_j)$ in $\overline{G}$. We deduce f is a restrained Roman dominating function of $\overline{G}$. Therefore f is a global restrained Roman dominating function of G, so $\gamma_{grR}(G) \leq \gamma_{rR}(G)$, by Observation 1 we have $\gamma_{grR}(G) = \gamma_{rR}(G)$. □

Theorem 12. *Let G be a graph of order $n \geq 6$ which contain only one pendant vertex, and $\Delta(G) = n-1$. If G has no vertex of degree $n-2$, then $\gamma_g(G) = 2$ and $\gamma_{grR}(G) = 4$.*

Proof. It is clear that for every nontrivial graph G, $\gamma_g(G) \geq 2$. Let u be a vertex of degree $n-1$ and v be the only leaf of G. Then $\{u, v\}$ is a $\gamma_g(G)$-set. Hence $\gamma_g(G) = 2$. Now define $f = (V_0, V_1, V_2)$ by $V_0 = V(G) - \{u, v\}$, $V_1 = \emptyset$, $V_2 = \{u, v\}$. It is easy to see that f is a global Restrained dominating function of G. Since $d(w) \neq 1$ and $d(w) \neq n-2$ for every $w \in V_0$, so w is not an isolated vertex in $G[V_0]$ and $\overline{G[V_0]}$. Therefore f is a global restrained Roman dominating function of G, $\gamma_{grR}(G) \leq 4$. Since $n \geq 6$, so by Proposition 1 $\gamma_{grR}(G) \geq 4$, Therefore $\gamma_{grR}(G) = 4$. □

Theorem 13. *Let $f = (V_0, V_1, V_2)$ be any $\gamma_{grR}(G)$-function. Then V_2 is a γ_{gr}-set of $H = G[V_0 \cup V_2]$.*

Proof. On the contrary suppose S is a global restrained dominating set of H, and $|S| < |V_2|$. Define $f' = (V_0', V_1', V_2')$ by $V_0' = (V_0 \cup V_2) - S$, $V_1' = V_1$, $V_2' = S$. The function f' is a global restrained Roman dominating function of G. However $w(f') = |V_1'| + 2|V_2'| < |V_1| + 2|V_2| = w(f) = \gamma_{grR}(G)$, which is a contradiction. □

Theorem 14. *Let G be a graph and $f = (V_0, V_1, V_2)$ be a γ_{rR}-function of G and $H = G[V_0 \cup V_2]$. If diam$(H) \geq 5$, then $\gamma_{grR}(G) = \gamma_{rR}(G)$.*

Proof. If $\gamma_{grR}(G) \neq \gamma_{rR}(G)$, then f is not a restrained Roman dominating function of $\overline{G}$. Therefore at least one of the following cases holds:
i) $\overline{G[V_0]}$ has an isolated vertex. Let u be an isolated vertex in $\overline{G[V_0]}$. So u is adjacent to all vertices of $V_0 - \{u\}$ in G. It is clear that for every two vertices $z, t \in V(H)$, $d_G(z, t) \leq 4$. Therefore diam$(H) \leq 4$, which is a contradiction.
ii) $\overline{G[V_0]}$ has a vertex, w, which is not adjacent to any vertex of V_2 in $\overline{G}$. So w is adjacent to all vertices of V_2 in G. It is clear that for every two vertices $z, t \in V(H)$, $d_G(z, t) \leq 4$. Therefore diam$(H) \leq 4$, which is a contradiction.

□

Corollary 2. *Let G be a graph and $f = (V_0, V_1, V_2)$ be a γ_{rR}-function of G. If $G[V_0 \cup V_2]$ is a disconnected graph, then $\gamma_{grR}(G) = \gamma_{rR}(G)$.*

Proof. Let $H = G[V_0 \cup V_2]$. If $\gamma_{grR}(G) \neq \gamma_{rR}(G)$, then by Theorem 14, diam$(H) \leq 4$, so H is a connected graph, which is a contradiction. □

The inverse of Corollary 2 is not true. For the path $P_7 = u_1 u_2 u_3 \ldots u_7$ let $V_0 = \{u_2, u_3, u_5, u_6\}$, $V_1 = \emptyset$, $V_2 = \{u_1, u_4, u_7\}$. Then the function $f = (V_0, V_1, V_2)$ is a γ_{rR}-function and also a γ_{grR}-function of P_7, so $\gamma_{grR}(P_7) = \gamma_{rR}(P_7)$, but $H = G[V_0 \cup V_2]$ is a connected graph.

Proposition 9. *Let G be a graph and $f = (V_0, V_1, V_2)$ be a γ_{rR}-function of G. If $\Delta(G) \leq min\{|V_0| - 1, |V_2|\}$, then $\gamma_{grR}(G) = \gamma_{rR}(G)$.*

Proof. If $V_0 = \emptyset$, then $V_2 = \emptyset$ and $V_1 = V(G)$, therefore $\gamma_{grR}(G) = \gamma_{rR}(G) = n$. Now let $V_0 \neq \emptyset$ and u be an arbitrary vertex of V_0. Since u is adjacent to at least one vertex of V_2 and $d(u) \leq \Delta(G) \leq |V_0| - 1$, so u is nonadjacent to at least one vertex of V_0. Also since u is adjacent to at least one vertex of V_0 and $d(u) \leq \Delta(G) \leq |V_2|$, so u is nonadjacent to at least one vertex of V_2. Therefor V_2 dominates V_0 in $\overline{G}$ and $\overline{G[V_0]}$ has not any isolated vertex, so f is a restrained Roman dominating set of $\overline{G}$, too. Therefore $\gamma_{grR}(G) = \gamma_{rR}(G)$. □

Proposition 10. *Let G be a graph of order $n \geq 6$ and $\gamma_{grR}(G) = 4$, Then $V(G)$ has two vertices u, v such that $d(u) + d(v) = n$ or $d(u) + d(v) = n - 2$.*

Proof. Let $f = (V_0, V_1, V_2)$ be a γ_{grR}-function of G. Since $\gamma_{grR}(G) < n$, so by Theorem 1, $|V_2| \geq 2$. Since $\gamma_{grR}(G) = 4$, so $|V_2| = 2$ and $V_1 = \emptyset$. Let $V_2 = \{u, v\}$. Every vertex of V_0 is adjacent to exactly one vertex of V_2. So if u and v are not adjacent, then $d(u)+d(v) = n-2$ and if u and v are adjacent, then $d(u)+d(v) = n$. □

Proposition 11. *Let G be a graph of order $n \geq 6$ and $\gamma_{grR}(G) = 5$, Then $V(G)$ has two vertices u, v such that $n - 3 \leq d(u) + d(v) \leq n + 1$.*

Proof. Let $f = (V_0, V_1, V_2)$ be a γ_{grR}-function of G. Since $\gamma_{grR}(G) < n$, so by Theorem 1, $|V_2| \geq 2$. Since $\gamma_{grR}(G) = 5$, so $|V_2| = 2$ and $|V_1| = 1$ and $|V_0| = n - 3$. Let $V_1 = \{z\}$ and $V_2 = \{u, v\}$. Every vertex of V_0 is adjacent to exactly one vertex of V_2. If z is not adjacent to any vertex of V_2 and u, v are not adjacent, then $d(u) + d(v) = n - 3$. If z is not adjacent to any vertex of V_2 and u, v are adjacent, then $d(u) + d(v) = n - 1$. If z is adjacent to one of the vertices of V_2 and u, v are not adjacent, then $d(u) + d(v) = n - 2$. If z is adjacent to one of the vertices of V_2 and u, v are adjacent, then $d(u) + d(v) = n$. If z is adjacent to u and v, and u, v are not adjacent, then $d(u) + d(v) = n - 1$. If z is adjacent to u and v, and u, v are adjacent, then $d(u) + d(v) = n + 1$. □

6. Small global restrained Roman domination number

Proposition 12. *Let G be a graph and $n \in \{1, 2, 3\}$. Then $\gamma_{grR}(G) = n$ if and only if $|V(G)| = n$.*

Proof. By Corollary 1 for any graph of order $1 \leq n \leq 3$, we have $\gamma_{grR}(G) = n$. Conversely, let $\gamma_{grR}(G) = n \in \{1, 2, 3\}$, and $f = (V_0, V_1, V_2)$ be a $\gamma_{grR}(G)$-function. It is clear if $V_2 \neq \emptyset$, then $|V_2| \geq 2$, hence $\gamma_{grR}(G) \geq 4$, which is a contradiction. Therefore $V_2 = \emptyset$, which deduces $V_0 = \emptyset$ and $V_1 = V(G)$, so $\gamma_{grR}(G) = n$. □

Let H be a graph without isolated vertex and $\Delta(H) \leq |V(H)| - 2$.
Let G_1 be a graph constructed from H by adding two vertices a, b and joining to at least one vertex and at most $|V(H)| - 1$ vertices in which $V(H) \subseteq N(a) \cup N(b)$.
Let G_2 be a graph constructed from H by adding two vertices a, b and joining one vertex say a to b and to all vertices of H, in which $N(b) \cap V(H) = \emptyset$.
Let G_3 be a graph constructed from H by adding two vertices a, b and joining one vertex say a to all vertices of H, in which the vertex b is an isolated vertex.

Proposition 13. *Let G be a graph. Then, $\gamma_{grR}(G) = 4$ if and only if $|V(G)| = 4$ or $G \in \{G_1, G_2, G_3\}$.*

Proof. Suppose that $\gamma_{grR}(G) = 4$ and $f = (V_0, V_1, V_2)$ be a $\gamma_{grR}(G)$-function. If $V_0 = \emptyset$, then $\gamma_{grR}(G) = |V(G)|$ by Proposition 1 and Corollary 1. Now assume that $V_0 \neq \emptyset$, since for $\gamma_{grR}(G) = 4$ we need at least two vertices of positive weight, $|V(G)| \geq 6$. There exist two vertices a and b in V_2 such that $H = G(V(G) \setminus \{a, b\})$ and $\overline{H}$ have no isolated vertex, in the other words H is a graph without isolated vertex and $\Delta(H) \leq |V(H)| - 2$.
If the vertices a, b are adjacent to at least one vertex and at most $|V(H)| - 1$ vertices in which $V(H) \subseteq N(a) \cup N(b)$, then $G = G_1$. If the vertex a is adjacent to all vertices of H and b, in which the $N(b) \cap V(H) = \emptyset$, then $G = G_3$.
If the vertex a is adjacent to all vertices of H, in which the vertex b is an isolated vertex, then $G = G_3$.
Conversely, If G is a graph of order 4, then by Corollary 1 any GRRD function f assigns 1 to each vertex of G. Let $G \in \{G_1, G_2, G_3\}$. Then by assignment 2 to the vertices of a, b we obtain a $\gamma_{grR}(G)$-function f with $w(f) = 4$. □

Let H be a graph without isolated vertex and $\Delta(H) \leq |V(H)| - 2$.
Let F_1 be a graph constructed from H by adding three vertices a, b, c and joining to at least one vertex and at most $|V(H)| - 1$ vertices to each a, b in which $V(H) \subseteq N(a) \cup N(b)$, whereas c is adjacent to only some vertices of a, b or only some vertices of H or is an isolated vertex.
Let F_2 be a graph constructed from H by adding three vertices a, b, c and joining one vertex say a to all vertices of H and b, in which the $N(b) \cap V(H) = \emptyset$, whereas c is adjacent to only some vertices of a, b or only some vertices of H or is an isolated vertex.
Let F_3 be a graph constructed from H by adding three vertices a, b, c and joining one vertex say a to all vertices of H, in which the vertex b is an isolated vertex, whereas c is adjacent to only some vertices of a, b or only some vertices of H or is an isolated vertex.

Proposition 14. *Let G be a graph. Then, $\gamma_{grR}(G) = 5$ if and only if $|V(G)| = 5$ or $G \in \{F_1, F_2, F_3\}$.*

Proof. Suppose that $\gamma_{grR}(G) = 5$ and $f = (V_0, V_1, V_2)$ be a $\gamma_{grR}(G)$-function. If $V_0 = \emptyset$, then $\gamma_{grR}(G) = |V(G)|$ by Proposition 1 and Corollary 1. Now assume that $V_0 \neq \emptyset$, since for $\gamma_{grR}(G) = 5$, we need at least three vertices of positive weight, so $|V(G)| \geq 7$. Therefore, there exist two vertices a and b in V_2 and one vertex $c \in V_1$ such that $H = G(V(G) \setminus \{a, b, c\})$ and $\overline{H}$ have no isolated vertex, in the other words H is a graph without isolated vertex and $\Delta(H) \leq |V(H)| - 2$.
If the vertices a, b are adjacent to at least one vertex and at most $|V(H)| - 1$ vertices in which $V(H) \subseteq N(a) \cup N(b)$, whereas c is adjacent to only some vertices of a, b or only some vertices of H or is an isolated vertex, then $G = F_1$.
If the vertex a is adjacent to all vertices of H, in which the vertex b is an isolated vertex, whereas c is adjacent to only some vertices of a, b or only some vertices of H or is an isolated vertex, then $G = F_2$.
If the vertex a is adjacent to all vertices of H, in which the vertex b is an isolated vertex, whereas c is adjacent to only some vertices of a, b or only some vertices of H or is an isolated vertex, then $G = F_2$.
Conversely, If G is a graph of order 5, then any GRRD function f assigns 1 to each vertex of G. Let $G \in \{F_1, F_2, F_3\}$. Then by assignment 2 to the vertices of a, b and 1 to c, we obtain a $\gamma_{grR}(G)$-function f with $w(f) = 5$. □

7. Characterization of graphs G with $\gamma_{grR}(G) = |V(G)|$

In this section we study the characterization of trees and graphs in terms of their orders.

Theorem 15. *Let T be a tree of order n. Then $\gamma_{grR}(T) = n$ if and only if diam$(T) \leq 5$.*

Proof. By Proposition 6, for the trees T which diam$(T) \geq 6$ we have $\gamma_{grR}(T) < n$, therefore, it is sufficient to verify the trees T which diam$(T) \leq 5$.
For diam$(T) = 0$ or diam$(T) = 1$, it is clear by Corollary 1. If diam$(T) = 2$, then T is a star and by Observation 2, $\gamma_{grR}(T) = n$.
Now let diam$(T) = 3$. Then T is a double star with two support vertices a, b. If $f = (V_0, V_1, V_2)$ is a γ_{grR}-function of T, then $L(T) \cap V_0 = \emptyset$, and $|V_0| \leq 2$. Now by Proposition 1, $\gamma_{grR}(T) = n$.
Let diam$(T) = 4$ and $f = (V_0, V_1, V_2)$ be a γ_{grR}-function of T. Suppose $C(T) = \{u\}$, then $V(T) = L(T) \cup S(T) \cup \{u\}$. Let $V_0 \neq \emptyset$, since $V_0 \cap L(T) = \emptyset$. Then $V_0 \subseteq S(T) \cup \{u\}$. If $u \notin V_0$, then $G[V_0]$ is an independent set which is a contradiction. If $u \in V_0$, then $V_0 \subseteq N[u]$. But it means that u is an isolated vertex in $\overline{G[V_0]}$, which is a contradiction. Therefor $V_0 = \emptyset$, and also $V_2 = \emptyset$. This shows that $V_1 = V(G)$, and $\gamma_{grR}(T) = n$.
Now let diam$(T) = 5$ and $f = (V_0, V_1, V_2)$ be a γ_{grR}-function of T. If $V_0 = \emptyset$, then $V_2 = \emptyset$ and $V_1 = V(G)$, so $\gamma_{grR}(T) = n$. Let $V_0 \neq \emptyset$. Since diam$(T) = 5$, T has $C(T) = \{u, v\}$ as centers and $S(T) \subseteq N(C(T))$ and so $V_0 \subseteq S(T) \cup \{u, v\}$. If $\{u, v\} \cap V_0 = \emptyset$, then $G[V_0]$ is an independent set which is a contradiction. Therefore

$u \in V_0$ or $v \in V_0$. If only one of the vertices of $\{u, v\}$, for example u belongs to V_0, then $V_0 \subseteq N[u]$, So u is an isolated vertex in $\overline{G[V_0]}$, which is a contradiction. Therefore $\{u, v\} \subseteq V_0$. It is clear that $G[V_0]$ is a connected graph and for every vertex of V_0, there is exactly one vertex of V_2, therefore $|V_0| = |V_2|$, that means $\gamma_{grR}(T) = n$. □

As an immediate result, by Proposition 6 and Theorem 15, we have.

Corollary 3. *Let T be a tree of order n. If $\gamma_{grR}(T) = n - 1$, then, $6 \leq diam(T) \leq 8$.*

In the follow, we define four families of graphs to obtain the final result.

• 1. Let G be a graph. For every $u \in S(G)$, we remove all leaves from $N(u)$ except one, then the resulted graph is called the pruned subgraph of G and denoted by G_p.

• 2. Let G be a graph of order at least 4 and $u, v, z, t \in V(G)$. G_{uv} is the family of graphs obtained from G by adding at least one new path of length 2 between u and v.
$G_{uv,zt}$ is the family of graphs obtained from G by adding at least one new path of length 2 between u and v and at least one new path of length 2 between z and t.

• 3. Let G be a graph of order n. f^*_G is defined as $f^*_G = (\emptyset, V(G), \emptyset)$. It is clear that $w(f^*_G) = n$.

• 4. Consider the graphs $G^{(1)}, G^{(2)}, \ldots, G^{(13)}$ depicted in Figure 3.
According to the • 2, let $\Phi = \{G^{(k)} : k \in \{1, 2, \ldots, 13\}\} \bigcup \{G^{(k)}_{ac} : k \in \{1, 2, \ldots, 13\} - \{6\}\} \bigcup \{G^{(2)}_{bd}\}$.

Theorem 16. *For any connected triangle free graph G of order n which is not a tree, $\gamma_{grR}(G) = n$ if and only if $G_p \in \Phi$.*

Proof. If $G_p \in \Phi$, then it is easy to check that $\gamma_{grR}(G) = n$. Now let $\gamma_{grR}(G) = n$ and G be a connected triangle free graph of order n and G is not a tree. To complete the proof we need the followings.

Lemma 1. *Let G be a triangle free graph. If G has a path of length 6 or a cycle with at least 6 vertices, then $\gamma_{grR}(G) \neq n$.*

Proof. If G has a path $P = u_0u_1u_2...u_6$ of length 6, then since G is a triangle free graph, the function $f = (\{u_1, u_2, u_4, u_5\}, V(G) - \{u_0, u_1, u_2, \ldots, u_6\}, \{u_0, u_3, u_6\})$ is a global restrained Roman dominating function of size $n - 1$, so $\gamma_{grR}(G) \leq n - 1$.
If G contains C_k, $k \geq 7$ as a subgraph, then G has paths of length 6 and so $\gamma_{grR}(G) \leq n - 1$.

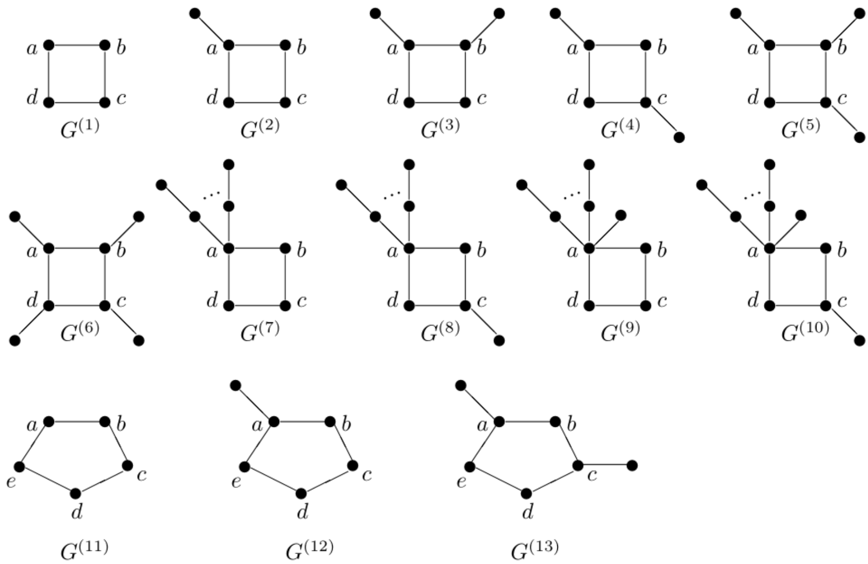

Figure 3. **The base graphs of the graphs in Φ**

Let G have a cycle C_6 with vertex set $\{a, b, c, d, e, f\}$. Since G does not have any path of length 6, so $V(G) = V(C_6)$, so G is isomorphic to one of the graphs $C_6 = abcdefa$, $abcdefa+ad$, $abcdefa+ad+cf$, $abcdefa+ad+cf+be$. For each of them the function $f = (\{b, c, e, f\}, \emptyset, \{a, d\})$ is a γ_{grR}-function of G, so $\gamma_{grR}(G) = 4 < n$. □

Lemma 2. *According to the assumption given in the Theorem 16, let G have a cycle with 5 vertices and $\gamma_{grR}(G) = n$. Then $G_p \in \{G^{(11)}, G^{(12)}, G^{(13)}\} \cup \{G^{(11)}_{ac}, G^{(12)}_{ac}, G^{(13)}_{ac}\}$*

Proof. Let G have a cycle C_5 with vertex set $\{a, b, c, d, e\}$ and have no cycle C_n $(n \geq 6)$. If G has a subgraph which is isomorphic to one of the graphs such as $G^{(11)}_{ac,ad}$ or G has a subgraph which is isomorphic to one of the graphs such as $G^{(11)}_{ac,bd}$, then G has a path of length 6, so $\gamma_{grR}(G) < n$. Now we consider the graph G which G has no subgraph isomorphic to one of the graphs of family $G^{(11)}_{ac,ad}$ or family $G^{(11)}_{ac,bd}$. If $G = C_5$, then $\gamma_{grR}(G) = 5 = n$, otherwise since G does not have any path of length 6, so for any vertex $u \in V(G) - V(C_5)$ we have $d(u, V(C_5)) = 1$ and G does not have any two adjacent support vertices. Therefore we have the following.

Case 1. If G has no support vertex, then the function f^*_G is a γ_{grR}-function of G and so $\gamma_{grR}(G) = n$.

Case 2. If G has only one support vertex, then we have the followings.
2.1. If G_p is isomorphic to $G^{(12)}$, then the function f^*_G is a γ_{grR}-function of G and

so $\gamma_{grR}(G) = n$.
2.2. If G_p is isomorphic to one of the graphs of family $G_{ac}^{(12)}$, then the function f_G^* is a γ_{grR}-function of G and so $\gamma_{grR}(G) = n$.
2.3. If G has a subgraph which is isomorphic to one the graphs of family $G_{ce}^{(12)}$, then G has a path of length 6, so $\gamma_{grR}(G) < n$.
2.4. If G has a subgraph which is isomorphic to one the graphs of family $G_{be}^{(12)}$, then G has a path of length 6, so $\gamma_{grR}(G) < n$.

Case 3. Let G have two nonadjacent support vertices. We have the followings.
3.1. If G_p is isomorphic to $G^{(13)}$, then f_G^* is a γ_{grR}-function of G and so $\gamma_{grR}(G) = n$.
3.2. If G_p is isomorphic to one of the graphs of family $G_{ac}^{(13)}$, then f_G^* is a γ_{grR}-function of G and so $\gamma_{grR}(G) = n$.
3.3. If G has a subgraph which is isomorphic to one of the graphs of family $G_{ad}^{(13)}$, then G has a path of length 6, so $\gamma_{grR}(G) < n$.
3.4. If G has a subgraph which is isomorphic to one of the graphs of family $G_{bd}^{(13)}$, then G has a path of length 6, so $\gamma_{grR}(G) < n$.

□

Lemma 3. *According to the assumption given in the Theorem 16, let G have a cycle with 4 vertices and $\gamma_{grR}(G) = n$. Then $G_p \in \{G^{(k)} : k \in \{1, 2, ..., 10\}\} \bigcup \{G_{ac}^{(k)} : k \in \{1, 2, ..., 10\} - \{6\}\} \bigcup \{G_{bd}^{(2)}\}$*

Proof. Let G have a cycle C_4 with vertex set $\{a, b, c, d\}$ and have no cycle C_n $(n \geq 5)$. If G has a subgraph which is isomorphic to one of the graphs such as $G_{ac,bd}^{(1)}$, then by letting $V_0 = \{u : d(u, a) = d(u, c) = 1\} \cup \{u : d(u, b) = d(u, d) = 1\} - \{a, b\}$, $V_1 = V(G) - (V_0 \cup \{a, b\})$, $V_2 = \{a, b\}$ the function $f = (V_0, V_1, V_2)$ is a global restrained Roman dominating function with $w(f) \leq n - 2$, thus $\gamma_{grR}(G) < n$.
Now we consider the graphs which don't have any subgraph such as $G_{ac,bd}^{(1)}$. Since G does not have any path of length 6, for any vertex $u \in V(G) - V(C_4)$ we have $d(u, V(C_4)) \leq 2$. So we have the following.
Case 1. If G does not have any support vertex, then f_G^* is a γ_{grR}-function of G and so $\gamma_{grR}(G) = n$.

Case 2. If G has only one support vertex $a \in V(C_4)$, then f_G^* is a γ_{grR}-function of G and so $\gamma_{grR}(G) = n$.

Case 3. If G has only two adjacent support vertices $a, b \in V(C_4)$, then f_G^* is a γ_{grR}-function of G and so $\gamma_{grR}(G) = n$.

Case 4. If G has only two nonadjacent support vertices $a, c \in V(C_4)$, then we have the followings.
4.1. If G_p is isomorphic to $G^{(4)}$, then f_G^* is a γ_{grR}-function of G and so $\gamma_{grR}(G) = n$.
4.2. If G has a subgraph which is isomorphic to one of the graphs of family $G_{ac}^{(4)}$,

then f^*_G is a γ_{grR}-function of G and so $\gamma_{grR}(G) = n$.
4.3. If G has a subgraph which is isomorphic to one of the graphs of family $G^{(4)}_{bd}$, then G has a path of length 6, so $\gamma_{grR}(G) < n$.

Case 5. If G has only three support vertices $a, b, c \in V(C_4)$, where b is adjacent to a and c. We have the followings.
5.1. If G_p is isomorphic to $G^{(5)}$, then f^*_G is a γ_{grR}-function of G and so $\gamma_{grR}(G) = n$.
5.2. If G_p is isomorphic to one of the graphs of family $G^{(5)}_{ac}$, then f^*_G is a γ_{grR}-function of G and so $\gamma_{grR}(G) = n$.
5.3. If G has a subgraph which is isomorphic to one of the graphs of family $G^{(5)}_{bd}$, then G has a path of length 6, so $\gamma_{grR}(G) < n$.

Case 6. If G has only four support vertices $a, b, c, d \in V(C_4)$, then we have the followings.
6.1. If G_p is isomorphic to $G^{(6)}$, then f^*_G is a γ_{grR}-function of G and so $\gamma_{grR}(G) = n$.
6.2. If G has a subgraph which is isomorphic to one of the graphs of family $G^{(6)}_{ac}$, then G has a path of length 6, so $\gamma_{grR}(G) < n$.

Case 7. If G has some support vertices which all of them are adjacent to a vertex of C_4, then we have the followings.
7.1. If G_p is isomorphic to one of the graphs of family $G^{(7)}$, then f^*_G is a γ_{grR}-function of G and so $\gamma_{grR}(G) = n$.
7.2. If G_p is isomorphic to one of the graphs of family $G^{(7)}_{ac}$, then f^*_G is a γ_{grR}-function of G and so $\gamma_{grR}(G) = n$.
7.3. If G has a subgraph which is isomorphic to one of the graphs of family $G^{(7)}_{bd}$, then G has a path of length 6, so $\gamma_{grR}(G) < n$.

Case 8. Let G have some support vertices which all of them are adjacent to a vertex $a \in V(C_4)$, and also G has another support vertex, $c \in V(C_4)$, such that $d(a, c) = 2$, then we have the followings.
8.1. If G_p is isomorphic to one of the graphs of family $G^{(8)}$, then f^*_G is a γ_{grR}-function of G and so $\gamma_{grR}(G) = n$.
8.2. If G_p is isomorphic to one of the graphs of family $G^{(8)}_{ac}$, then f^*_G is a γ_{grR}-function of G and so $\gamma_{grR}(G) = n$.
8.3. If G has a subgraph which is isomorphic to one of the graphs of family $G^{(8)}_{bd}$, then G has a path of length 6, so $\gamma_{grR}(G) < n$.

Case 9. If G has one support vertex $a \in V(C_4)$, and also G has some support vertices which are adjacent to vertex a, then we have the followings.
9.1. If G_p is isomorphic to one of the graphs of family $G^{(9)}$, then f^*_G is a γ_{grR}-function of G and so $\gamma_{grR}(G) = n$.
9.2. If G_p is isomorphic to one of the graphs of family $G^{(9)}_{ac}$, then f^*_G is a γ_{grR}-function of G and so $\gamma_{grR}(G) = n$.

9.3. If G has a subgraph which is isomorphic to one of the graphs of family $G_{bd}^{(9)}$, then G has a path of length 6, so $\gamma_{grR}(G) < n$.

Case 10. If G has two support vertices $a, c \in V(C_4)$, and $d(a,c) = 2$ and also G has some support vertices which are adjacent to vertex a, then we have the followings.
10.1. If G_p is isomorphic to one of the graphs of family $G^{(10)}$, then f_G^* is a γ_{grR}-function of G and so $\gamma_{grR}(G) = n$.
10.2. If G_p is isomorphic to one of the graphs of family $G_{ac}^{(9)}$, then f_G^* is a γ_{grR}-function of G and so $\gamma_{grR}(G) = n$.
10.3. If G has a subgraph which is isomorphic to one of the graphs of family $G_{bd}^{(10)}$, then G has a path of length 6, so $\gamma_{grR}(G) < n$. □

Now, according to Lemmas 1, 2 and 3, if $\gamma_{grR}(G) = n$ then $G_p \in \Phi$. Thus the proof is easily established. □

8. Concluding remark

In Theorem 16, we characterized the triangle free graphs G with $\gamma_{grR}(G) = n$. Now we may have the below problems.
Problem 1. Let G be an arbitrary graph (G may have a triangle). Characterize graphs G with $\gamma_{grR}(G) = n$.
Problem 2. Characterize graphs G with $\gamma_{grR}(G) = n - 1$.

Conflict of interest. The authors declare that they have no conflict of interest.

Data Availability. Data sharing is not applicable to this article as no datasets were generated or analyzed during the current study.

References

[1] H. Abdollahzadeh Ahangar and S.R. Mirmehdipour, *Bounds on the restrained Roman domination number of a graph*, Commun. Comb. Optim. **1** (2016), no. 1, 75–82
https://doi.org/10.22049/cco.2016.13556.

[2] M. Alishahi and D.A. Mojdeh, *Global outer connected domination number of a graph*, Algebra Discrete Math. **25** (2018), no. 1, 18–26.

[3] J. Amjadi, B. Samadi, and L. Volkmann, *Total restrained Roman domination*, Commun. Comb. Optim. **8** (2023), no. 3, 575–587
https://doi.org/10.22049/cco.2022.27628.1303.

[4] G. Chartrand and L. Lesniak, *Graphs and Digraphs, 4th ed.*, CRC press, Boca Raton, 2010.

[5] M. Chellali, N. Jafari Rad, S.M. Sheikholeslami, and L. Volkmann, *Roman domination in graphs*, Topics in Domination in Graphs (T.W. Haynes, S.T. Hedetniemi, and M.A. Henning, eds.), Springer, Berlin/Heidelberg, 2020, pp. 365–409 https://doi.org/10.1007/978-3-030-51117-3_11.

[6] ———, *Varieties of Roman domination II*, AKCE Int. J. Graphs Comb. **17** (2020), no. 3, 966–984 https://doi.org/10.1016/j.akcej.2019.12.001.

[7] ———, *Varieties of Roman domination*, Structures of Domination in Graphs (T.W. Haynes, S.T. Hedetniemi, and M.A. Henning, eds.), Springer, Berlin/Heidelberg, 2021, p. 273–307.

[8] E.J. Cockayne, P.A. Dreyer Jr, S.M. Hedetniemi, and S.T. Hedetniemi, *Roman domination in graphs*, Discrete Math. **278** (2004), no. 1-3, 11–22 https://doi.org/10.1016/j.disc.2003.06.004.

[9] P. Dankelmann, J.H. Hattingh, M.A. Henning, and H.C. Swart, *Trees with equal domination and restrained domination numbers*, J. Global Optim. **34** (2006), 597–607 https://doi.org/10.1007/s10898-005-8565-z.

[10] D. Delić and C. Wang, *The global connected domination in graphs*, Ars Combin. **114** (2014), 105–110.

[11] G.S. Domke, J.H. Hattingh, S.T. Hedetniemi, R.C. Laskar, and L.R. Markus, *Restrained domination in graphs*, Discrete Math. **203** (1999), no. 1-3, 61–69 https://doi.org/10.1016/S0012-365X(99)00016-3.

[12] G.S. Domke, J.H. Hattingh, and M.A. Henning, *Restrained domination in graphs with minimum degree two*, **35** (2000), 239–254.

[13] G.S. Domke, J.H. Hattingh, M.A. Henning, and L.R. Markus, *Restrained domination in trees*, Discrete Math. **211** (2000), no. 1-3, 1–9 https://doi.org/10.1016/S0012-365X(99)00036-9.

[14] X. Fu, Y. Yang, and B. Jiang, *Roman domination in regular graphs*, Discrete Math. **309** (2009), no. 6, 1528–1537 https://doi.org/10.1016/j.disc.2008.03.006.

[15] T.W. Haynes, S. Hedetniemi, and P. Slater, *Fundamentals of Domination in Graphs (1st ed.)*, CRC press, 1998.

[16] M.A. Henning, *Defending the Roman Empire from multiple attacks*, Discrete Math. **271** (2003), no. 1-3, 101–115 https://doi.org/10.1016/S0012-365X(03)00040-2.

[17] N. Jafari Rad and L. Volkmann, *Roman domination perfect graphs*, An. St. Univ. Ovidius Constanta, Ser. Mat. **19** (2011), no. 3, 167–174.

[18] R. Kala and T.R.N. Vasantha, *Global Restrained Domination Number of a Graph*, pp. 39–49, Narosa Publishing House Pvt. Limited, New Dehli, 2009.

[19] V.R. Kulli and B. Janakiram, *The total global domination number of a graph*, Indian J. Pure Appl. Math. **27** (1996), no. 6, 537–542.

[20] C.H. Liu and G.J. Chang, *Roman domination on strongly chordal graphs*, J. Comb. Optim. **26** (2013), no. 3, 608–619 https://doi.org/10.1007/s10878-012-9482-y.

[21] A.C. Martinez, A. Martinez Arias, and M.M. Castillo, *A characterization relating domination, semitotal domination and total Roman domination in trees*, Commun. Comb. Optim. **6** (2021), no. 2, 197–209 https://doi.org/10.22049/cco.2020.26892.1157.

[22] D.A. Mojdeh and M. Alishahi, *Outer independent global dominating set of trees and unicyclic graphs*, Electron. J. Graph Theory Appl. **7** (2019), no. 1, 121–145 https://doi.org/10.5614/ejgta.2019.7.1.10.

[23] D.A. Mojdeh, G. Hao, I. Masoumi, and A. Parsian, *Unique response strong Roman dominating functions of graphs*, Electron. J. Graph Theory Appl. **9** (2021), no. 2, 469–484
https://dx.doi.org/10.5614/ejgta.2021.9.2.18.

[24] P.R.L. Pushpam and S. Padmapriea, *Restrained Roman domination in graphs*, Trans. Comb. **4** (2015), no. 1, 1–17
https://doi.org/10.22108/toc.2015.4395.

[25] ______, *Global Roman domination in graphs*, Discrete Appl. Math. **200** (2016), 176–185
https://doi.org/10.1016/j.dam.2015.07.014.

[26] I. Stewart, *Defend the Roman empire!*, Sci. Am. **281** (1999), no. 6, 136–138.

[27] D.S. Vladimir, *Roman domination excellent graphs: trees*, Commun. Comb. Optim. **3** (2018), no. 1, 1–24
https://doi.org/10.22049/cco.2017.25806.1041.