Communications in Combinatorics and Optimization
Vol. xx, No. x (xxxx), pp. 1-16
https://doi.org/10.22049/cco.2023.28166.1465

CCO
Commun. Comb. Optim.

Research Article

On the essential dot product graph of a commutative ring

Asma Ali[†] **and Bakhtiyar Ahmad**[*]

Department of Mathematics, Aligarh Muslim University, Aligarh
[†]asma_ali@rediffmail.com
[*]bakhtiyarahmad2686@gmail.com

Received: 17 December 2022; Accepted: 30 October 2023
Published Online: 15 November 2023

Abstract: Let $\mathcal{B}$ be a commutative ring with unity $1 \neq 0$, $1 \leq m < \infty$ be an integer and $\mathcal{R} = \mathcal{B} \times \mathcal{B} \times \cdots \times \mathcal{B}$ (m times). The total essential dot product graph $ETD(\mathcal{R})$ and the essential zero-divisor dot product graph $EZD(\mathcal{R})$ are undirected graphs with the vertex sets $\mathcal{R}^* = \mathcal{R} \setminus \{(0, 0, \ldots, 0)\}$ and $Z(\mathcal{R})^* = Z(\mathcal{R}) \setminus \{(0, 0, \ldots, 0)\}$ respectively. Two distinct vertices $w = (w_1, w_2, \ldots, w_m)$ and $z = (z_1, z_2, \ldots, z_m)$ are adjacent if and only if $ann_{\mathcal{B}}(w \cdot z)$ is an essential ideal of $\mathcal{B}$ (where $w \cdot z = w_1z_1 + w_2z_2 + \cdots + w_mz_m \in \mathcal{B}$). In this paper, we prove some results on connectedness, diameter and girth of $ETD(\mathcal{R})$ and $EZD(\mathcal{R})$. We classify the ring $\mathcal{R}$ such that $EZD(\mathcal{R})$ and $ETD(\mathcal{R})$ are planar, outerplanar, and of genus one.

Keywords: essential graph, dot product graph, planar graph, genus, reduced ring, essential ideal.

AMS Subject classification: 13A15, 05C99, 05C10.

1. Introduction

Assignment of a graph to a commutative ring was first started and studied by Beck [9]. He was mainly interested in coloring of the graph. Let $\mathcal{R}$ be a ring. In 1998, Anderson et al. in [4] introduced the zero-divisor graph $\Gamma(\mathcal{R})$ in which they considered the vertex set to be the set of nonzero zero-divisors of $\mathcal{R}$, denoted by $Z(\mathcal{R})^*$. Two distinct vertices are adjacent if their product is zero. Anderson et al. studied about connectedness, diameter and girth of the graph $\Gamma(\mathcal{R})$. Since then, a lot of work has been done to explore the structure of a ring with respect to its zero-divisor graph, as one can see [1–5, 10, 11]. Furthermore, the concept of essential graph $EG(\mathcal{R})$ was introduced

* *Corresponding author*

by Nikmeher et al. [13]. They considered $Z(\mathcal{R})^*$ to be vertex set of the graph and two distinct vertices $w, z \in Z(\mathcal{R})^*$ are adjacent if and only if $ann_{\mathcal{R}}(wz) \leq_e \mathcal{R}$. Nikmeher et al. observed that $EG(\mathcal{R})$ is extended graph of zero-divisor graph $\Gamma(\mathcal{R})$. They studied about connectedness, diameter and girth of essential graph $EG(\mathcal{R})$. By considering this concept of a essential graph, many authors have shown their interest for further study [6, 12, 14, 15].

Badawi [7] in his paper, considered the ring of the form $\mathcal{R} = \mathcal{B} \times \cdots \times \mathcal{B}$ (m times), $1 \leq m < \infty$, where $\mathcal{B}$ is a commutative ring with $1 \neq 0$. He introduced the total dot product graph $TD(\mathcal{R})$ and zero-divisor dot product graph $ZD(\mathcal{R})$, in which vertices are taken from $\mathcal{R}^*$ and $Z(\mathcal{R})^*$ respectively and two distinct vertices $w = (w_1, w_2, \ldots, w_m)$ and $z = (z_1, z_2, \ldots, z_m)$ are adjacent if their normal dot product is zero. Motivated by the idea of Badawi and concept of essential graph given by Nikmeher et al., we introduce the "essential dot product graph". In this paper, we introduce two types of graphs, "total essential dot product graph" and "essential zero-divisor dot product graph", in which vertices are taken from $\mathcal{R}^* = \mathcal{R} \setminus \{(0, 0, \ldots, 0)\}$ and $Z(\mathcal{R})^* = Z(\mathcal{R}) \setminus \{(0, 0, \ldots, 0)\}$ respectively. Two distinct vertices w and z are adjacent if and only if $ann_{\mathcal{B}}(w \cdot z) \leq_e \mathcal{B}$. We denote total essential dot product graph by $ETD(\mathcal{R})$ and essential zero-divisor dot product graph by $EZD(\mathcal{R})$. It is easy to observe that $ETD(\mathcal{R})$ and $EZD(\mathcal{R})$ are extended graphs of $TD(\mathcal{R})$ and $ZD(\mathcal{R})$ respectively.

Throughout the paper, we consider ring to be a commutative ring with unity. We denote $Z(\mathcal{R})$, $Z(\mathcal{B})$, $\mathcal{R}^\times$ and $\mathcal{B}^\times$ set of zero-divisor elements of $\mathcal{R}$, set of zero-divisor elements of $\mathcal{B}$, unit elements of $\mathcal{R}$ and unit elements of $\mathcal{B}$ respectively. Let us recall some basic definition of graph and ring regarding the present paper. A ring $\mathcal{R}$ is said to be reduced if $N(\mathcal{R}) = \{0\}$, where $N(\mathcal{R})$ is the set of nilpotent elements of $\mathcal{R}$. An ideal $\mathcal{I}$ of $\mathcal{R}$ is said to be an essential ideal, if it has nonzero intersection with every nonzero ideal of $\mathcal{R}$. We denote $\leq_e \mathcal{R}$ to be an essential ideal of $\mathcal{R}$ and $\leq_e \mathcal{B}$ to be an essential ideal of $\mathcal{B}$. A graph $G = (V, E)$ is defined as the set of vertices and its edges. If every pair of distinct vertices of G are joined by a path, then G is said to be connected. We define $d(r, s)$ to be the length of a shortest path from r to s, where r and s are vertices of a graph G. We say $d(r, s) = \infty$, if there is no path between r and s, where $r, s \in V$. The diameter of graph is denoted and given by $diam(G) = max\{d(r, s) : \ r, s \in V\}$. The girth of graph is the length of shortest cycle in G, it is denoted by $gr(G)$. We denote $gr(G) = \infty$, if it contains no cycle. If for every distinct $w, z \in V(G)$ are adjacent, then the graph is said to be *complete* graph. A *complete bipartite* graph is a graph in which the vertex set V is divided into two vertex sets say V_1 and V_2 such that for every $x \in V_1$ is adjacent to every $y \in V_2$, and no two distinct vertices in the same set are adjacent, we denote this graph by $K_{m,n}$, where $m, n \in \mathbb{N} \setminus \{0\}$. An outerplanar graph G is a graph such that it can be drawn on a plane in a way that no vertex lies on the bounded region of the plane.

Let S_n be a sphere with n handles, where $n \in \mathbb{N}$, i.e. S_n is an orientable with n handles. The minimum n such that G can be orientable in S_n is defined as the genus of the graph, we denote the genus of a graph G by $\gamma(G)$. A graph G with $\gamma(G) = 0$ is called a *planar* graph. A graph G with $\gamma(G) = 1$ is called a *toroidal* graph. A minor

G' of a graph G is obtained by contracting the edges of G and isolated vertices in G. We symbolize the contracted edges by the vertex $[u, z]$, where $u, z \in V(G)$. If G has minor G', then $\gamma(G') \leq \gamma(G)$.
In the second section, we study about the connectedness, diameter and girth of $ETD(\mathcal{R})$ and $EZD(\mathcal{R})$ for $m = 2$, $m \geq 2$ and $m \geq 3$. In the third section, we establish the relation between $TD(\mathcal{R})$, $ZD(\mathcal{R})$, $ETD(\mathcal{R})$ and $EZD(\mathcal{R})$. In the fourth section, we classify the ring $\mathcal{R}$ such that $ETD(\mathcal{R})$ and $EZD(\mathcal{R})$ to be a planar graph. In the last section, we classify the ring $\mathcal{R}$ such that $ETD(\mathcal{R})$ and $EZD(\mathcal{R})$ to be of genus one.

2. Properties of $ETD(\mathcal{R})$ and $EZD(\mathcal{R})$

In this section, we discuss some results about connectedness, diameter and girth of $ETD(\mathcal{R})$ and $EZD(\mathcal{R})$ and establish the affinity between graph $EZD(\mathcal{R})$ and $EG(\mathcal{R})$.

Lemma 1. *[[13], Lemma 3.1] Let $\mathcal{R}$ be a nonreduced ring. Then the following statements hold:*

(i) For every $u \in N(\mathcal{R})^*$, u is adjacent to all other vertices of $EG(\mathcal{R})$.

(ii) $EG(\mathcal{R})[N(\mathcal{R})^*]$ is a (induced) complete subgraph of $EG(\mathcal{R})$.

Lemma 2. *Let $\mathcal{B}$ be a ring with $1 \neq 0$ and $\mathcal{R} = \mathcal{B} \times \mathcal{B} \times \cdots \times \mathcal{B}$ (m times), where $2 \leq m < \infty$. Then the following hold:*

(i) For every $w \in N(\mathcal{R})^*$, w is adjacent to all other vertices of $EZD(\mathcal{R})$ (respectively, $ETD(\mathcal{R})$).

(ii) $EZD(\mathcal{R})[N(\mathcal{R})^*]$ (respectively, $ETD(\mathcal{R})[N(\mathcal{R})^*]$) is a (induced) complete subgraph of $EZD(\mathcal{R})$ (respectively, $ETD(\mathcal{R})$).

Proof. (i) Let $w = (w_1, w_2, \dots, w_m) \in N(\mathcal{R})^*$. Then for any $z = (z_1, z_2, \dots, z_m) \in \mathcal{R}^*$, we have $w \cdot z = w_1 z_1 + \cdots + w_m z_m = k \in N(\mathcal{B})$ (sum of nilpotent elements is nilpotent). Now, we have to show that $ann_{\mathcal{B}}(k) \leq_e \mathcal{B}$. We can assume that $k \neq 0$. Let $I_{\mathcal{B}}$ be a nonzero ideal of $\mathcal{B}$. Let $b \in I_{\mathcal{B}} \setminus \{0\}$. Since $k \in N(\mathcal{B})$, it is possible to find $n \geq 1$ such that $bk^{(n-1)} \neq 0$ but $bk^n = 0$. Hence, $bk^{(n-1)} \in ann_{\mathcal{B}}(k) \cap I_{\mathcal{B}}$ and so, $ann_{\mathcal{B}}(k) \cap I_{\mathcal{B}} \neq (0)$. Therefore, $ann_{\mathcal{B}}(k)$ is an essential ideal of $\mathcal{B}$. Hence w is adjacent to all other vertices of $EZD(\mathcal{R})$ (respectively, $ETD(\mathcal{R})$).

(ii) It is clear from part (i). □

Lemma 3. *Let $\mathcal{B}$ be a ring with $1 \neq 0$ and $\mathcal{R} = \mathcal{B} \times \mathcal{B} \times \cdots \times \mathcal{B}$ (m times), $1 \leq m < \infty$. If $ann_{\mathcal{R}}(wz) \leq_e \mathcal{R}$ for some $w, z \in \mathcal{R}^*$, then $ann_{\mathcal{B}}(w \cdot z) \leq_e \mathcal{B}$.*

Proof. Suppose that $ann_{\mathcal{R}}(wz) \leq_e \mathcal{R}$, for some $w = (w_1, w_2, \ldots, w_m)$, $z = (z_1, z_2, \ldots, z_m) \in \mathcal{R}^*$. Then we have to show that $ann_{\mathcal{B}}(w \cdot z) \leq_e \mathcal{B}$. Now, consider the following cases:

Case (a) If $wz = (0, 0, \ldots, 0)$, then $w \cdot z = 0$ and hence $ann_{\mathcal{B}}(w \cdot z) \leq_e \mathcal{B}$.

Case (b) $wz \neq (0, 0, \ldots, 0)$.
If $m = 1$, then $wz = w \cdot z$, $\mathcal{R} = \mathcal{B}$, and from $ann_{\mathcal{R}}(wz) \leq_e \mathcal{R}$, it follows that $ann_{\mathcal{B}}(w \cdot z) \leq_e \mathcal{B}$. Hence, we can assume that $m \geq 2$. As $wz \neq (0, 0, \ldots, 0)$, it follows that $w_i z_i \neq 0$ for at least one $i \in \{1, 2, \ldots, m\}$. We can assume without loss of generality that $w_1 z_1 \neq 0, \ldots, w_r z_r \neq 0$, whereas $w_j z_j = 0$ for all $j \in \{1, 2, \ldots, m\} \setminus \{1, \ldots, r\}$. Note that $w \cdot z = \sum_{k=1}^{r} w_i z_i$. Observe that $ann_{\mathcal{R}}(wz) = I_1 \times I_2 \times \cdots \times I_m$ with $I_i = ann_{\mathcal{B}}(w_i z_i)$ for each $i \in \{1, 2, \ldots, m\}$. Let $I_{\mathcal{B}}$ be a nonzero ideal of $\mathcal{B}$. Let J be the ideal of $\mathcal{R}$ defined by $J = I_{\mathcal{B}} \times (0) \times \cdots \times (0)$. Since $ann_{\mathcal{R}}(wz) \leq_e \mathcal{R}$, it follows that $I_{\mathcal{B}} \cap ann_{\mathcal{B}}(w_1 z_1) \neq (0)$. Suppose that $r \geq 2$. Consider the ideal J' of $\mathcal{R}$ defined by $J' = (0) \times (I_{\mathcal{B}} \cap ann_{\mathcal{B}}(w_1 z_1)) \times (0) \times \cdots \times (0)$. From $I_1 \times I_2 \times \cdots \times I_m$, it follows that $(I_{\mathcal{B}} \cap ann_{\mathcal{B}}(w_1 z_1)) \cap ann_{\mathcal{B}}(w_2 z_2) \neq (0)$. The above argument can be repeated and we arrive that there exists $b \in I_{\mathcal{B}} \setminus \{0\}$ such that $b w_i z_i = 0$ for each $i \in \{1, \ldots, r\}$. Hence, $b(w_1 z_1 + w_2 z_2 + \cdots + w_m z_m) = 0 = b(w \cdot z)$. Therefore, $I_{\mathcal{B}} \cap ann_{\mathcal{B}}(w \cdot z) \neq (0)$. This proves that $ann_{\mathcal{B}}(w \cdot z) \leq_e \mathcal{B}$. □

Remark 1. Converse of the Lemma 3 need not be true in general.

Example 1. Let $\mathcal{B} = \mathbb{Z}_{12}$ and $R = \mathbb{Z}_{12} \times \mathbb{Z}_{12}$, then for $a = (1, 1), b = (3, 3) \in \mathcal{R}^*$, $ann_{\mathcal{R}}((1, 1)(3, 3)) = ann_{\mathcal{R}}(3, 3) \nleq_e \mathcal{R}$ but $ann_{\mathcal{B}}((1, 1) \cdot (3, 3)) = ann_{\mathcal{B}}(6) \leq_e \mathcal{B}$ (since $6 \in N(\mathcal{B})$).

Lemma 4. *Let $\mathcal{B}$ be a ring with $1 \neq 0$ and $\mathcal{R} = \mathcal{B} \times \mathcal{B} \times \cdots \times \mathcal{B}$ (m times), where $2 \leq m < \infty$. If $w - z$ is an edge of $EG(\mathcal{R})$, then $w - z$ is also an edge of $EZD(\mathcal{R})$, where $w, z \in Z(\mathcal{R})^*$.*

Proof. Let $w, z \in Z(\mathcal{R})^*$ and $w - z$ is an edge of $EG(\mathcal{R})$, then $ann_{\mathcal{R}}(wz) \leq_e \mathcal{R}$. Therefore, from Lemma 3, $ann_{\mathcal{B}}(w \cdot z) \leq_e \mathcal{B}$. Hence $w - z$ is also an edge of $EZD(\mathcal{R})$. □

Theorem 1. *Let $\mathcal{B}$ be a ring with $1 \neq 0$ and $\mathcal{R} = \mathcal{B} \times \mathcal{B} \times \cdots \times \mathcal{B}$ (m times), where $2 \leq m < \infty$. Then $EZD(\mathcal{R})$ is connected and $diam(EZD(\mathcal{R})) \leq 3$. Moreover, if $EZD(\mathcal{R})$ contains a cycle, then $gr(EZD(\mathcal{R})) \leq 4$.*

Proof. It follows from Lemma 4 that $EG(\mathcal{R})$ is a spanning subgraph of $EZD(\mathcal{R})$. Hence, by [[13], Theorem 2.1], we get that $EZD(\mathcal{R})$ is connected, $diam(EZD(\mathcal{R}))$ ≤ 3. By assumption, $EZD(\mathcal{R})$ contains a cycle. Suppose that $m \geq 3$. For each $i \in \{1, 2, 3, \ldots, m\}$, let e_i denote the element of $\mathcal{R}$ whose i-th coordinate equal 1 and j-th coordinate equals 0 for all $j \in \{1, 2, \ldots, m\} \setminus \{i\}$. Note $e_1 - e_2 - e_3 - e_1$ is a cycle of length 3 in $EZD(\mathcal{R})$. Therefore, $gr(EZD(\mathcal{R})) = 3$ if $m \geq 3$. Assume that $m = 2$.

If $\mathcal{B}$ is not reduced, then there exists $b \in \mathcal{B} \setminus \{0\}$ such that $b^2 = 0$. Observe that $(b,0)-(0,b)-(1,0)-(b,0)$ is a cycle of length 3 in $EZD(\mathcal{R})$. Hence $gr(EZD(\mathcal{R})) = 3$. Suppose that $\mathcal{B}$ is reduced. Then either $\mathcal{B}$ is an integral domain or $\mathcal{B}$ is not an integral domain. If $\mathcal{B}$ is an integral domain, then $Z(\mathcal{R})^* = V_1 \cup V_2$, where $V_1 = \{(b,0) | b \in \mathcal{B}^*\}$ and $V_2 = \{(0,c) | c \in \mathcal{B}^*\}$. It is not hard to verify that $EZD(\mathcal{R}) = \Gamma(\mathcal{R})$ is a complete bipartite graph with vertex partition $Z(\mathcal{R})^* = V_1 \cup V_2$. Since $EZD(\mathcal{R})$ contains a cycle by assumption, it follows that $gr(EZD(\mathcal{R})) = 4$. Assume that $\mathcal{B}$ is reduced ring but not an integral domain. Then there exist $a, b \in \mathcal{B} \setminus \{0\}$ such that $ab = 0$. Then $a \neq b$ and $(a,0) - (b,0) - (0,a) - (a,0)$ is a cycle of length 3 in $EZD(\mathcal{R})$. Hence, $gr(EZD(\mathcal{R})) = 3$. Therefore, if $EZD(\mathcal{R})$ contains a cycle, then $gr(EZD(\mathcal{R})) \leq 4$. □

Theorem 2. *Let $\mathcal{B}$ be a nonreduced ring with $1 \neq 0$ and $\mathcal{R} = \mathcal{B} \times \mathcal{B} \times \cdots \times \mathcal{B}$ (m times), where $2 \leq m < \infty$. Then the following hold:*

(i) $ETD(\mathcal{R})$ is connected and $diam(ETD(\mathcal{R})) = 2$.

(ii) $EZD(\mathcal{R})$ is connected and $diam(EZD(\mathcal{R})) = 2$.

Proof. (i) Since $\mathcal{B}$ is a non-reduced ring, then there exist $n = (c, c, \ldots, c) \in N(\mathcal{R})^*$, where $c \in N(\mathcal{B})^*$. We know from Lemma 2-(i) that n is adjacent to all the vertices of $ETD(\mathcal{R})$. Hence, we obtain that $ETD(\mathcal{R})$ is connected and $diam(ETD(\mathcal{R})) \leq 2$. As $c \in N(\mathcal{B})^*$, it follows that $1 + c \in \mathcal{B}^\times$ and $1 + c \neq 1$. Let $x = (1, 0, \ldots, 0)$ and $y = (1 + c, 0, \ldots, 0)$. It is clear that $x \cdot y = 1 + c$ and so, $(0) = ann_{\mathcal{B}}(x \cdot y) \nleq_e \mathcal{B}$. Therefore, x and y are not adjacent in $ETD(\mathcal{R})$. Hence, $diam(ETD(\mathcal{R})) \geq 2$ and so, $diam(ETD(\mathcal{R})) = 2$.

(ii) It can be shown as in the proof of (i) that $diam(EZD(\mathcal{R})) = 2$. □

Lemma 5. *Let $\mathcal{B}$ be a ring with $1 \neq 0$ and $\mathcal{R} = \mathcal{B} \times \mathcal{B} \times \cdots \times \mathcal{B}$ (m times), where $2 \leq m < \infty$. Then the following hold:*

(i) $TD(\mathcal{R})$ is a spanning subgraph of $ETD(\mathcal{R})$.

(ii) $ZD(\mathcal{R})$ is a spanning subgraph of $EZD(\mathcal{R})$.

Proof. (i) Let $w, z \in \mathcal{R}^*$ be such that $w - z$ is an edge of $TD(\mathcal{R})$. Then $w \cdot z = 0$ and so, $ann_{\mathcal{B}}(w \cdot z) \leq_e \mathcal{B}$. Therefore, $w - z$ is an edge of $ETD(\mathcal{R})$. As $V(TD(\mathcal{R})) = V(ETD(\mathcal{R})) = \mathcal{R}^*$, it follows that $TD(\mathcal{R})$ is a spanning subgraph of $ETD(\mathcal{R})$.

(ii) This can be proved using arguments similar to those that are used in the proof of (i). □

Remark 2. Converse of the Lemma 5 need not be true in general.

Example 2. Let $\mathcal{B} \cong \mathbb{Z}_4$ and $\mathcal{R} = \mathcal{B} \times \mathcal{B}$. Consider $(2,0)$ and $(1,0) \in \mathcal{R}^*$. Then $(2,0)\cdot(1,0) = 2 \neq 0$ and $2 \in N(\mathcal{B})$. Hence $(2,0) - (1,0)$ is an edge in $ETD(\mathcal{R})$ (respectively, $EZD(\mathcal{R})$) but not an edge in $TD(\mathcal{R})$ (respectively, $ZD(\mathcal{R})$).

Remark 3. Let $\mathcal{B}$ be a ring with $1 \neq 0$ and $\mathcal{R} = \mathcal{B} \times \cdots \times \mathcal{B}$ (m times), $2 \leq m < \infty$. It is shown in Lemma 5 that $TD(\mathcal{R})$ (respectively, $ZD(\mathcal{R})$) is a spanning subgraph of $ETD(\mathcal{R})$ (respectively, $EZD(\mathcal{R})$). Assume that $\mathcal{B}$ is a reduced ring. Then it is not hard to verify that for any any $b \in \mathcal{B}$, $ann_{\mathcal{B}}(b) \cap \mathcal{B}b = (0)$. Let $w, z \in \mathcal{R}^*$ be such that $w - z$ is an edge $ETD(\mathcal{R})$. Then $ann_{\mathcal{B}}(w \cdot z) \leq_e \mathcal{B}$. From $ann_{\mathcal{B}}(w \cdot z) \cap \mathcal{B}(w \cdot z) = (0)$, it follows that $w \cdot z = 0$. Therefore, $w - z$ is an edge of $TD(\mathcal{R})$. Hence, $ETD(\mathcal{R})$ is spanning subgraph of $TD(\mathcal{R})$ and so, $EZD(\mathcal{R}) = ZD(\mathcal{R})$. Similarly, $EZD(\mathcal{R})$ is spanning subgraph of $ZD(\mathcal{R})$ and so, $EZD(\mathcal{R}) = ZD(\mathcal{R})$. Thus if $\mathcal{B}$ is a reduced ring, $ETD(\mathcal{R}) = TD(\mathcal{R})$ and $EZD(\mathcal{R}) = ZD(\mathcal{R})$.

Theorem 3. *Let $\mathcal{B}$ (not an integral domain) be a reduced ring with $1 \neq 0$ and $\mathcal{R} = \mathcal{B} \times \mathcal{B}$. Then*

(i) $ETD(\mathcal{R})$ is a connected graph and $diam(ETD(\mathcal{R})) = 3$.

(ii) $diam(EZD(\mathcal{R})) = 3$.

(iii) $gr(ETD(\mathcal{R})) = gr(EZD(\mathcal{R})) = 3$.

Proof. By hypothesis, $\mathcal{B}$ is a reduced ring but not an integral domain. We Know from Remark 3 that $ETD(\mathcal{R})= TD(\mathcal{R})$ and $EZD(\mathcal{R})= ZD(\mathcal{R})$.

(i) This follows from [[7], Theorem 2.3(1)].

(ii) This follows from [[7], Theorem 2.3(2)].

(iii) This follows from [[7], Theorem 2.3(3)].

□

Theorem 4. *Let $\mathcal{B}$ be a ring with $1 \neq 0$ and $R = \mathcal{B} \times \mathcal{B} \times \cdots \times \mathcal{B}$ (m times), $3 \leq m < \infty$. Then $ETD(\mathcal{R})$ is connected and diam($ETD(\mathcal{R})) = 2$.*

Proof. From [[7], Theorem 2.4], $TD(\mathcal{R})$ is connected and $diam(TD(\mathcal{R})) = 2$. By Lemma 5-(i), $TD(\mathcal{R})$ is a subgraph of $ETD(\mathcal{R})$. Therefore, $ETD(\mathcal{R})$ is connected and $diam(ETD(\mathcal{R})) \leq diam(TD(\mathcal{R})) = 2$. Now, it remains to prove that $diam(ETD(\mathcal{R})) = 2$. For this, we have to show that there exists $a, b \in \mathcal{R}^*$ such that $d(a,b) = 2$. Let $a = (1,1,1,0,\ldots,0), b = (1,0,0,0,\ldots,0) \in \mathcal{R}^*$, then $ann_{\mathcal{B}}(a \cdot b) \not\leq_e \mathcal{B}$ and hence $d(a,b) > 1$. Also, we have $d(a,b) \leq 2$. Therefore, diam$(ETD(\mathcal{R})) = 2$. □

Theorem 5. *Let $\mathcal{B}$ be a reduced ring with $1 \neq 0$ and $\mathcal{R} = \mathcal{B} \times \mathcal{B} \times \mathcal{B}$. Then the following statements hold:*

(i) $diam(EZD(\mathcal{R})) = 3$ if $\mathcal{B}$ is an integral domain.

(ii) $diam(EZD(\mathcal{R})) = 2$ if $\mathcal{B}$ is not an integral domain.

Proof. We know from Theorem 1 that $diam(EZD(\mathcal{R})) \leq 3$. Let $w = (1,1,0)$ and $z = (0,1,1)$. It is clear that $w, z \in Z(\mathcal{R})^*$ and $w \neq z$. Observe that $ann_{\mathcal{B}}(w \cdot z) \nleq_e \mathcal{B}$. Therefore, $d(w,z) \geq 2$ in $EZD(\mathcal{R})$. Hence, $diam(EZD(\mathcal{R})) \geq 2$.

(i) Assume that $\mathcal{B}$ is an integral domain. We know from Remark 3 that $EZD(\mathcal{R}) = ZD(\mathcal{R})$. Hence, we obtain from [[7], Theorem 2.5(1)] that $diam(EZD(\mathcal{R})) = 3$.

(ii) Assume that $\mathcal{B}$ is not an integral domain. We know from [[7], Theorem 2.5(2)] that $diam(ZD(\mathcal{R})) = 2$. As $ZD(\mathcal{R})$ is a spanning subgraph of $EZD(\mathcal{R})$, by Lemma 5-(ii), we obtain that $diam(EZD(\mathcal{R})) \leq 2$. It is noted that at the beginning of the proof of this theorem that $diam(EZD(\mathcal{R})) \geq 2$ and so, $diam(EZD(\mathcal{R})) = 2$. $\square$

Theorem 6. *Let $\mathcal{B}$ be a reduced ring with $1 \neq 0$ and $R = \mathcal{B} \times \mathcal{B} \times \cdots \times \mathcal{B}$ (m times), where $2 \leq m < \infty$. Then the followings statements are equivalent:*

(i) $EZD(\mathcal{R})$ is complete.

(ii) $\mathcal{B} \cong \mathbb{Z}_2$.

(iii) $EZD(\mathcal{R}) = K_{1,1}$

Proof. $(i) \implies (ii)$ Assume that $EZD(\mathcal{R})$ is complete graph. If $m \geq 3$, then $a = (1,0,0,\dots,0)$ and $b = (1,1,0,\dots,0) \in Z(\mathcal{R})^*$ are such that $ann_{\mathcal{B}}(a \cdot b) = ann_{\mathcal{B}}(1) = (0) \nleq_e \mathcal{B}$. This is a contradiction and so, $m = 2$. We claim that $|\mathcal{B}| = 2$. If $|\mathcal{B}| \geq 3$, then it is possible to find $w \in \mathcal{B} \setminus \{0,1\}$. Let $x = (1,0)$ and $y = (w,0)$. It is clear that $x, y \in Z(\mathcal{R})^*$ and $x \neq y$. Note that $x \cdot y = w$. Since $\mathcal{B}$ is reduced, $ann_{\mathcal{B}}(w) \cap \mathcal{B}w = (0)$. Hence, $ann_{\mathcal{B}}(w) \nleq_e \mathcal{B}$. Therefore, x and y are not adjacent in $EZD(\mathcal{R})$. This is a contradiction and so, $|\mathcal{B}| = 2$. Therefore, $m = 2$ and $\mathcal{B} \cong \mathbb{Z}_2$.
$(ii) \implies (iii)$ and $(iii) \implies (i)$ are clear. $\square$

Theorem 7. *Let $\mathcal{B}$ be a ring with $1 \neq 0$ and $\mathcal{R} = \mathcal{B} \times \mathcal{B} \times \cdots \times \mathcal{B}$ (m times), where $3 \leq m < \infty$. Then $gr(ETD(\mathcal{R})) = gr(EZD(\mathcal{R})) = 3$.*

Proof. The proof of theorem is contained in the proof of Theorem 1. $\square$

3. Relationship between $TD(\mathcal{R})$ and $ETD(\mathcal{R})$ (respectively, $ZD(\mathcal{R})$ and $EZD(\mathcal{R})$)

Let $\mathcal{B}$ be a ring with $1 \neq 0$ and $\mathcal{R} = \mathcal{B} \times \mathcal{B} \times \cdots \times \mathcal{B}$ (m times), $2 \leq m < \infty$. If $\mathcal{B}$ is a reduced ring, then it is noted in Remark 3 that $TD(\mathcal{R}) = ETD(\mathcal{R})$ and $ZD(\mathcal{R}) = EZD(\mathcal{R})$. In this section, with the help of results from [7] and [13], we deduce several corollaries and we characterize $\mathcal{R}$ such that $EG(\mathcal{R}) = EZD(\mathcal{R})$.

Corollary 1. *Let $\mathcal{B}$ be an integral domain and $\mathcal{R} = \mathcal{B} \times \mathcal{B}$. Then the following hold:*

(i) $EG(\mathcal{R}) = EZD(\mathcal{R})$,

(ii) $ETD(\mathcal{R})$ *is disconnected.*

Proof. (i) We know from Remark 3 that $ZD(\mathcal{R}) = EZD(\mathcal{R})$. Note that $\Gamma(\mathcal{R}) = ZD(\mathcal{R})$ by [[7], Theorem 2.1] and since $\mathcal{R}$ is reduced $\Gamma(\mathcal{R}) = EG(\mathcal{R})$ by [[13], Theorem 2.2]. Therefore, $EG(\mathcal{R}) = EZD(\mathcal{R})$.

(ii) Note that $TD(\mathcal{R}) = ETD(\mathcal{R})$ by Remark 3 and we know from [[7], Theorem 2.1] that $TD(\mathcal{R})$ is disconnected. Therefore, $ETD(\mathcal{R})$ is disconnected. □

Corollary 2. *Let $\mathcal{B}$ be a reduced ring with $1 \neq 0$ and $\mathcal{R} = \mathcal{B} \times \mathcal{B} \times \cdots \times \mathcal{B}$ (m times), where $2 \leq m < \infty$. Then $EZD(\mathcal{R}) = EG(\mathcal{R})$ if and only if $\mathcal{R} \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ or $m = 2$ and $\mathcal{B}$ is an integral domain.*

Proof. Since $\mathcal{B}$ is a reduced ring by hypothesis, $ZD(\mathcal{R}) = EZD(\mathcal{R})$ by Remark 3. As $\mathcal{R}$ is reduced, $\Gamma(\mathcal{R}) = EG(\mathcal{R})$ by [[13], Theorem 2.2]. Therefore, $EZD(\mathcal{R}) = EG(\mathcal{R})$ if and only if $ZD(\mathcal{R}) = \Gamma(\mathcal{R})$. Hence, we obtain from [[7], Theorem 2.2] that $EZD(\mathcal{R}) = EG(\mathcal{R})$ if and only if $\mathcal{R} \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ or $m = 2$ and $\mathcal{B}$ is an integral domain. □

Theorem 8. *Let $\mathcal{B}$ be a nonreduced ring with $1 \neq 0$ and $\mathcal{R} = \mathcal{B} \times \mathcal{B} \times \cdots \times \mathcal{B}$ (m times), where $2 \leq m < \infty$. Then the following hold:*

(i) $TD(\mathcal{R}) \neq ETD(\mathcal{R})$.

(ii) $ZD(\mathcal{R}) \neq EZD(\mathcal{R})$.

Proof. (i) To show that $TD(\mathcal{R}) \neq ETD(\mathcal{R})$, we have to show that there exists an edge in $ETD(\mathcal{R})$, which is not an edge of $TD(\mathcal{R})$. Since $\mathcal{B}$ is nonreduced ring, then there exists a nonzero nilpotent element i.e $f \in N(\mathcal{B})^*$. Now, consider $(f, 1, 0, \ldots, 0)$ and $(1, 0, 0, \ldots, 0) \in \mathcal{R}$. Clearly, $((f, 1, 0, \ldots, 0) \cdot (1, 0, 0, \ldots, 0)) = f \in N(\mathcal{B})^*$. From Lemma 2-(i), $ann_{\mathcal{B}}((f, 1, 0, \ldots, 0) \cdot (1, 0, 0, \ldots, 0)) \leq_e \mathcal{B}$. This implies that $(f, 1, 0, \ldots, 0) - (1, 0, 0, \ldots, 0)$ is an edge in $ETD(\mathcal{R})$ but not in $TD(\mathcal{R})$.

(ii) Proof is similar as above. □

4. Planarity and outerplanarity of $ETD(\mathcal{R})$ and $EZD(\mathcal{R})$

Lemma 6. *(Kuratowski Theorem) [16] A graph G is planar if and only if it contains no subdivision of K_5 or $K_{3,3}$.*

Lemma 7. *[16] A graph G is outerplanar if and only if it contains no subdivision of K_4 or $K_{2,3}$.*

Theorem 9. *Let $\mathcal{B}$ be a ring with $1 \neq 0$ and $\mathcal{R} = \mathcal{B} \times \mathcal{B} \times \cdots \times \mathcal{B}$ (m times), where $2 \leq m < \infty$. If $EZD(\mathcal{R})$ is planar if and only if $\mathcal{R}$ is isomorphic to one of the rings from the collection $\{ \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2, \mathbb{Z}_2 \times \mathbb{Z}_2, \mathbb{Z}_3 \times \mathbb{Z}_3\}$.*

Proof. Assume that $EZD(\mathcal{R})$ is planar. Let $i \in \{1, 2, \ldots, m\}$. It is convenient to denote the element of $\mathcal{R}$ whose $i - th$ coordinate equals 1 and $j - th$ coordinates equals 0 for all $j \in \{1, 2, \ldots, m\} \setminus \{i\}$ by e_i. It is clear that $e_i e_j = (0, 0, \ldots, 0)$ for all distinct $i, j \in \{1, 2, \ldots, m\}$. We first claim that $m \geq 3$. Suppose that $m \geq 4$. Let $V_1 = \{e_1, e_2, e_1 + e_2\}$ and $V_2 = \{e_3, e_4, e_3 + e_4\}$. It is clear that $V_1 \cup V_2 \subset Z(\mathcal{R})^* = V(EZD(\mathcal{R}))$. It is clear that $V_1 \cap V_2 = \emptyset$. For any $x \in V_1$ and $y \in V_2$, $xy = (0, 0, 0, 0, \ldots, 0)$. Therefore, x and y are adjacent in $EZD(\mathcal{R})$. Note that the subgraph of $EZD(\mathcal{R})$ induced by $V_1 \cup V_2$ contains $K_{3,3}$ as a subgraph. This is impossible by Lemma 6, since $EZD(\mathcal{R})$ is planar by assumption. Therefore, $m \leq 3$.
We claim that $\mathcal{B}$ is reduced. If $\mathcal{B}$ is not reduced, then it is possible to find $b \in \mathcal{B} \setminus \{0\}$ such that $b^2 = 0$. Let $W = \{e_1, e_2, (b, 0), (0, b), (b, b)\}$ in the case $m = 2$ (respectively, $W = \{e_1, e_2, (b, 0, 0), (0, b, 0), (b, b, 0)\}$ in the case $m = 3$). It is not hard to verify that the subgraph of $EZD(\mathcal{R})$ induced by W is a clique with 5 vertices. This is possible by Lemma 6, since $EZD(\mathcal{R})$ is planar by assumption. Therefore, $\mathcal{B}$ is reduced.
Suppose that $m = 3$. If $|\mathcal{B}| \geq 3$, then there exists $b \in \mathcal{B} \setminus \{0, 1\}$. Let $V_1 = \{e_1, (b, 0, 0), e_2\}$, $V_2 = \{e_3, (0, 0, b), (0, b, 0)\}$, and $V_3 = \{e_1 + e_3\}$. Note that $V_1 \cup V_2 \cup V_3 \subset Z(\mathcal{R})^*$. Observe that for each $y \in V_2$, $e_1 y = (b, 0, 0)y = (0, 0, 0)$, $e_2 e_3 = e_2(0, 0, b) = (0, 0, 0)$, $e_2 - e_1 + e_3 - (0, b, 0)$ is a path of length two in $EZD(\mathcal{R})$. Consider the subgraph H of $EZD(\mathcal{R})$ induced by $V_1 \cup V_2 \cup V_3$. The edges $e_2 - e_1 + e_3$ and $e_1 + e_3 - (0, b, 0)$ are in series of this subgraph (that is, their common end vertex is of degree two in H). By merging these edges in series, we obtain a subgraph of H isomorphic to $K_{3,3}$. This is impossible by Lemma 6, since $EZD(\mathcal{R})$ is planar by assumption. Therefore, $|\mathcal{B}| \leq 2$ and so, $\mathcal{R} \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$.
Suppose that $m = 2$. In this case, we claim that $|\mathcal{B}| \leq 3$. If $|\mathcal{B}| \geq 4$, then there exist distinct $b_1, b_2 \in \mathcal{B} \setminus \{0, 1\}$. Let $V_1 = \{e_1, (b_1, 0), (b_2, 0)\}$ and $V_2 = \{e_2, (0, b_1), (0, b_2)\}$. Note that $V_1 \cap V_2 = \emptyset$. For any $x \in V_1$ and $y \in V_2$, $x \cdot y = 0$. Therefore, the subgraph of $EZD(\mathcal{R})$ induced by $V_1 \cup V_2$ contains $K_{3,3}$ as a subgraph. This is impossible by Lemma 6, since $EZD(\mathcal{R})$ is planar by assumption. Therefore, $|\mathcal{B}| \leq 3$. Hence, $\mathcal{R}$ is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$ or $\mathbb{Z}_3 \times \mathbb{Z}_3$.
Thus if $EZD(\mathcal{R})$ is planar, then $\mathcal{R}$ is isomorphic to one of the rings from the collection $\{\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2, \mathbb{Z}_2 \times \mathbb{Z}_2, \mathbb{Z}_3 \times \mathbb{Z}_3\}$.
Conversely, assume that $\mathcal{R}$ is isomorphic to one of the rings from the collection $\{\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2, \mathbb{Z}_2 \times \mathbb{Z}_2, \mathbb{Z}_3 \times \mathbb{Z}_3\}$, then from Figure 1, Figure 2 and Figure 3 that $EZD(R)$ is planar. □

Theorem 10. *Let $\mathcal{B}$ be a ring with $1 \neq 0$ and $\mathcal{R} = \mathcal{B} \times \mathcal{B} \times \cdots \times \mathcal{B}$ (m times), where $2 \leq m < \infty$. If $ETD(\mathcal{R})$ is planar if and only if $\mathcal{R}$ is isomorphic to one of the rings from the collection $\{ \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2, \mathbb{Z}_2 \times \mathbb{Z}_2, \mathbb{Z}_3 \times \mathbb{Z}_3\}$.*

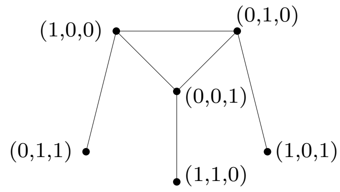

Figure 1. **Planar embedding of** $EZD(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2)$

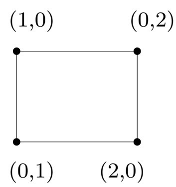

Figure 2. **Planar embedding of** $EZD(\mathbb{Z}_3 \times \mathbb{Z}_3)$

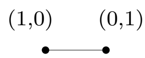

Figure 3. **Planar embedding of** $EZD(\mathbb{Z}_2 \times \mathbb{Z}_2)$

Proof. Assume that $ETD(\mathcal{R})$ is planar. Hence, its subgraph $EZD(\mathcal{R})$ is planar. Therefore, we obtain from Theorem 9 that $\mathcal{R}$ is isomorphic to one of the rings from the collection $\{\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2, \mathbb{Z}_2 \times \mathbb{Z}_2, \mathbb{Z}_3 \times \mathbb{Z}_3\}$.
Conversely, if $\mathcal{R}$ is isomorphic to one of the rings from the collection $\{ \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2, \mathbb{Z}_3 \times \mathbb{Z}_3, \mathbb{Z}_2 \times \mathbb{Z}_2\}$, then it is clear from Figure 4, Figure 5, and Figure 6 that $ETD(\mathcal{R})$ is planar. $\square$

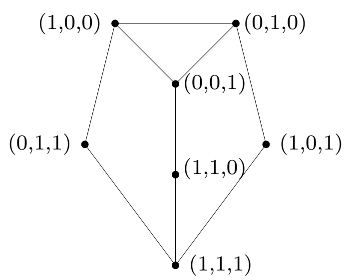

Figure 4. **Planar embedding of** $ETD(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2)$

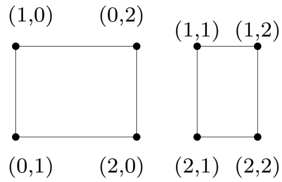

Figure 5. **Planar embedding of** $ETD(\mathbb{Z}_3 \times \mathbb{Z}_3)$

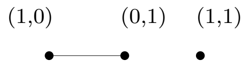

Figure 6. **Planar embedding of** $ETD(\mathbb{Z}_2 \times \mathbb{Z}_2)$

Theorem 11. *Let $\mathcal{B}$ be a ring with $1 \neq 0$ and $\mathcal{R} = \mathcal{B} \times \mathcal{B} \times \cdots \times \mathcal{B}$ (m times), where $2 \leq m < \infty$. Then $EZD(\mathcal{R})$ is outerplanar if and only if $\mathcal{R}$ is isomorphic to one of the rings from the collection $\{\ \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2,\ \mathbb{Z}_2 \times \mathbb{Z}_2,\ \mathbb{Z}_3 \times \mathbb{Z}_3\}$.*

Proof. Assume $EZD(\mathcal{R})$ is outerplanar. Then $EZD(\mathcal{R})$ is necessarily planar. Hence, we obtain from Theorem 9, $\mathcal{R}$ is isomorphic to one of the rings from the collection $\{\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2, \mathbb{Z}_2 \times \mathbb{Z}_2, \mathbb{Z}_3 \times \mathbb{Z}_3\}$.
Conversely, if $\mathcal{R}$ is isomorphic to one of the rings from the collection $\{\ \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2,$ $\mathbb{Z}_2 \times \mathbb{Z}_2, \mathbb{Z}_3 \times \mathbb{Z}_3\}$, then it is clear from Figure 1, Figure 2 and Figure 3 that $EZD(\mathcal{R})$ is outerplanar.

□

Theorem 12. *Let $\mathcal{B}$ be a ring with $1 \neq 0$ and $\mathcal{R} = \mathcal{B} \times \mathcal{B} \times \cdots \times \mathcal{B}$ (m times), where $2 \leq m < \infty$. Then $ETD(\mathcal{R})$ is outerplanar if and only if $\mathcal{R}$ is isomorphic to either $\mathbb{Z}_3 \times \mathbb{Z}_3$ or $\mathbb{Z}_2 \times \mathbb{Z}_2$.*

Proof. Assume that $ETD(\mathcal{R})$ is outerplanar. Then $ETD(\mathcal{R})$ is planar. Hence, we obtain from Theorem 10 that $\mathcal{R}$ is isomorphic to one of the rings from the collection $\{\ \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2,\ \mathbb{Z}_2 \times \mathbb{Z}_2,\ \mathbb{Z}_3 \times \mathbb{Z}_3\}$. Consider $T = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$. It is clear from Figure 4, that the edges of $ETD(T)$, $(1,0,0) - (0,1,1)$ and $(0,1,1) - (1,1,1)$ are in series, $(0,0,1) - (1,1,0)$ and $(1,1,0) - (1,1,1)$ are in series, and $(0,1,0) - (1,0,1)$ and $(1,0,1) - (1,1,1)$ are in series. By merging the above three pairs of edges in series, the resulting graph is K_4. Hence, $ETD(T)$ is not outerplanar by Lemma 7. Thus if $ETD(\mathcal{R})$ is outerplanar, then $\mathcal{R}$ is isomorphic to either $\mathbb{Z}_3 \times \mathbb{Z}_3$ or $\mathbb{Z}_2 \times \mathbb{Z}_2$.
Conversely, if $\mathcal{R}$ is isomorphic to either $\mathbb{Z}_3 \times \mathbb{Z}_3$ or $\mathbb{Z}_2 \times \mathbb{Z}_2$, then it is clear from Figures 5, 6 that $ETD(\mathcal{R})$ is outerplanar. □

5. $EZD(\mathcal{R})$ and $ETD(\mathcal{R})$ of genus one

In this section, we classify the ring $\mathcal{R}$ such that $EZD(\mathcal{R})$ and $ETD(\mathcal{R})$ are of genus one by using the Euler characteristic formula and a method of insertion and deletion. We use some results that is needed for classification of the genus of the $EZD(\mathcal{R})$ and $ETD(\mathcal{R})$.

Lemma 8. *[17] For $\alpha, \beta \geq 2$, $\gamma(K_{\alpha,\beta}) = \lceil(\alpha-2)(\beta-2)/4\rceil$. In particular, $\gamma(K_{4,4}) = \gamma(K_{3,\beta}) = 1$, if $\beta = 3, 4, 5, 6$ and $\gamma(K_{5,4}) = \gamma(K_{6,4}) = \gamma(K_{\alpha,3}) = 2$, if $\alpha = 7, 8, 9, 10$.*

Theorem 13. *[8] The genus of a graph is the sum of the genera of its blocks.*

Example 3. Let $\mathcal{B}$ be a ring with $1 \neq 0$ and $\mathcal{R} = \mathcal{B} \times \mathcal{B} \times \cdots \times \mathcal{B}$ ($m \geq 4$ *times*). Then $\gamma(EZD(R)) \geq 2$.

Proof. Let us consider the set $H = \{(1,0,0,0,\ldots,0),\ (0,0,0,1,0,\ldots,0),\ (0,0,1,0,\ldots,0),\ (1,1,0,0,\ldots,0),\ (1,1,0,0,\ldots,0),\ (0,1,0,1,0,\ldots,0),\ (1,0,1,0,\ldots,0),\ (0,1,1,0,\ldots,0),\ (1,0,0,1,0,\ldots,0),\ (1,1,1,0,\ldots,0),\ (0,1,1,1,0,\ldots,0),\ (1,0,1,1,0,\ldots,0),\ (1,1,0,1,0,\ldots,0)\}$. Then $H \subseteq V(EZD(\mathcal{R}))$. On merging the edges $(1,1,0,0,\ldots,0) - (0,0,1,1,0,\ldots,0)$, $(0,1,0,1,0,\ldots,0) - (1,0,1,0,\ldots,0)$ and $(0,1,1,0,\ldots,0) - (1,0,0,1,0,\ldots,0)$, then the graph obtained from the set H contain an induced subgraph $K_{3},_{7}$ (see Figure 7). Therefore, from Lemma 8, $\gamma(EZD(\mathcal{R})) \geq 2$. □

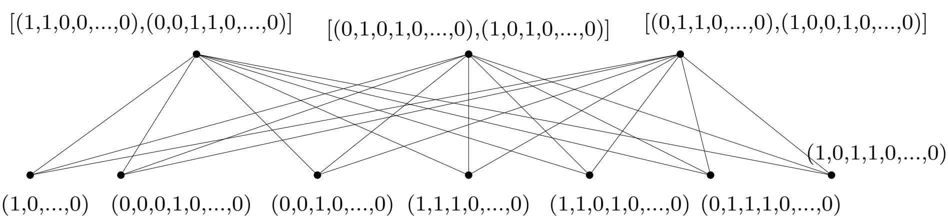

Figure 7. **Minor subgraph $K_{3,7}$**

Example 4. Let $\mathcal{B} \cong \mathbb{Z}_3$ and $\mathcal{R} = \mathcal{B} \times \mathcal{B} \times \mathcal{B}$. Then $\gamma(EZD(\mathcal{R})) \geq 2$.

Proof. Consider the set $V = \{(1,1,0), (0,0,1), (0,1,0)(0,0,2), (1,2,0), (2,1,0), (1,1,0)\}$ and $V' = \{(1,0,0), (0,1,1), (2,0,0), (0,2,0), (0,2,1), (0,1,2), (0,2,2)\}$. On merging $(0,1,0) - (0,1,1)$ in V, then the graph obtained with vertex set V contains a induced subgraph $K_{3,3}$ (see Figure 8). On merging $(2,0,0) - (0,2,0)$ in V', then the graph contains $K_{3,3}$ as a induced subgraph (see Figure 9). As we obtain the minor of the graph containing two disjoint copies of $K_{3,3}$. Therefore, from Theorem 13 and the Lemma 8, $\gamma(EZD(\mathcal{R})) \geq 2$. □

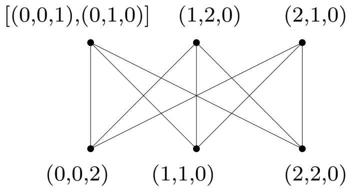

Figure 8. **Minor subgraph of** $EZD(\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3)$

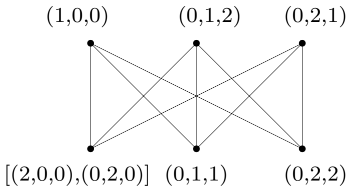

Figure 9. **Minor subgraph of** $EZD(\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3)$.

Main results of this section

Theorem 14. *Let $\mathcal{B}$ be a ring $1 \neq 0$ and $\mathcal{R} = \mathcal{B} \times \mathcal{B} \times \cdots \times \mathcal{B}$ (m times), where $2 \leq m < \infty$. Then $\gamma(EZD(\mathcal{R})) = 1$ if and only if $\mathcal{R}$ is isomorphic to $\mathbb{F}_4 \times \mathbb{F}_4$ or $\mathbb{Z}_5 \times \mathbb{Z}_5$.*

Proof. If $m \geq 4$, then by Example 3 we have $\gamma(EZD(\mathcal{R})) \geq 2$. If $m = 3$ and $\mathcal{B} \cong \mathbb{Z}_3$, then by Example 4 we have $\gamma(EZD(\mathcal{R})) \geq 2$. If $m = 3$ and $|\mathcal{B}| \geq 4$, then let $1, a, b \in \mathcal{B} \setminus \{0\}$ be distinct elements of $\mathcal{B}$ and consider the sets $H = \{(1,0,0), (a,0,0), (b,0,0), (0,1,1), (0,a,a), (0,b,b)\}$ and $H' = \{(0,0,1), (0,0,a), (0,0,b), (1,1,0), (a,a,0), (b,b,0)\}$. Clearly the graph obtained from the set H and H' contain an induced subgraph $K_{3,3}$. As we obtain the minor of the graph containing two disjoint copies of $K_{3,3}$. Therefore, from Theorem 13 and the Lemma 8, $\gamma(EZD(\mathcal{R})) \geq 2$. Now, if $m = 2$ and $\mathcal{B}$ is nonreduced ring, then $|\mathcal{B}| \geq 4$. Let us take $z \in N(\mathcal{B})^*$ and $1 \neq k \in \mathcal{B}^{\times}$. Consider the set $H = \{a, b, c, d, e, u, v, w, x, y\}$, where $a = (z, 0)$, $b = (z, z)$, $c = (0, z)$, $d = (1, z)$, $e = (z, 1)$, $u = (1, 0)$, $v = (0, 1)$, $w = (k, 0)$, $x = (0, k)$ and $y = (k, z)$. On contracting edge $d - e$, then the minor of the graph obtained from the set H contain the induced subgraph, which is isomorphic to $K_{4,5}$ (see Figure 10). As H is the least vertex set that contain in every vertex of a graph $EZD(\mathcal{R})$, if $\mathcal{B}$ is nonreduced ring. Therefore, $\gamma(EZD(\mathcal{R})) > 1$.
If $m = 2$ and $|\mathcal{B}| \geq 6$. Then one can easily observe that $EZD(\mathcal{R})$ has a induced subgraph $K_{5,5}$, so by Lemma 8, $\gamma(EZD(\mathcal{R})) > 1$. Now, remaining ring to be check; $\mathbb{Z}_2 \times \mathbb{Z}_2$, $\mathbb{Z}_3 \times \mathbb{Z}_3$, $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$, $\mathbb{F}_4 \times \mathbb{F}_4$ and $\mathbb{Z}_5 \times \mathbb{Z}_5$. If we consider the rings $\mathbb{Z}_2 \times \mathbb{Z}_2$, $\mathbb{Z}_3 \times \mathbb{Z}_3$ and $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$, then from Theorem 9, $EZD(\mathcal{R})$ are planar graph. So, the possible ring $\mathcal{R}$ such that $EZD(\mathcal{R})$ to be genus one are $\mathbb{F}_4 \times \mathbb{F}_4$ or $\mathbb{Z}_5 \times \mathbb{Z}_5$.

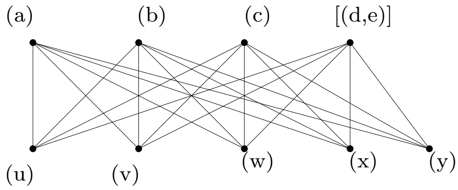

Figure 10. **Minor subgraph** $K_{4,5}$

Conversely, if $\mathcal{R}$ is isomorphic to $\mathbb{F}_4 \times \mathbb{F}_4$ or $\mathbb{Z}_5 \times \mathbb{Z}_5$. Then the graph obtained from $\mathbb{F}_4 \times \mathbb{F}_4$ and $\mathbb{Z}_5 \times \mathbb{Z}_5$ are isomorphic to $K_{3,3}$ and $K_{4,4}$, so from Lemma 8, $\gamma(EZD(\mathcal{R})) = 1$.

$\square$

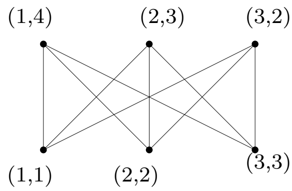

Figure 11. **Unit subgraph of** $ETD(\mathbb{F}_5 \times \mathbb{F}_5)$

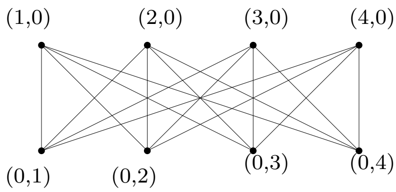

Figure 12. **Zero-divisor subgraph of** $ETD(\mathbb{F}_5 \times \mathbb{F}_5)$

Theorem 15. *Let $\mathcal{B}$ be a ring with $1 \neq 0$ and $\mathcal{R} = \mathcal{B} \times \mathcal{B} \times \cdots \times \mathcal{B}$ (m times), where $2 \leq m < \infty$. Then the genus of $ETD(\mathcal{R})$ is atleast 2.*

Proof. Since $EZD(\mathcal{R})$ is an induced subgraph of $ETD(\mathcal{R})$, then from Theorem 14, the only possibility of $ETD(\mathcal{R})$ to be of genus one are $\mathbb{F}_4 \times \mathbb{F}_4$ or $\mathbb{Z}_5 \times \mathbb{Z}_5$. If we consider $\mathbb{Z}_5 \times \mathbb{Z}_5$, then it contain the subgraph which have disjoint copies of $K_{3,3}$ and $K_{4,4}$ (see Figures 11 and 12). So, from Lemma 8 and Theorem 13, $\gamma(ETD(\mathbb{Z}_5 \times \mathbb{Z}_5)) \geq 2$. Similarly, if we consider $\mathbb{F}_4 \times \mathbb{F}_4$, then it contain two disjoint copies of $K_{3,3}$. Therefore, from Lemma 8 and Theorem 13, $\gamma(ETD(\mathbb{F}_4 \times \mathbb{F}_4) \geq 2$. Hence, genus of $ETD(\mathcal{R})$ is atleast 2. $\square$

Acknowledgements. We extend our heartfelt gratitude to Dr. S. Visweswaran and the referee for their valuable suggestions, which significantly improved the article.

Conflict of interest. The authors declare that they have no conflict of interest.

Data Availability. Data sharing is not applicable to this article as no datasets were generated or analyzed during the current study.

References

[1] S. Akbari and A. Mohammadian, *On the zero-divisor graph of a commutative ring*, J. Algebra **274** (2004), no. 2, 847–855
https://doi.org/10.1016/S0021-8693(03)00435-6.

[2] ______, *Zero-divisor graphs of non-commutative rings*, J. Algebra **296** (2006), no. 2, 462–479
https://doi.org/10.1016/j.jalgebra.2005.07.007.

[3] D.F. Anderson, R. Levy, and J. Shapiro, *Zero-divisor graphs, von neumann regular rings, and boolean algebras*, J. Pure Appl. Algebra **180** (2003), no. 3, 221–241
https://doi.org/10.1016/S0022-4049(02)00250-5.

[4] D.F. Anderson and P.S. Livingston, *The zero-divisor graph of a commutative ring*, J. Algebra **217** (1999), no. 2, 434–447
https://doi.org/10.1006/jabr.1998.7840.

[5] D.F. Anderson and M. Naseer, *Beck's coloring of a commutative ring*, J. Algebra **159** (1993), no. 2, 500–514
https://doi.org/10.1006/jabr.1993.1171.

[6] A. Azadi, M.J. Nikmehr, and B. Soleymanzadeh, *Perfect essential graphs*, Le Matematiche **74** (2019), no. 2, 279–287.

[7] A. Badawi, *On the dot product graph of a commutative ring*, Comm. Algebra **43** (2015), no. 1, 43–50
https://doi.org/10.1080/00927872.2014.897188.

[8] J. Battle, F. Harary, and Y. Kodama, *Additivity of the genus of a graph*, Bull. Amer. Math. Soc. **68** (1962), no. 6, 565–568.

[9] I. Beck, *Coloring of commutative rings*, J. Algebra **116** (1988), no. 1, 208–226
https://doi.org/10.1016/0021-8693(88)90202-5.

[10] J.D. LaGrange, *Complemented zero-divisor graphs and boolean rings*, J. Algebra **315** (2007), no. 2, 600–611
https://doi.org/10.1016/j.jalgebra.2006.12.030.

[11] T.G. Lucas, *The diameter of a zero divisor graph*, J. Algebra **301** (2006), no. 1, 174–193
https://doi.org/10.1016/j.jalgebra.2006.01.019.

[12] M. Nazim and N. ur Rehman, *On the essential annihilating-ideal graph of commutative rings.*, Ars Math. Contemp. **22** (2022), no. 3, #P3.05
https://doi.org/10.26493/1855-3974.2645.8fc.

[13] M.J. Nikmehr, R. Nikandish, and M. Bakhtyiari, *On the essential graph of a commutative ring*, J. Algebra Appl. **16** (2017), no. 7, Article ID: 1750132
https://doi.org/10.1142/S0219498817501328.

[14] S. Payrovi, S. Babaei, and E.S. Sevim, *On the compressed essential graph of a commutative ring*, Bull. Belg. Math. Soc. Simon Stevin **26** (2019), no. 3, 421–429
https://doi.org/10.36045/bbms/1568685656.

[15] K. Selvakumar, M. Subajini, and M.J. Nikmehr, *On the genus of essential graph of commutative rings.*, Australas. J Comb. **74** (2019), no. 1, 74–85.

[16] D.B. West, *Introduction to Graph Theory*, Prentice hall Upper Saddle River, New Jersey, 1996.

[17] A.T. White, *Graphs, Groups and Surfaces*, Elsevier, New York, 1985.