Communications in Combinatorics and Optimization
Vol. xx, No. x (xxxx), pp. 1-15
https://doi.org/10.22049/cco.2023.28810.1730

CCO
Commun. Comb. Optim.

Research Article

$L(2, 1)$-labeling of some zero-divisor graphs associated with commutative rings

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Received: 5 July 2023; Accepted: 9 November 2023
Published Online: 25 November 2023

Abstract: Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be a simple graph, an $L(2, 1)$-labeling of $\mathcal{G}$ is an assignment of labels from non-negative integers to vertices of $\mathcal{G}$ such that adjacent vertices get labels which differ by at least by two, and vertices which are at distance two from each other get different labels. The λ-number of $\mathcal{G}$, denoted by $\lambda(\mathcal{G})$, is the smallest positive integer ℓ such that $\mathcal{G}$ has an $L(2, 1)$-labeling with all labels as members of the set $\{0, 1, \ldots, \ell\}$. The zero-divisor graph of a finite commutative ring R with unity, denoted by $\Gamma(R)$, is the simple graph whose vertices are all zero divisors of R in which two vertices u and v are adjacent if and only if $uv = 0$ in R. In this paper, we investigate $L(2, 1)$-labeling of some zero-divisor graphs. We study the *partite truncation*, a graph operation that allows us to obtain a reduced graph of relatively small order from a graph of significantly larger order. We establish the relation between λ-numbers of the graph and its partite truncated one. We make use of the operation *partite truncation* to contract the zero-divisor graph of a reduced ring to the zero-divisor graph of a Boolean ring.

Keywords: zero-divisor graph, $L(2, 1)$-labeling, λ-number, partite truncation.

AMS Subject classification: 05E40, 13A99

1. Introduction

Given a simple undirected finite graph $\mathcal{G}$, and two positive integers j, k, an $L(j, k)$-labeling of $\mathcal{G}$ is a function $f : V(\mathcal{G}) \longrightarrow \mathbb{Z}_{\geq 0}$ such that $|f(v_1) - f(v_2)| \geq j$ whenever v_1 is adjacent to v_2 and $|f(v_1) - f(v_2)| \geq k$ whenever v_1 and v_2 are at distance two apart.

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The difference between the maximum and minimum values of f is called *span* of f, denoted by $span(f)$. If f is an $L(j,k)$-labeling of $\mathcal{G}$ with minimum value say δ, then the function g defined on vertices of $\mathcal{G}$ by $g(v) = f(v) - \delta$ is also an $L(j,k)$-labeling with minimum value 0 and the maximum value as $span(f) = span(g)$. As a result, we assume that every $L(j,k)$-labeling has 0 as its minimum value and its span as the maximum value. The minimum span over all $L(j,k)$-labelings of $\mathcal{G}$ is called the $L(j,k)$-labeling number, denoted by, $\lambda_{j,k}(\mathcal{G})$. An $L(j,k)$-labeling f of $\mathcal{G}$ is said to be *minimal labeling* if $span(f) = \lambda_{j,k}(\mathcal{G})$. The $L(j,k)$-labeling problem has been studied extensively for the case $j = 2$ and $k = 1$. The $L(2,1)$-labeling number of a graph $\mathcal{G}$ is called its λ-*number*, denoted by $\lambda(\mathcal{G})$.

The difficulty of assigning channel frequencies to transmitters without interference motivates the study of $L(j,k)$-labeling problem of a graph. Roberts [20] proposed the challenge of efficiently assigning radio channels to transmitters at several sites using non-negative integers to denote channels so that adjacent locations receive distinct channels, and extremely close locations receive channels that are at least at differ by two. As a result, there would be no interference between these channels. In 1992, Griggs and Yeh [9] formalised the concept of $L(j,k)$-labeling and demonstrated that the $L(2,1)$-labeling problem is NP-complete for general graphs. The $L(j,k)$-labeling problem and $L(2,1)$-labeling problem have been studied in [7, 8, 12, 21].

On the other hand, the fundamental objective of associating a graph to an algebraic structure is to investigate the relationship between algebraic properties and combinatorial properties of algebraic structures. The graphs such as Cayley graph [4], power graph [11], zero-divisor graph [2], group-annhilator [14] and torsion graph [22], are some graphs emerging from algebraic structures such as groups, rings and modules. Beck [3] introduced the concept of an undirected zero-divisor graph $\Gamma'(R)$ which is called the *Beck's zero-divisor graph* of a commutative ring R. In his investigation, all elements of a ring R are vertices of the graph $\Gamma'(R)$ with two distinct vertices u and v adjacent in $\Gamma'(R)$ if and only if $uv = 0$. Anderson and Livingston [2] also studied the combinatorial properties of a commutative ring R. They associated a graph $\Gamma(R)$, called the *zero-divisor graph*, to R with vertices as elements of $Z^*(R) = Z(R) \setminus \{0\}$, that is, the non-trivial zero-divisors of R with two vertices $u, v \in Z^*(R)$ being adjacent in $\Gamma(R)$ if and only if $uv = 0$. For more on zero-divisor graphs, please see [1, 6, 15–19].

Let R be a finite commutative ring with unity. A ring R is said to be a *local ring* if it has a unique maximal ideal. Let $R = R_1 \times \cdots \times R_r \times F_1 \times \cdots \times F_s$, be the Artinian decomposition of R, where each R_i is a commutative local ring with unity, and each F_j is a field. A ring R is said to be a *mixed ring* if either $r \geq 1$ and $s \geq 1$ or $r \geq 2$ and $s = 0$ in its Artinian decomposition. A ring R is said to be a *reduced ring* if $r = 0$ and $s \geq 1$ in its Artinian decomposition.

This research article is organised as follows. In Section 2, we present some preliminary results and definitions related to $L(2,1)$-labeling of graphs. In Section 3, we study *partite truncation* operation in graphs, and obtain exact value of the λ-number of zero-divisor graphs realized by some classes of reduced rings.

2. Preliminaries

In this section, we discuss some results related to $L(2,1)$-labeling of some classes of graphs.

For a positive real number d, an $L_d(2,1)$-*labeling* of a simple graph $\mathcal{G}$ is a positive real valued function f defined on $V(\mathcal{G})$ such that for $u_1, u_2 \in V(\mathcal{G}), |f(u_1) - f(u_2)| \geq 2d$ whenever u_1 is adjacent to u_2 and $|f(u_1) - f(u_2)| \geq d$ whenever distance between u_1 and u_2 is two. The $L_d(2,1)$-labeling number of $\mathcal{G}$, denoted by $\lambda(\mathcal{G}, d)$, is the smallest number m such that $\mathcal{G}$ has an $L_d(2,1)$-labeling f with $\max\limits_{v \in V(\mathcal{G})}\{f(v)\} = m$. $L_d(2,1)$-labeling introduced by Griggs and Yeh [9] is a natural generalisation of $L(2,1)$-labeling. Note that $\lambda(\mathcal{G}, 1) = \lambda(\mathcal{G})$. The authors confirmed that to determine $\lambda(\mathcal{G}, d)$, it suffices to study the case for $\lambda(\mathcal{G})$. They provided bounds on λ-number for a variety of graphs, including trees, cycles, 3-connected graphs and hypercubes. They proved following interesting results in which they obtain upper bounds of $\lambda(\mathcal{G})$ in terms of graph invariants such as chromatic number $\chi(\mathcal{G})$ and maximum degree $\Delta(\mathcal{G})$.

Theorem 1 ([9], Theorem 4.1). *If $\mathcal{G}$ is a graph with n vertices, then $\lambda(\mathcal{G}) \leq n + \chi(\mathcal{G}) - 2$.*

Theorem 2 ([9], Theorem 6.2). *For any graph $\mathcal{G}$, $\lambda(\mathcal{G}) \leq \Delta^2(\mathcal{G}) + 2\Delta(\mathcal{G})$.*

The authors in [8] introduced the notion of holes, multiplicities and gaps of an $L(2,1)$-labeling. Let f be an $L(2,1)$-labeling of $\mathcal{G}$, and given a positive integer h, $0 < h < span(f)$, let $f_h = \{v \in V(\mathcal{G}) : f(v) = h\}$. If f_h is empty, then h is called a *hole* of f and if $|f_h| \geq 2$, then h is called *multiplicity* of f. h is called a *gap* of f if h is a hole with $|f_{h-1}| = |f_{h+1}| = 1$ and $\{v^{h-1}, v^{h+1}\} \in E(\mathcal{G})$, where $f(v^{h-1}) = h - 1$ and $f(v^{h+1}) = h + 1$. $H(f), M(f)$ and $G(f)$ represents collection of holes, multiplicities and gaps of f respectively. Let $h(f)$ and $g(f)$ denote cardinalities of $H(f)$ and $G(f)$. The function f is called the *minimum* $L(2,1)$-*labeling* of $\mathcal{G}$ if it is minimal and has minimum number of holes over the set $L_{(2,1)}(\mathcal{G})$, where $L_{(2,1)}(\mathcal{G})$ denotes the set of all minimal $L(2,1)$-labelings of $\mathcal{G}$. It should be noted that a minimal $L(2,1)$-labeling of a graph $\mathcal{G}$ is not unique whereas the minimum $L(2,1)$-labeling of a graph $\mathcal{G}$ is always unique. For example, if P_6 is a path on six vertices, then the labeling $(4,0,3,1,5,2)$ with span as 5 is an $L(2,1)$-labeling but not the minimum one, since it is not a minimal $L(2,1)$-labeling for P_6.

The graph invariant called as *path covering number* is the least number of vertex-disjoint paths required to cover vertices of the graph. The relationship between $\lambda(\mathcal{G})$ and path covering number $c(\mathcal{G})$ of a graph $\mathcal{G}$ proved in [8] is given as follows.

Lemma 1 ([8], Lemma 2.2). *Let f be a minimum $L(2,1)$-labeling of $\mathcal{G}$. If h is a hole of f, then $|f_{h-1}| = |f_{h+1}| > 0$. Furthermore, if $|f_{h-1}| = |f_{h+1}| = 1$, then h is a gap.*

Lemma 2 ([8], Lemma 2.3). *If f is a minimum $L(2,1)$-labeling of $\mathcal{G}$, then $G(f)$ is empty or $M(f)$ is empty.*

Theorem 3 ([8], Theorem 1.1). *Let $\mathcal{G}$ be a simple graph of order n, and let $\mathcal{G}^c$ be its complement. Then the following hold,*
(i) $\lambda(\mathcal{G}) \leq n-1$ if and only if $c(\mathcal{G}^c) = 1$.
(ii) Let $r \geq 2$ be an integer. Then $\lambda(\mathcal{G}) = n+r-2$ if and only if $c(\mathcal{G}^c) = r$.

A *clique* of a graph $\mathcal{G}$ is a subset of $V(\mathcal{G})$ such that any two distinct vertices of this subset are adjacent in $\mathcal{G}$. The maximum cardinality of a clique in $\mathcal{G}$ is known as its *clique number*, denoted by $\omega(\mathcal{G})$. An *independent set* of a graph $\mathcal{G}$ is a subset of $V(\mathcal{G})$ such that no two vertices in the subset are adjacent. The maximum cardinality of an independent subset in $\mathcal{G}$ is called its *independence number*, denoted by $\alpha(\mathcal{G})$. In [13], the authors investigated the relationships between λ-number and clique number, and also between λ-number and independence number of $\mathcal{G}$. They obtained some interesting results which are presented below.

Theorem 4 ([13], Proposition 4.1). *Let C be a clique of a graph $\mathcal{G}$ such that $|C| = \omega(\mathcal{G})$. Then $\lambda(\mathcal{G}) = 2\omega(\mathcal{G}) - 2$ if there exist partitions*

$$\{A_1, A_2, \ . \ . \ . \ , A_s\} \text{ and } \{C_1, C_2, \ . \ . \ . \ , C_s, C_{s+1}\}$$

of $V(\mathcal{G}) \setminus C$ and C respectively satisfying the following conditions for each $i \in \{1, 2, \ldots, s\}$

1. *$|A_i| \leq |C_i| - 1$.*
2. *Every vertex in A_i and every vertex in C_i are non-adjacent in $\mathcal{G}$.*

Theorem 5 ([13], Proposition 2.4). *If $\mathcal{G}$ is a graph of order n, then $\lambda(\mathcal{G}) \leq 2n - \alpha(\mathcal{G}) - 1$.*

A p-group is a group in which order of every element is a power of p, where p is a prime number. A finite group is a p-group if and only if its order is a power of a prime p. The *Power Graph* Γ_G [5, 11] of a finite group G is the simple graph with vertex set G in which two vertices are adjacent if and only if one is the power of the other. The exact value of the λ-number of power graphs of dihedral group D_{2n}, p-group, and generalised quaternion group Q_{4n} was computed in [13]. For some certain classes of groups of order n, they obtained an upper bound related to λ-number of a power graph.

Theorem 6 ([13], Theorem 4.1). *If G is a group of order n, where n is not a prime power, then $\lambda(\Gamma_G) \leq 2n-4$. The bound is achieved if and only if G is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$ or $\mathbb{Z}_{2q}$, where q is a prime greater than 2.*

Furthermore, the $L(j,k)$-labeling of Cayley graphs has been studied in [23], and Kelarev, Ras, and Zhou [10] revealed a relationship between a semigroup structure and the minimum distance labeling spans of its Cayley graph.

3. Partite truncation of a graph

In this section, we study a graph operation called as *partite truncation* of a graph. This operation allows us to obtain a graph of relatively smaller order from a graph of significantly larger order. We prove an interesting result in which we determine the λ-number and a corresponding minimum labeling of Beck's zero-divisor graph $\Gamma'(R)$ from the graph $\Gamma(R)$ of a ring R, provided that $diam(\Gamma(R)) < 3$.

Definition 1. Let $\mathcal{G}$ be an n-partite graph with $\mathcal{V}_1, \mathcal{V}_2, \ldots, \mathcal{V}_n$ as partite sets. We define **partite truncated graph** $\hat{\mathcal{G}}$ of $\mathcal{G}$ with vertices $\hat{v_1}, \hat{v_2}, \ldots, \hat{v_n}$, where $\hat{v_i}$ corresponds to the whole partite set $\mathcal{V}_i$, $1 \le i \le n$, and two distinct vertices $\hat{v_i}$ and $\hat{v_j}$ are adjacent in $\hat{\mathcal{G}}$ if and only if there is a vertex in $\mathcal{V}_i$ adjacent to some vertex in $\mathcal{V}_j$.

Thus, in a partite truncation of an n-partite graph, each partite set is truncated to one vertex, and edges (if any) connecting two partite sets are truncated to a single edge. It is clear from the definition that if $\mathcal{G}$ is connected, so is $\hat{\mathcal{G}}$, and if there is a path between two partite sets $\mathcal{V}_i$ and $\mathcal{V}_j$ of $\mathcal{G}$, then there must be a path between the two corresponding vertices $\hat{v_i}$ and $\hat{v_j}$ in $\hat{\mathcal{G}}$. If $diam(\mathcal{G}) = 1$, then $\mathcal{G} \cong \mathcal{K}_n \cong \hat{\mathcal{G}}$. Also, if the cardinality of each partite set of $\mathcal{G}$ is equal to one, then $\mathcal{G} \cong \hat{\mathcal{G}}$. We consider n-partite graphs $\mathcal{G}$ with $diam(\mathcal{G}) > 1$ in which there is at least one partite set $\mathcal{V}$ in $\mathcal{G}$ of cardinality greater than one.

Let $\mathcal{H}$ denote the n-partite graph with the property that the bipartite subgraph induced by any two distinct partite sets $\mathcal{V}$ and $\mathcal{W}$ is either complete bipartite or empty. If $diam(\mathcal{H}) = 2$, then there are some $u, w \in V(\mathcal{H})$ such that $d(u,w) = diam(\mathcal{H})$. If u, w are in two distinct partite sets $\mathcal{V}, \mathcal{W}$ of $\mathcal{H}$, then $d(\hat{v}, \hat{w}) = diam(\hat{\mathcal{H}}) = 2$. However, if there are no such vertices u, w lying in two distinct partite sets $\mathcal{V}, \mathcal{W}$ of $\mathcal{H}$, then $diam(\hat{\mathcal{H}}) = 1$. Consequently, $diam(\hat{\mathcal{H}}) \le 2 = diam(\mathcal{H})$. If $diam(\mathcal{H}) \ge 3$, then it is easy to verify that $diam(\mathcal{H}) = diam(\hat{\mathcal{H}})$.

In the following result, we see that there is a shift in the λ-number of a graph with a diameter less than three, when a particular number of isolated vertices (set of independent vertices), and then a dominant vertex (a vertex adjacent to all these isolated vertices and to all other vertices of the graph) are added to it. This subsequently aids in determining the λ-number of Beck's zero-divisor graph $\Gamma'(R)$ of a ring R.

Proposition 1. *Let $\mathcal{G}$ be a graph with $diam(\mathcal{G}) < 3$, and let the minimum labeling of $\mathcal{G}$ have s many holes. If $\mathcal{G}_{m+1}$ is the graph obtained from $\mathcal{G}$ when m isolated vertices, and subsequently a dominant vertex are added to it, then the following holds,*

$$\lambda(\mathcal{G}_{m+1}) = \begin{cases} \lambda(\mathcal{G}) + m - s + 2, & m > s, \\ \lambda(\mathcal{G}) + 2, & m \le s. \end{cases}$$

Proof. Let $u_1, u_2, \ldots, u_m$ denote the isolated vertices, and u_0 be the dominant vertex. Also, let f be the minimum labeling of $\mathcal{G}$ with holes $h_1, h_2, \ldots, h_s$. We define $L(2,1)$-labeling g on $\mathcal{G}_{m+1}$ as, $g(u_0) = 0,\ \ g(v) = f(v) + 2,$ for all $v \in V(\mathcal{G})$ and $g(u_i) = h_i + 2,$ for all $i,\ 1 \le i \le s$. For $m > s,\ g(u_{s+j}) = \lambda(\mathcal{G}) + j + 2,$ for all $j,\ 1 \le j \le m - s$.
Therefore,

$$span(g) = \begin{cases} \lambda(\mathcal{G}) + m - s + 2, & m > s, \\ \lambda(\mathcal{G}) + 2, & m \le s. \end{cases}$$

Since $diam(\mathcal{G}) < 3$, therefore f must be one-to-one. This implies, g is well defined $L(2,1)$-labeling of $\mathcal{G}_{m+1}$. Thus, $\lambda(\mathcal{G}_{m+1}) \le span(g)$. If we assume that there exist another $L(2,1)$-labeling h of $\mathcal{G}_{m+1}$ with $span(h) < span(g)$, then the restriction $h \restriction_{V(\mathcal{G})}$ is a well defined $L(2,1)$-labeling of $\mathcal{G}$ with $span(h \restriction_{V(\mathcal{G})}) < span(f)$, a contradiction. □

Corollary 1. *Let R be a finite ring of order n such that $diam(\Gamma(R)) < 3$, $\lambda(\Gamma(R)) = k$, and the minimum labeling of $\Gamma(R)$ have s many holes. Then the following holds,*

$$\lambda(\Gamma'(R)) = \begin{cases} k + n - m - s + 1, & n - m - 1 > s, \\ k + 2, & n - m - 1 \le s, \end{cases}$$

where m is the order of $\Gamma(R)$.

Proof. Any two distinct vertices $x, y \in R$ are adjacent in $\Gamma'(R)$ if and only if $xy = 0$. Therefore, from the structure of $\Gamma'(R)$, it is clear that the set of $n - m - 1$ many units of R represents the isolated vertices, and 0 represents the dominant vertex added to $\Gamma(R)$. Therefore, by Proposition 1,

$$\lambda(\Gamma'(R)) = \begin{cases} k + n - m - s + 1, & n - m - 1 > s, \\ k + 2, & n - m - 1 \le s. \end{cases}$$

□

Let $\hat{\mathcal{H}}$ be the partite truncation of the n-partite graph $\mathcal{H}$ with $diam(\mathcal{H}) \ge 3$. Suppose $\lambda(\hat{\mathcal{H}}) = k$ with the corresponding minimum labeling as f. Consider the set,

$$\mathcal{F} = \{f(v) : v \in V(\hat{\mathcal{H}})\} = \{i_0, i_1, i_2, \ldots, i_m\},$$

where $f(v) = i_j,\ 0 = i_0 < i_1 < \cdots < i_m = k$. Let s_j denotes the multiplicities of i_j, for all $j,\ 0 \le j \le m$. For each j, consider the subset $W_j = \{v_{j,1}, v_{j,2}, \ldots, v_{j,s_j}\} = f^{-1}(i_j)$ of vertices of $\hat{\mathcal{H}}$. Furthermore, for each $v_{j,l} \in W_j, 1 \le l \le s_j$, let $\mathcal{V}_{j,l}$ be its corresponding partite set in $\mathcal{H}$. Define the set $\mathcal{C}$, a collection of partite sets of $\mathcal{H}$ by,

$$\mathcal{C} \ = \ \{\mathcal{V}_0, \mathcal{V}_1, \mathcal{V}_2, \ldots \mathcal{V}_m\},$$

where,

$$\mathcal{V}_j \in \{\mathcal{V}_{j,l} : 1 \leq l \leq s_j\} \text{ and } |\mathcal{V}_j| = \max_{1 \leq l \leq s_j} \{|\mathcal{V}_{j,l}|\}. \tag{3.1}$$

Now, we prove an important result related to λ-number of graphs $\mathcal{H}$ and $\hat{\mathcal{H}}$. We determine the λ-number of $\mathcal{H}$ if the λ-number of its partite truncation $\hat{\mathcal{H}}$ is given. This result also provides an algorithm for finding the minimal $L(2,1)$-labeling of $\mathcal{H}$ given the minimum $L(2,1)$-labeling of $\hat{\mathcal{H}}$.

Theorem 7. *If $\mathcal{H}$ is an n-partite graph and $\hat{\mathcal{H}}$ is its partite truncation, then the following holds,*

$$\lambda(\mathcal{H}) = \begin{cases} |V(\mathcal{H})| + \lambda(\hat{\mathcal{H}}) - n, & \text{if } diam(\mathcal{H}) = 2, \\ \sum\limits_{\mathcal{V}_j \in \mathcal{C}} |V_j| + \lambda(\hat{\mathcal{H}}) - |\mathcal{C}|, & \text{if } diam(\mathcal{H}) \geq 3. \end{cases}$$

Proof. Let f be the minimum $L(2,1)$-labeling of the graph $\hat{\mathcal{H}}$. We consider the following two cases.

Case I. $diam(\mathcal{H}) = 2$**.**

In this case, f is injective. Therefore,

$$\mathcal{F} = \{f(v) : v \in V(\hat{\mathcal{H}})\} = \{0 = i_0 < i_1 < i_2 < \cdots < i_m = \lambda(\hat{\mathcal{H}})\}.$$

Here $s_j = 1$ for all j, $0 \leq j \leq n-1$. Let v_j denote the vertex in $\hat{\mathcal{H}}$ such that $f(v_j) = i_j$, and let $\mathcal{V}_j$ be its corresponding partite set in $\mathcal{H}$ of cardinality m_j. Denote the vertices of $\mathcal{H}$ lying in the partite set $\mathcal{V}_j$ by $v_{j,i}$, where $1 \leq i \leq m_j$, that is,

$$\begin{aligned} \mathcal{V}_0 &= \{v_{0,1}, v_{0,2}, \ldots v_{0,m_0}\}, \\ \mathcal{V}_1 &= \{v_{1,1}, v_{1,2}, \ldots v_{1,m_1}\}, \\ &\vdots \\ \mathcal{V}_{n-1} &= \{v_{n-1,1}, v_{n-1,2}, \ldots v_{n-1,m_{n-1}}\}. \end{aligned}$$

Define $L(2,1)$-labeling $g : V(\mathcal{H}) \longrightarrow \mathbb{Z}_{\geq 0}$ on $\mathcal{H}$ by,

$$\begin{aligned} g(v_{0,i}) &= i - 1, \\ g(v_{j,i}) &= i + (i_j - i_{j-1}) + \max_{v \in \mathcal{V}_{j-1}} \{g(v)\} - 1, \end{aligned}$$

where,

$$\max_{v \in \mathcal{V}_{j-1}} \{g(v)\} = m_0 + m_1 + \cdots + m_{j-1} + i_{j-1} - j.$$

Therefore, $span(g) = |V(\mathcal{H})| + \lambda(\hat{\mathcal{H}}) - n$. Since f is well defined $L(2,1)$-labeling of $\hat{\mathcal{H}}$, vertices of $\mathcal{H}$ lying in the same partite set are at distance two apart, and the distance between two vertices of $\mathcal{H}$ lying in distinct partite sets is same as the distance between their corresponding vertices in $\hat{\mathcal{H}}$. This implies, g is a well defined $L(2,1)$-labeling of $\mathcal{H}$, and consequently $\lambda(\mathcal{H}) \leq |V(\mathcal{H})| + \lambda(\hat{\mathcal{H}}) - n$.
Suppose g is not a minimal $L(2,1)$-labeling, and let h be a minimal $L(2,1)$-labeling of $\mathcal{H}$ with $Span(h) < span(g)$. The operation partite truncation contracts each partite set to one vertex, and since the vertices lying in the same partite set are at distance two from each other, therefore,

$$\begin{aligned}
\lambda(\hat{\mathcal{H}}) &= \lambda(\mathcal{H}) - \sum_{0 \leq j \leq n-1} (|\mathcal{V}_j| - 1) \\
&= \lambda(\mathcal{H}) - |V(\mathcal{H})| + n \\
&< |V(\mathcal{H})| + \lambda(\hat{\mathcal{H}}) - n - |V(\mathcal{H})| + n \\
&= \lambda(\hat{\mathcal{H}})
\end{aligned}$$

which clearly is a contradiction. Thus, $\lambda(\mathcal{H}) = |V(\mathcal{H})| + \lambda(\hat{\mathcal{H}}) - n$.
Case II. $diam(\mathcal{H}) \geq 3$**.**
We have, $diam(\hat{\mathcal{H}}) = diam(\mathcal{H}) \geq 3$. This implies, f may not be injective, which in turn implies that $s_j \geq 1$ for all $j,\ 0 \leq j \leq m$. As above,

$$\mathcal{F} = \{f(v) : v \in V(\hat{\mathcal{H}})\} = \{0 = i_0 < i_1 < i_2 < \cdots < i_m = \lambda(\hat{\mathcal{H}})\}.$$

and $\mathcal{C} = \{\mathcal{V}_0, \mathcal{V}_1, \mathcal{V}_2, \ldots \mathcal{V}_m\}$. Define $L(2,1)$-labeling g on $V(\mathcal{H})$ as follows,

1. Label the partite sets in $\mathcal{C}$ in the same way as in case-I.
2. For each j, provide same labels for vertices in the remaining partite sets $\mathcal{V}_{j,l}$ (if any) as given to vertices in $\mathcal{V}_j \in \mathcal{C}$ (see (3.1) for the definition of $\mathcal{V}_{j,l}$).

On the similar lines as in case-I, it can be shown that g is a well defined minimal $L(2,1)$-labeling of $\mathcal{H}$, and we conclude that $\lambda(\mathcal{H}) = \sum_{\mathcal{V}_j \in \mathcal{C}} |\mathcal{V}_j| + \lambda(\hat{\mathcal{H}}) - |\mathcal{C}|$. □

Remark 1. The set $\mathcal{C}$ can be also defined by defining an equivalence relation $\sim$ on $V(\hat{\mathcal{H}})$ by $v_1 \sim v_2$ if and only if $f(v_1) = f(v_2)$, where f is a minimal $L(2,1)$-labeling of $\hat{\mathcal{H}}$. Since there is one-to-one correspondence between n vertices of $\hat{\mathcal{H}}$ and n partite sets of $\mathcal{H}$, we have an equivalence relation on n-partite sets of $\mathcal{H}$ as well. We choose the representatives of each equivalence class in a way that it has maximum cardinality among other other partite sets in its class. The collection of all such partite sets among all equivalence classes defines $\mathcal{C}$.

The subsequent example illustrates the previously mentioned Remark 1.

Example 1. Consider a 5-partite graph $\mathcal{G}$, where the partite sets are denoted by, $V_1, V_2, \ldots, V_5$. Moreover, assume that $\hat{\mathcal{G}}$ is isomorphic to P_5, where P_5 represents the path on 5 vertices $v_1, v_2, \ldots, v_5$, and each set V_i satisfies $|V_i| = 2i + 1$ for $1 \leq i \leq 5$. Let $\{0, 2, 4, 0, 2\}$ be an $L(2,1)$-labeling of P_5. As a consequence of Remark 1, the equivalence classes are given by, $A_1 = \{v_1, v_4\}, A_2 = \{v_2, v_5\}$, and $A_3 = \{v_3\}$. Consequently, the set $\mathcal{C}$ can be described as $\mathcal{C} = \{V_3, V_4, V_5\}$.

Next, we provide some examples in which we compute the $\lambda-$number of a complete n-partite graph, and λ-number of the zero-divisor graph of a ring $\mathbb{F}_p \times \mathbb{F}_q$, where $\mathbb{F}_p$ and $\mathbb{F}_q$ are finite fields of order p and q respectively.

Example 2. Let $\mathbb{K}_{m_1, m_2, \ldots, m_n}$ be a complete n-partite graph, and let $\mathcal{V}_i$ with $|\mathcal{V}_i| = m_i, 1 \leq i \leq n$, be its partite sets. We assume that there exists at least one m_i such that $m_i > 1$. This implies, $diam(\mathbb{K}_{m_1, m_2, \ldots, m_n}) = 2$. Note, that the partite truncation of $\mathbb{K}_{m_1, m_2, \ldots, m_n}$ is isomorphic to $\mathbb{K}_n$, and a bipartite subgraph induced by any two distinct partite sets $\mathcal{V}_\rangle$ and $\mathcal{V}_|$ is a complete bipartite graph. It is easy to verify that $\lambda(\mathbb{K}_n) = 2n - 2$. Thus, by Theorem 7, $\lambda(\mathbb{K}_{m_1, m_2, \ldots, m_n}) = \sum\limits_{i=1}^{n} m_i + n - 2$.

Example 3. Let $\mathbb{F}_p$ and $\mathbb{F}_q$ be two finite fields of order p and q, respectively. The set of non-trivial zero divisors of $\mathbb{F}_p \times \mathbb{F}_q$ and hence the vertices of $\Gamma(\mathbb{F}_p \times \mathbb{F}_q)$ are given as $\mathcal{V}_1 = \{(a, 0) : a \in \mathbb{F}_p \text{ and } a \neq 0\}$, $\mathcal{V}_2 = \{(0, b) : b \in \mathbb{F}_q \text{ and } b \neq 0\}$. This implies, $|\mathcal{V}_1| = p - 1$ and $|\mathcal{V}_2| = q - 1$. There are no two vertices in sets $\mathcal{V}_1$ or $\mathcal{V}_2$ that are adjacent to each other, and each vertex in $\mathcal{V}_1$ is adjacent to every vertex in $\mathcal{V}_2$. Thus, $\Gamma(\mathbb{F}_p \times \mathbb{F}_q) \cong \mathbb{K}_{p-1, q-1}$, and consequently, $\lambda(\mathcal{G}) = p + q - 2$.

In the following result, we demonstrate that by applying the operation "partite truncation" to the zero-divisor graph of a reduced ring we obtain the zero-divisor graph associated with a Boolean ring.

Proposition 2. *For $1 \leq i \leq n$, let $\mathbb{F}_{q_i}$ be finite fields of order q_i and let $\mathcal{G} \cong \Gamma(\prod\limits_{i=1}^{n} \mathbb{F}_{q_i})$. Then $\mathcal{G}$ is a $(2^n - 2)$-partite graph with $\hat{\mathcal{G}} \cong \Gamma(\prod\limits^{n} \mathbb{Z}_2)$.*

Proof. We prove the result in two parts parts. First, we show that $\mathcal{G}$ is $(2^n - 2)$-partite graph, and then we show its partite truncation is isomorphic to $\Gamma(\prod\limits^{n} \mathbb{Z}_2)$.

Part-I.

Note that each non-trivial zero divisor of $\Gamma(\prod\limits_{i=1}^{n} \mathbb{F}_{q_i})$, and hence every vertex of $\mathcal{G}$ is an n-tuple which has at least one zero and one non-zero entry in it. For $1 \leq t < n$, we partition the vertices of $\mathcal{G}$ as follows,

$$\mathcal{U}_t = \{v_t : v_t \text{ has exactly } t \text{ non-zero entries in it}\}.$$

For each such t, the number of ways to choose t entries out of n entries is $\binom{n}{t}$. Furthermore, for each t, we partition the set $\mathcal{U}_t$ into $\binom{n}{t}$ sets as, $\mathcal{U}_t = \big\{\mathcal{V}_{l,t} : 1 \leq l \leq$

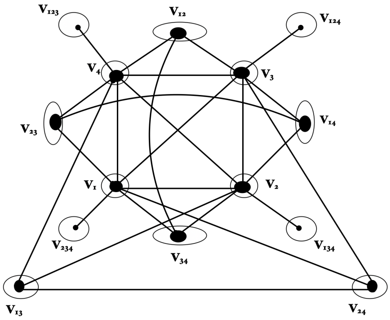

Figure 1. **A Rough Sketch of** $\Gamma(\prod_{i=1}^{4}\mathbb{F}_{q_i})$

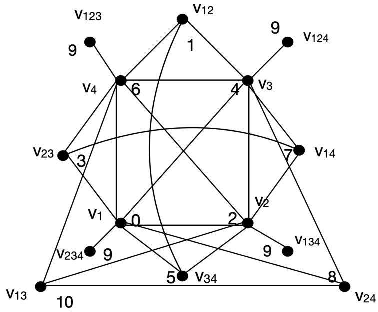

Figure 2. $\Gamma((\mathbb{Z}_2)^4)$

$\binom{n}{t}\big\}$, where $\mathcal{V}_{l,t}$ consists of those vertices of $\mathcal{U}_t$ that have t non-zero entries in the same position. For each l and t, no two vertices in $\mathcal{V}_{l,t}$ are adjacent to each other. On the other hand, if a vertex of one such set say $\mathcal{V}_{l,r}$ is adjacent to some vertex of another set say $\mathcal{V}_{l',r'}$, where $1 \leq r \leq r' < n$, then each vertex of $\mathcal{V}_{l,r}$ is adjacent to every vertex of $\mathcal{V}_{l',r'}$. The total number of all such sets in $V(\mathcal{G})$ is given as, $\sum\limits_{1\leq t<n}\binom{n}{t} = 2^n - 2$. This implies, $\mathcal{G}$ is a $(2^n - 2)$-partite graph.

Part-II.

As in Part-I, we partition vertex set of the graph $\Gamma(\prod^{n}\mathbb{Z}_2)$ into $\big\{\mathcal{V}_{l,t} : 1 \leq l \leq \binom{n}{t}\big\}$. Since there is only one non-zero term in $\mathbb{Z}_2$, therefore, for each l and t, $|\mathcal{V}_{l,t}| = 1$. It follows that the order of $\Gamma(\prod^{n}\mathbb{Z}_2)$ is equal to the number of partite sets in $\Gamma(\prod_{i=1}^{n}\mathbb{F}_{q_i})$, which by Part-I is equal to $2^n - 2$. The map $\hat{v_{l,t}} \longrightarrow \mathcal{V}_{l,t}$ defines an isomorphism between $\hat{\mathcal{G}}$ and $\Gamma(\prod^{n}\mathbb{Z}_2)$. $\square$

The following example illustrates the concept of partite truncation of a graph as well as Theorem 7 and Proposition 2.

Example 4. Let $\mathbb{F}_{q_i}$ be finite fields of order q_i, where $1 \leq i \leq 4$ and $q_1 \leq q_2 \leq q_3 \leq q_4$. As illustrated in Proposition 2, we partition vertex set of the graph $\Gamma(\prod_{i=1}^{4}\mathbb{F}_{q_i})$ as, $\mathcal{U}_1 = \{\mathcal{V}_1, \mathcal{V}_2, \mathcal{V}_3, \mathcal{V}_4\}, \mathcal{U}_2 = \{\mathcal{V}_{12}, \mathcal{V}_{13}, \mathcal{V}_{14}, \mathcal{V}_{23}, \mathcal{V}_{24}, \mathcal{V}_{34}\}$ and $\mathcal{U}_3 = \{\mathcal{V}_{123}, \mathcal{V}_{124}, \mathcal{V}_{134}, \mathcal{V}_{234}\}$, and obtain $\Gamma(\prod_{i=1}^{4}\mathbb{F}_{q_i})$ as a 14-partite graph. A rough drawing of $\Gamma(\prod_{i=1}^{4}\mathbb{F}_{q_i})$ is shown in Figure 1, where an edge between two partite sets means that each vertex of one set is adjacent to every vertex of the other. If we replace every non-zero entry of a vertex in the above partite sets with 1, we obtain the partite truncation of the graph $\Gamma(\prod_{i=1}^{4}\mathbb{F}_{q_i})$, which is isomorphic to $\Gamma((\mathbb{Z}_2)^4)$. A minimal $L(2,1)$-labeling of the zero-divisor graph $\Gamma((\mathbb{Z}_2)^4)$ is shown in Figure 2. If we suppose that the labeling is minimal, then any other labeling with a shorter span would imply that it has more than one multiplicity, which is impossible to achieve. Thus, $\lambda(\Gamma((\mathbb{Z}_2)^4)) = 10$. Given the minimal labeling (as shown in Figure 2) on the partite truncation of $\Gamma(\prod_{i=1}^{4}\mathbb{F}_{q_i})$, we obtain the equivalence relation on the partite sets of $\Gamma(\prod_{i=1}^{4}\mathbb{F}_{q_i})$ as discussed in Remark 1. The representatives of the equivalence classes are given as, $\mathcal{C}=\{\mathcal{V}_1,\mathcal{V}_2,\mathcal{V}_3,\mathcal{V}_4,\mathcal{V}_{12},\mathcal{V}_{13},\mathcal{V}_{14},\mathcal{V}_{23},\mathcal{V}_{24},\mathcal{V}_{34},\mathcal{V}_{234}\}$. Therefore, by Theorem 7,

$$\begin{aligned}\lambda(\Gamma(\prod_{i=1}^{4}\mathbb{F}_{q_i})) &= \sum_{\mathcal{V}\in\mathcal{C}}|\mathcal{V}|+\lambda(\Gamma((\mathbb{Z}_2)^4))-|\mathcal{C}| \\ &= \sum_{i=1}^{4}(q_i-1)+\sum_{1\le i<j\le 4}\big((q_i-1)(q_j-1)\big)+(q_2-1)(q_3-1)(q_4-1)-1.\end{aligned}$$

Note that Proposition 1 cannot be applied to determine the λ-number of Beck's zero-divisor graph, $\Gamma'(\prod_{i=1}^{n}\mathbb{F}_{q_i})$, since $diam(\Gamma(\prod_{i=1}^{n}\mathbb{F}_{q_i}))\ge 3$ for $n\ge 3$. Therefore, we establish the following Proposition, which bears resemblance to Proposition 2 regarding Beck's zero-divisor graph of $\Gamma'(\prod_{i=1}^{n}\mathbb{F}_{q_i})$ and $\Gamma'(\prod^{n}\mathbb{Z}_2)$.

Proposition 3. *For $1\le i\le n$, let $\mathbb{F}_{q_i}$ be finite fields of order q_i and let $\mathcal{H}\cong\Gamma'(\prod_{i=1}^{n}\mathbb{F}_{q_i})$. Then $\mathcal{H}$ is a 2^n-partite graph with $\hat{\mathcal{H}}\cong\Gamma'(\prod^{n}\mathbb{Z}_2)$.*

Proof. Partition the non-zero zero-divisors of $\prod_{i=1}^{n}\mathbb{F}_{q_i}$ in the same manner as outlined in Proposition 2. Furthermore, include two additional sets in the partition, denoted as $\mathcal{W}_1$ and $\mathcal{W}_2$, defined as $\mathcal{W}_1=\{0\in\prod_{i=1}^{n}\mathbb{F}_{q_i}\}$ and $\mathcal{W}_2=\{u\in\prod_{i=1}^{n}\mathbb{F}_{q_i}\,|\,u \text{ is a unit}\}$. This partition effectively partitions the vertex set of $\mathcal{H}$ into 2^n distinct sets. Note that no two vertices within the same set are adjacent to each other. Consequently, $\mathcal{H}$ is a 2^n-partite graph.
On similar lines as in Proposition 2, truncating the partite sets containing zero divisors of $\prod_{i=1}^{n}\mathbb{F}_{q_i}$ corresponds to the zero-divisors of $\prod^{n}\mathbb{Z}_2$. Furthermore, truncating $\mathcal{W}_1$ corresponds to the element $0\in\prod^{n}\mathbb{Z}_2$, while truncating $\mathcal{W}_2$ corresponds to the element $(1,1,\dots,1)\in\prod^{n}\mathbb{Z}_2$. Consequently, we establish the isomorphism $\hat{\mathcal{H}}\cong\Gamma'(\prod^{n}\mathbb{Z}_2)$. □

The following example illustrates Proposition 3 as well as Theorem 7 for the graphs $\Gamma'(\prod_{i=1}^{3}\mathbb{F}_{q_i})$ and $\Gamma'(\prod^{3}\mathbb{Z}_2)$.

Example 5. Let $\mathbb{F}_{q_i}$ be finite fields of order q_i, where $1\le i\le 3$, and $q_1\le q_2\le q_3$. As illustrated in Proposition 3, partition the vertex set of $\Gamma'(\prod_{i=1}^{3}\mathbb{F}_{q_i})$ as

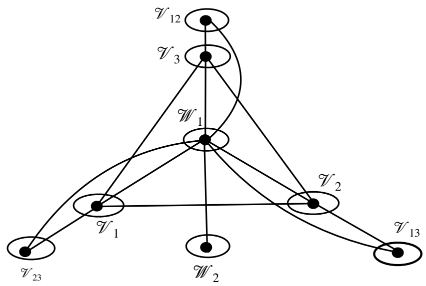

Figure 3. **A Rough Sketch of** $\Gamma'(\prod_{i=1}^{3} \mathbb{F}_{q_i})$

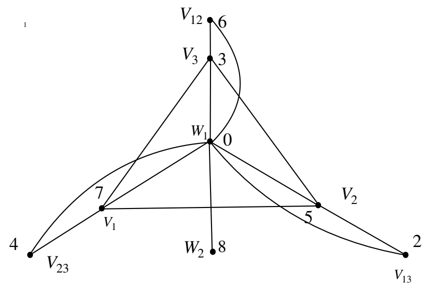

Figure 4. $\Gamma'((\mathbb{Z}_2)^3)$

$\{\mathcal{V}_1, \mathcal{V}_2, \mathcal{V}_3, \mathcal{V}_{12}, \mathcal{V}_{23}, \mathcal{V}_{13}, \mathcal{W}_1, \mathcal{W}_2\}$. The sizes of these sets are given by $|\mathcal{V}_1| = q_1 - 1$, $|\mathcal{V}_2| = q_2 - 1$, $|\mathcal{V}_3| = q_3 - 1$, $|\mathcal{V}_{12}| = (q_1 - 1)(q_2 - 1)$, $|\mathcal{V}_{23}| = (q_2 - 1)(q_3 - 1)$, $|\mathcal{V}_{13}| = (q_1 - 1)(q_3 - 1)$, $|\mathcal{W}_1| = 1$, and $|\mathcal{W}_2| = (q_1 - 1)(q_2 - 1)(q_3 - 1)$. By replacing every non-zero term in each vertex of $\Gamma'(\prod_{i=1}^{3} \mathbb{F}_{q_i})$, we obtain the partite truncation of the graph, which is $\Gamma'(\prod^{3} \mathbb{Z}_2)$. Note that $\Gamma'(\prod_{i=1}^{3} \mathbb{F}_{q_i})$ is an 8-partite graph.

A visual depiction of the graph $\Gamma'(\prod_{i=1}^{3} \mathbb{F}_{q_i})$ is presented in Figure 3, highlighting the interconnectivity between its partite sets. The presence of an edge between two partite sets in this figure indicates that each vertex in one set is adjacent to every vertex in the other set. Additionally, Figure 4 showcases a minimal $L(2,1)$-labeling of the zero-divisor graph $\Gamma'((\mathbb{Z}_2)^3)$. The labeling is minimal because every $L(2,1)$-labeling of $\Gamma'((\mathbb{Z}_2)^3)$ must have at least one hole. By considering the minimal labeling, we can determine the equivalence classes of the partite sets in $\Gamma'(\prod_{i=1}^{3} \mathbb{F}_{q_i})$ as discussed in Remark 1. The representatives of these equivalence classes are $\mathcal{C} = \{\mathcal{V}_1, \mathcal{V}_2, \mathcal{V}_3, \mathcal{V}_{12}, \mathcal{V}_{23}, \mathcal{V}_{13}, \mathcal{W}_1, \mathcal{W}_2\}$. Therefore, based on Theorem 7,

$$\begin{aligned}
\lambda(\Gamma'(\prod_{i=1}^{3} \mathbb{F}_{q_i})) &= \sum_{\mathcal{V} \in \mathcal{C}} |\mathcal{V}| + \lambda(\Gamma((\mathbb{Z}_2)^3)) - |\mathcal{C}| \\
&= \sum_{i=1}^{3} (q_i - 1) + \sum_{1 \le i < j \le 3} \big((q_i - 1)(q_j - 1)\big) + (q_1 - 1)(q_2 - 1)(q_3 - 1) + 1.
\end{aligned}$$

It would be very interesting to determine the λ-number and its corresponding minimal $L(2,1)$-labeling of the graphs $\Gamma(\prod_{i=1}^{n} \mathbb{F}_{q_i})$ and $\Gamma'(\prod_{i=1}^{n} \mathbb{F}_{q_i})$. So, we conclude the paper with the following open problem.

Problem 1. Determine λ-number of graphs $\Gamma(\prod^{n}(\mathbb{Z}_2))$ and $\Gamma'(\prod^{n}(\mathbb{Z}_2))$ with their corresponding minimal $L(2,1)$-labeling.

4. Conclusion

In this research article, we studied partite truncation, a graph operation that aids to obtain a smaller order graph from relatively larger order n-partite graph. This contraction of the order allow us to determine the λ-number and corresponding minimal labeling of the n-partite graph from the contracted graph. We exhibited that the zero-divisor graph associated with a Boolean ring is a partite truncated graph of the zero-divisor graph realized by a reduced ring. We also proved an interesting result that illustrates the deduction of λ-number of Beck's zero-divisor graph $\Gamma'(R)$ from λ-number of the zero-divisor graph $\Gamma(R)$. We concluded this paper with an open problem for future research work.

Acknowledgements. The first author's research is supported by the University Grants Commission, Govt. of India under UGC-Ref. No. 191620023547, and the second author's research work is funded by the Department of Atomic Energy, Govt. of India under S. No. 02011/15/2023NBHM(R.P)/R&D II /5866.

Conflict of interest. The authors declare that they have no conflict of interest.

Data Availability. Data sharing is not applicable to this article as no datasets were generated or analyzed during the current study.

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