Communications in Combinatorics and Optimization
Vol. xx, No. x (xxxx), pp. 1-5
https://doi.org/10.22049/cco.2024.28939.1790

CCO
Commun. Comb. Optim.

Short Note

A new construction of regular and quasi-regular self-complementary graphs

Lata Kamble[1,*], Charusheela Deshpande[2,†] and Bhagyshree Athawale[2,‡]

[1]Department of Mathematics, M.E.S's Abasaheb Garware College, Pune-411004, India
[*]lata7429@gmail.com

[2]Department of Mathematics, College of Engineering Pune, Pune-411006, India
[†]dcm.maths@coep.ac.in
[‡]bhagyashriathawale@gmail.com

Received: 19 August 2023; Accepted: 4 January 2024
Published Online: 10 January 2024

Abstract: A graph G with a vertex set V and an edge set E is called regular if the degree of every vertex is the same. A quasi-regular graph is a graph whose vertices have one of two degrees r and $r-1$, for some positive integer r. A graph G is said to be self-complementary if G is isomorphic to it's complement $\bar{G}$. In this paper we give a new method for construction of regular and quasi-regular self-complementary graph.

Keywords: self-complementary graph, regular graph, quasi-regular graph.

AMS Subject classification: 05C07, 05C60

1. Introduction

The study of self-complementary graphs was initiated by Sachs in 1962 [5] and later but independently by Ringel [4]. Each presents a construction algorithm for self-complementary graphs. Sachs and Ringel also gave a construction algorithm for regular and quasi-regular self-complementary graphs. In 1972 R. Gibbs [3] gave a new algorithm for construction of self-complementary graphs. This algorithm provides a method for constructing all self-complementary graphs having a given complementing permutation σ with cycles of lengths that are powers of 2.

In this paper we present a new method for construction of regular self-complementary and quasi-regular self-complementary graphs. In section 2, we give some preliminary definitions and known results. In section 3, we introduce a new method for construction of regular self-complementary graphs and in section 4, we provide a new method for construction of quasi-regular self-complementary graphs.

* *Corresponding Author*

2. Preliminary Definitions and Results

In this section we give some preliminary definitions and results. Graphs considered here are simple.

Definition 1. **(Regular graph)** A graph G with a vertex set V and an edge set E is called regular if the degree of every vertex is the same.

If G is a graph in which the degree of every vertex is k, then G is said to be a k-regular graph.

Definition 2. **(Bi-regular graph)** A graph G is said to be bi-regular if there exist two distinct positive integers d_1 and d_2 such that the degree of each vertex is either d_1 or d_2.

Definition 3. **(Quasi-regular graph)** A graph G is said to be quasi-regular if the degree of each vertex is either r or $r-1$ for some positive integer r.

Definition 4. **(Self-complementary graph)** A graph $G = (V;\ E)$ is called self-complementary if there exists a permutation $\sigma : V \to V$, called a complementing permutation, such that for every edge e of G, $e \in E$ if and only if $\sigma(e) \notin E$.

We state below the most basic results on self-complementary graphs and regular graphs, ones included even in introductory courses on graph theory.

Result 1. [2, 3, 5] If G is a self-complementary graph on n vertices, then $n \equiv 0$ or $1 \pmod 4$.

Result 2. [2, 3, 5] A graph G is k-regular graph on n vertices if and only if kn is even.

Result 3. [1, 2] If $d_1 \geq d_2 \geq \cdots \geq d_n$ is the degree sequence of a self-complementary graph G then $d_i + d_{n+1-i} = n-1$.

3. Constructing regular self-complementary graph

Theorem 4 gives a well known result for regular self-complementary graphs due to Sachs [5]. His proof involves first constructing a self-complementary graph G' on $4m$ vertices $v_1, v_2, \ldots, v_{4m}$, and then by adding a new vertex v_{4m+1} in the graph G' to get the required regular self-complementary graph G on the vertices $v_1, v_2, \ldots, v_{4m}, v_{4m+1}$. Two distinct vertices v_i, v_j in G' are joined if $i + j \equiv 0$ or $1 \pmod 4$ for $i, j = 1, 2, 3, \ldots, 4m$. In G', $d(v_i) = 2m$ if i is odd and $d(v_i) = 2m-1$ if i is even. Now the vertex v_{4m+1} is joined to all vertices v_i, with even i, $i = 1, 2, 3, \ldots, 4m$. The graph G so obtained is a regular self-complementary graph on the vertex set $V = \{v_1, v_2, \ldots, v_{4m}, v_{4m+1}\}$ with a complementing permutation, $\sigma : V \to V$ defined as $\sigma(v_i) = v_{i+1}, i = 1, 2, \ldots, 4m-1,\ \sigma(v_{4m}) = v_1, \sigma(v_{4m+1}) = v_{4m+1}$. There are several proofs for the theorem, and we introduce a new one.

Theorem 4. *There exists a regular self-complementary graph of order n if and only if $n \equiv 1 \pmod 4$.*

Proof. If G is a regular graph on n vertices, then from Result 3, degree of every vertex must be $r = \frac{n-1}{2}$. For r to be an integer, $n-1$ must be even, and since G is self-complementary, by Result 1, we get $n \equiv 1 \pmod 4$.

To prove the converse, we construct a regular self-complementary graph G of order n, where n is congruent to 1 modulo 4.
Let m be a positive integer and $V = \{u\} \cup V_0 \cup V_1 \cup V_2 \cup V_3$, where $V_i = \{v_j^i : j \in \mathbb{Z}_m\}$ for all $i \in \mathbb{Z}_4$. For pairwise distinct $i, i' \in \mathbb{Z}_4$, we define the following subsets of $V^{(2)}$, where $V^{(2)}$ denotes the set of all 2-subsets of V:

$$E_i = V_i^{(2)}, \quad E_{(i,i')} = \{\{v_{j_1}^i, v_{j_2}^{i'}\} : j_1, j_2 \in \mathbb{Z}_m\}, \quad E_i^u = \{\{u, v_j^i\} : j \in \mathbb{Z}_m\}.$$

Let $E = \bigcup\limits_{i=0,1} (E_i \cup E_i^u) \cup E_{(0,3)} \cup E_{(2,3)} \cup E_{(1,2)}$ and let G be the graph with vertex set V and edge set E as defined above, having $n = 4m+1$ vertices.

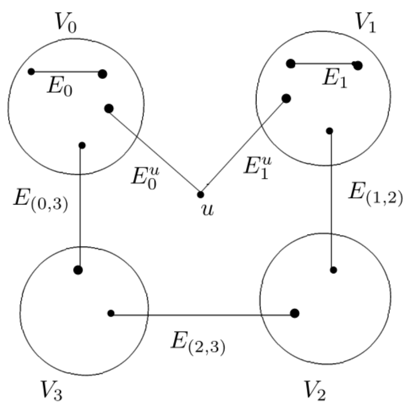

Figure 1. **The types of edges of the graph G**

Figure 1 explains the construction of the graph G in another way.
First we show that G is regular. Take any vertex v_j^i. Then, for fixed i, the vertex v_j^i lies in $m-1$ subsets of E_i, m subsets of $E_{(i,i')}$ and one subset of E_i^u.
Hence, for every vertex v_j^i in G with $i \in \{0,1\}$, we have $\mathrm{deg(v_j^i) = m - 1 + m + 1 = 2m}$, and for every vertex v_j^i in G with $i \in \{2,3\}$, we have $\mathrm{deg(v_j^i) = m + m = 2m}$. Furthermore, $\mathrm{deg(u) = m + m = 2m}$. We conclude that G is regular.
Define a bijection $\phi : V \to V$ as $\phi(u) = u, \phi(v_j^0) = v_j^3, \phi(v_j^1) = v_j^2, \phi(v_j^2) = v_j^0$, and

$\phi(v_j^3) = v_j^1$, for all $j \in \mathbb{Z}_m$. It can be easily checked that G is self-complementary, with ϕ as its complementing permutation. □

4. Constructing quasi-regular self-complementary graph

The known result for quasi-regular self-complementary graphs due to Sachs [5] is as follows. There are several proofs for the theorem, and we provide a new one.

Theorem 5. *There exists a quasi-regular self-complementary graph of order n if and only* $n \equiv 0 \pmod 4$.

Proof. Let G be a quasi-regular self-complementary graph on n vertices. By Result 3, $s + (s-1) = 2s - 1 = n - 1$. Then $n = 2s$, and since G is self-complementary, by Result 1, it follows that $n \equiv 0 \pmod 4$.

To prove the converse, we construct a graph G of order congruent to 0 modulo 4, which is quasi-regular and self-complementary.
Let m be a positive integer and $V = V_0 \cup V_1 \cup V_2 \cup V_3$, where $V_i = \{v_j^i : j \in \mathbb{Z}_m\}$ for all $i \in \mathbb{Z}_4$. For pairwise distinct $i, i' \in \mathbb{Z}_4$, define the following subsets of $V^{(2)}$ where $V^{(2)}$ denotes the set of all 2-subsets of V:

$$E_i = V_i^{(2)}, \quad E_{(i,i')} = \{\{v_{j_1}^i, v_{j_2}^{i'}\} : j_1, j_2 \in \mathbb{Z}_m\}.$$

Let $E = \bigcup_{i=0,1} (E_i) \cup E_{(0,3)} \cup E_{(2,3)} \cup E_{(1,2)}$ and let G be the graph with vertex set V

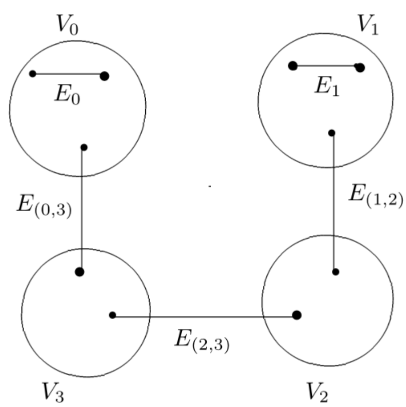

Figure 2. **The types of edges of the graph G**

and edge set E as defined above having $n = 4m$ vertices.
Figure 2 explains the construction of the graph G in another way.
First we show that G is quasi-regular. Take any vertex v_j^i. Then, for fixed i, the

vertex v_j^i lies in $m-1$ subsets of E_i and m subsets of $E_{(i,i')}$. Hence, for every vertex v_j^i in G with $i \in \{0,1\}$, we have $\mathrm{deg(v_j^i) = m - 1 + m = 2m - 1}$, and for every vertex v_j^i in G with $i \in \{2,3\}$, we have $\mathrm{deg(v_j^i) = m + m = 2m}$. Therefore, there are $2m$ vertices having degree $2m-1$ and $2m$ vertices of degree $2m$.
We conclude that G is quasi-regular.
Define a bijection $\phi : V \to V$ as $\phi(v_j^0) = v_j^3, \phi(v_j^1) = v_j^2, \phi(v_j^2) = v_j^0$, and $\phi(v_j^3) = v_j^1$, for all $j \in \mathbb{Z}_m$. It can be easily checked that G is self-complementary, with ϕ as its complementing permutation. □

Acknowledgements. The authors are thankful to Prof. Gutman and the anonymous referees for their careful reading and the suggestions that improved the article.

Conflict of interest. The authors declare no conflict of interest.

Data Availability. Data sharing is not applicable to this article as no datasets were generated or analyzed during the current study.

Authors Contribution. The authors confirm sole responsibility of the research work presented in this article.

References

[1] C.R.J Clapham and D.J. Kleitman, *The degree sequences of self-complementary graphs*, J. Comb. Theory. Ser. B **20** (1976), no. 1, 67–74.
https://doi.org/10.1016/0095-8956(76)90068-X.

[2] A. Farrugia, *Self-complementary graphs and generalisations: a comprehensive reference manual*, Ph.D. thesis, University of Malta, 1999.

[3] R.A. Gibbs, *Self-complementary graphs*, J. Comb. Theory. Ser. B **16** (1974), no. 2, 106–123.
https://doi.org/10.1016/0095-8956(74)90053-7.

[4] G. Ringel, *Selbstkomplementäre graphen*, Arch. Math. **14** (1963), no. 1, 354–358
https://doi.org/10.1007/BF01234967.

[5] H. Sachs, *Über selbstkomplementäre graphen*, Publ. Math. Drecen **9** (1962), no. 3–4, 270–288.