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Research Article

Restrained double Roman domatic number

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Abstract: Let G be a graph with vertex set $V(G)$. A double Roman dominating function (DRDF) on a graph G is a function $f : V(G) \rightarrow \{0,1,2,3\}$ having the property that if $f(v) = 0$, then the vertex v must have at least two neighbors assigned 2 under f or one neighbor w with $f(w) = 3$, and if $f(v) = 1$, then the vertex v mus have at least one neighbor u with $f(u) \geq 2$. If f is a DRDF on G, then let $V_0 = \{v \in V(G) : f(v) = 0\}$. A restrained double Roman dominating function is a DRDF f having the property that the subgraph induced by V_0 does not have an isolated vertex. A set $\{f_1, f_2, \dots, f_d\}$ of distinct restrained double Roman dominating functions on G with the property that $\sum_{i=1}^{d} f_i(v) \leq 3$ for each $v \in V(G)$ is called a restrained double Roman dominating family (of functions) on G. The maximum number of functions in a restrained double Roman dominating family on G is the restrained double Roman domatic number of G, denoted by $d_{rdR}(G)$. We initiate the study of the restrained double Roman domatic number, and we present different sharp bounds on $d_{rdR}(G)$. In addition, we determine this parameter for some classes of graphs.

Keywords: Restrained double Roman domination, restrained double Roman domatic number.

AMS Subject classification: 05C69.

1. Introduction

For definitions and notations not given here we refer to [6]. We consider simple and finite graphs G with vertex set $V = V(G)$ and edge set $E = E(G)$. The *order* of G is $n = n(G) = |V|$. The *neighborhood* of a vertex v is the set $N(v) = N_G(v) = \{u \in V(G) \mid uv \in E\}$. The *degree* of vertex $v \in V$ is $d(v) = d_G(v) = |N(v)|$. The *maximum degree* and *minimum degree* of G are denoted by $\Delta = \Delta(G)$ and $\delta = \delta(G)$, respectively. The *complement* of a graph G is denoted by $\overline{G}$. For a subset D of vertices in a graph G, we denote by $G[D]$ the subgraph of G induced by D. A set of pairwise independent edges of G is called a *matching* in G, while a matching of maximum cardinality is a *maximum matching* in G. A *leaf* is a vertex of degree one, and its neighbor is called a *support vertex*. We write P_n for the *path* of order n, C_n for the *cycle* of length n, K_n for the *complete graph* of order n. Also, let $K_{n_1,n_2,\ldots,n_p}$ denote the *complete p-partite graph* with vertex set $S_1 \cup S_2 \cup \ldots \cup S_p$ where $|S_i| = n_i$ for $1 \leq i \leq p$. For $n \geq 2$, the *star* $K_{1,n-1}$ has one vertex of degree $n-1$ and $n-1$ leaves.

A set $S \subseteq V(G)$ is called a *dominating set* if every vertex is either an element of S or is adjacent to an element of S. The *domination number* $\gamma(G)$ of a graph G is the minimum cardinality of a dominating set of G. A *minimal dominating set* in a graph G is a dominating set that contains no dominating set as a proper subset.

In this paper we continue the study of Roman dominating functions and Roman domatic numbers in graphs (see, for example, the survey articles [2–5]). If $f : V(G) \longrightarrow \{0, 1, 2, 3\}$ is a function, then let (V_0, V_1, V_2, V_3) be the ordered partition of $V(G)$ induced by f, where $V_i = \{v \in V(G) : f(v) = i\}$ for $i \in \{0, 1, 2, 3\}$. There is a 1-1 correspondence between the function f and the ordered partition (V_0, V_1, V_2, V_3). So we also write $f = (V_0, V_1, V_2, V_3)$. A *double Roman dominating function* (DRDF) on a graph G is defined in [1] as a function $f : V(G) \longrightarrow \{0, 1, 2, 3\}$ having the property that if $f(v) = 0$, then the vertex v must have at least two neighbors in V_2 or one neighbor in V_3, and if $f(v) = 1$, then the vertex v must have at least one neighbor in $V_2 \cup V_3$. The weight of a DRDF f is the value $f(V(G)) = \sum_{u \in V(G)} f(u)$. The *double Roman domination number* $\gamma_{dR}(G)$ is the minimum weight of a DRDF on G, and a double Roman dominating function of G with weight $\gamma_{dR}(G)$ is called a $\gamma_{dR}(G)$-*function* of G.

A set $\{f_1, f_2, \ldots, f_d\}$ of distinct double Roman dominating functions on G with the property that $\sum_{i=1}^{d} f_i(v) \leq 3$ for each $v \in V(G)$ is called in [10] a *double Roman dominating family* (of functions) on G. The maximum number of functions in a double Roman dominating family on G is the *double Roman domatic number* of G, denoted by $d_{dR}(G)$.

Mojdeh, Masoumi and Volkmann [7] defined the *restrained double Roman dominating function* (RDRDF) as a double Roman dominating function f with the property that the subgraph induced by V_0 does not have an isolated vertex. The *restrained double Roman domination number* $\gamma_{rdR}(G)$ equals the minimum weight of an RDRDF on G. An RDRDF on G with weight $\gamma_{rdR}(G)$ is called a $\gamma_{rdR}(G)$-function.

A set $\{f_1, f_2, \ldots, f_d\}$ of distinct restrained double Roman dominating functions on G with the property that $\sum_{i=1}^{d} f_i(v) \leq 3$ for each $v \in V(G)$ is called a *restrained double Roman dominating family* (of functions) on G. The maximum number of functions in a restrained double Roman dominating family on G is the *restrained double Roman domatic number* of G, denoted by $d_{rdR}(G)$. The definitions lead to $\gamma_{dR}(G) \leq \gamma_{rdR}(G)$ and $d_{rdR}(G) \leq d_{dR}(G)$.

We initiate the study of the restrained double Roman domatic number, and we present different sharp bounds on $d_{rdR}(G)$. In addition, we determine this parameter for some classes of graphs. Furthermore, if G is a connected graph of order $n \geq 3$, then we show that $6 \leq \gamma_{rdR}(G) + d_{rdR}(G) \leq \frac{3n}{2} + 2$.

We make use of the following results.

Proposition 1. *[10] If G is a graph, then $d_{dR}(G) \le \delta(G) + 1$.*

Since $d_{rdR}(G) \le d_{dR}(G)$, the next corollary is immediate.

Corollary 1. *If G is a graph of order n, then $d_{rdR}(G) \le \delta(G) + 1 \le n$.*

Proposition 2. *[10] Let C_n be a cycle of order $n \ge 3$. Then $d_{dR}(C_n) = 3$, when $n \equiv 0 \pmod 3$ and $d_{dR}(C_n) = 2$, when $n \equiv 1, 2 \pmod 3$.*

Proposition 3. *[10] Let G be a graph of order $n \ge 2$. If $\Delta(G) \le n-2$, then $d_{dR}(G) \le \frac{n}{2}$.*

Proposition 4. *[10] If G is a graph of order n, then $d_{dR}(G) + d_{dR}(\overline{G}) \le n + 1$, with equality if and only if $G = K_n$ or $\overline{G} = K_n$.*

Proposition 5. *[7] If G is a connected graph of order $n \ge 2$, then $\gamma_{rdR}(G) \le \frac{3n}{2}$.*

Proposition 6. *If G is a graph of order $n \ge 3$, then $\gamma_{rdR}(G) \ge 3$, with equality if and only if $\Delta(G) = n - 1$ and G contains a vertex w of maximum degree such that $\delta(G[N_G(w)]) \ge 1$.*

Proof. Since $n \ge 3$, it is easy to see that $\gamma_{rdR}(G) \ge 3$. Assume that G contains a vertex w with $d_G(w) = n - 1$ such that $\delta(G[N_G(w)]) \ge 1$. Define the function f by $f(w) = 3$ and $f(x) = 0$ for $x \in V(G) \setminus \{w\}$. Since $G[N_G(w)]$ does not contain an isolated vertex, we observe that f is an RDRDF on G of weight 3 and so $\gamma_{rdR}(G) = 3$. Conversely, assume that $\gamma_{rdR}(G) = 3$. Let f be a $\gamma_{rdR}(G)$-function. Since $n \ge 3$, there exists a vertex w with $f(w) = 3$ such that the remaining $n - 1$ vertices with value 0 are adjacent to w and $\delta(G[N_G(w)]) \ge 1$. □

Proposition 7. *[8] If G is a graph without isolated vertices and S is a minimal dominating set of G, then $V(G) \setminus S$ is a dominating set of G.*

Proposition 8. *[7] If $p, q \ge 2$ are integers, then $\gamma_{rdR}(K_{p,q}) = 6$.*

Proposition 9. *[9] Let $G = K_{n_1,n_2,\ldots,n_p}$ be a complete p-partite graph with $p \ge 2$ and $n_1 \le n_2 \le \ldots \le n_p$. If $n = n_1 + n_2 + \ldots + n_p$ and M is a maximum matching, then $|M| = \min\{n - n_p, \lfloor \frac{n}{2} \rfloor\}$.*

2. Properties and bounds

In this section we present basic properties and bounds on the restrained double Roman domatic number.

Theorem 1. *If G is a graph without isolated vertices, then $d_{rdR}(G) \ge 2$.*

Proof. Let T be a spanning forest of G without isolated vertices, and let X and Y be a bipartion of T. Define the functions f and g by $f(x) = 1$, $f(y) = 2$ and $g(x) = 2$, $g(y) = 1$ for $x \in X$ and $y \in Y$. Since T has no isolated vertices, f and g are distinct restrained double Roman dominating functions on T and also on G such that $f(u) + g(u) = 3$ for each $u \in V(G)$. Therefore $\{f, g\}$ is a restrained double Roman dominating family on G and thus $d_{rdR}(G) \geq 2$. □

We deduce from Corollary 1 and Theorem 1 the next resutl immediately.

Corollary 2. *Let G be a graph without isolated vertices. If G has a leaf, then $d_{rdR}(G) = 2$. In particular, if T is a nontrivial tree, then $d_{rdR}(T) = 2$.*

Corollary 3. *Let C_n be a cycle of order $n \geq 3$. Then $d_{rdR}(C_n) = 3$, when $n \equiv 0 \pmod 3$ and $d_{rdR}(C_n) = 2$, when $n \equiv 1, 2 \pmod 3$.*

Proof. If $n \equiv 1, 2 \pmod 3$, then $d_{rdR}(C_n) \geq 2$ by Theorem 1, and Proposition 2 implies $d_{rdR}(C_n) \leq d_{dR}(C_n) \leq 2$. This leads to $d_{rdR}(C_n) = 2$ in this case.
Let now $n = 3t$ for an integer $t \geq 1$, and let $C_n = v_1v_2 \dots v_nv_1$. We deduce from Corollary 1 that $d_{rdR}(C_n) \leq 3$. Now define f_1, f_2 and f_3 by $f_1(v_{3i-2}) = 3$ for $1 \leq i \leq t$ and $f_1(x) = 0$ otherwise, $f_2(v_{3i-1}) = 3$ for $1 \leq i \leq t$ and $f_2(x) = 0$ otherwise and $f_3(v_{3i}) = 3$ for $1 \leq i \leq t$ and $f_3(x) = 0$ otherwise. Then $\{f_1, f_2, f_3\}$ is a restrained double Roman dominating family on C_{3t} and thus $d_{rdR}(C_{3t}) \geq 3$. Therefore $d_{rdR}(C_n) = 3$, when $n \equiv 0 \pmod 3$. □

Theorem 2. *If G is a graph, then $\gamma_{rdR}(G) \cdot d_{rdR}(G) \leq 3n$. Moreover, if we have the equality $\gamma_{rdR}(G) \cdot d_{rdR}(G) = 3n$, then for each restrained double Roman dominating family $\{f_1, f_2, \dots, f_d\}$ on G with $d = d_{rdR}(G)$, each f_i is a $\gamma_{rdR}(G)$-function and $\sum_{i=1}^{d} f_i(v) = 3$ for all $v \in V(G)$.*

Proof. Let $\{f_1, f_2, \dots, f_d\}$ be a restrained double Roman dominating family on G with $d = d_{rdR}(G)$. Then

$$
\begin{aligned}
d \cdot \gamma_{rdR}(G) &= \sum_{i=1}^{d} \gamma_{rdR}(G) \leq \sum_{i=1}^{d} \sum_{v \in V(G)} f_i(v) \\
&= \sum_{v \in V(G)} \sum_{i=1}^{d} f_i(v) \leq \sum_{v \in V(G)} 3 = 3n.
\end{aligned}
$$

If $\gamma_{rdR}(G) \cdot d_{rdR}(G) = 3n$, then the two inequalities occuring in the proof become equalities. Hence for the restrained double Roman dominating family $\{f_1, f_2, \dots, f_d\}$ on G and for each i, $\sum_{v \in V(G)} f_i(v) = \gamma_{rdR}(G)$. Thus each f_i is a $\gamma_{rdR}(G)$-function, and $\sum_{i=1}^{d} f_i(v) = 3$ for each $v \in V(G)$. □

Theorem 3. *Let G be a graph of order $n \geq 3$. If G has $1 \leq p \leq n-1$ vertices of degree $n-1$, then $d_{rdR}(G) \geq p+1$.*

Proof. Let $\{v_1, v_2, \ldots, v_n\}$ be the vertex set of G and let $v_1, v_2, \ldots, v_p$ be the vertices of degree $n-1$. If $p = 1$, then Theorem 1 implies $d_{rdR}(G) \geq 2 = p+1$. Let now $p \geq 2$. Define the functions f_i by $f_i(v_i) = 3$ and $f_i(x) = 0$ for $x \neq v_i$ for $1 \leq i \leq p$ and f_{p+1} by $f_{p+1}(v_n) = f_{p+1}(v_{n-1}) = \ldots = f_{p+1}(v_{p+1}) = 3$ and $f_{p+1}(v_i) = 0$ for $1 \leq i \leq p$. Since $p \geq 2$, $f_1, f_2, \ldots, f_{p+1}$ are disdinct RDRD functions on G such that $f_1(x) + f_2(x) + \ldots + f_{p+1}(x) = 3$ for each $x \in V(G)$. Therefore $\{f_1, f_2, \ldots, f_{p+1}\}$ is a restrained double Roman dominating family on G and so $d_{rdR}(G) \geq p+1$. □

Corollary 4. *Let G be a graph of order n. Then $d_{rdR}(G) \leq n$ with equality if and only if G is complete.*

Proof. Corollary 1 implies $d_{rdR}(G) \leq n$. Let now G be complete. If $n = 1$, then obviously $d_{rdR}(G) = 1 = n$. If $n = 2$, then it follows from Corollary 2 that $d_{rdR}(G) = 2 = n$. Let now $n \geq 3$. Then Theorem 3 with $p = n-1$ leads to $d_{rdR}(G) \geq n$ and so $d_{rdR}(G) = n$.
Conversely assume that $d_{rdR}(G) = n$. If G is not complete, then $\delta(G) \leq n-2$ and Corollary 1 leads to the contradiction $n = d_{rdR}(G) \leq \delta(G) + 1 \leq n-1$. □

Example 1. Let $\{v_1, v_2, \ldots, v_n\}$ be the vertex set of the complete graph K_n ($n \geq 3$), and let k be an integer with $1 \leq k \leq n-2$. Define the graph $G = K_n - \{v_1v_n, v_2v_n \ldots, v_kv_n\}$. Then $\delta(G) = n-k-1$, and it follows from Corollary 1 that $d_{rdR}(G) \leq n-k$. Since $v_{k+1}, v_{k+2}, \ldots, v_{n-1}$ are vertices of degree $n-1$, we deduce from Theorem 3 that $d_{rdR}(G) \geq n-k$ and thus $d_{rdR}(G) = n-k = \delta(G)+1$.

This example shows that Corollary 1 is sharp. Since $d_{rdR}(G) \leq d_{dR}(G)$, Proposition 3 implies the next bound.

Corollary 5. *Let G be a graph of order $n \geq 2$. If $\Delta(G) \leq n-2$, then $d_{rdR}(G) \leq \frac{n}{2}$.*

Corollary 6. *If G is a graph of order n, then $d_{rdR}(G) + d_{rdR}(\overline{G}) \leq n+1$, with equality if and only if $G = K_n$ or $\overline{G} = K_n$.*

Proof. Proposition 4 implies $d_{rdR}(G) + d_{rdR}(\overline{G}) \leq n+1$ and $d_{rdR}(G) + d_{rdR}(\overline{G}) \leq n$ when $G \neq K_n$ and $\overline{G} \neq K_n$. If, without loss of generality, $G = K_n$, then we deduce from Corollary 4 that $d_{rdR}(G) + d_{rdR}(\overline{G}) = n+1$. □

Theorem 4. *If G is a graph of order $n \geq 3$ without isolated vertices, then*

$$6 \leq \gamma_{rdR}(G) + d_{rdR}(G) \leq \frac{3n}{2} + 2.$$

Proof. First we prove the lower bound. Proposition 6 implies $\gamma_{rdR}(G) \geq 3$. Assume that $\gamma_{rdR}(G) = 3$. Then it follows from Proposition 6 that $\Delta(G) = n-1$, and G contains a vertex w of maximum degree such that $\delta(G[N_G(w)]) \geq 1$. Now let S be a minimal dominating set of $G[N_G(w)]$. According to Proposition 7 $N_G(w) \setminus S$ is also a dominating set of $G[N_G(w)]$. Now define the functions f_1, f_2, f_3 by $f_1(w) = 3$ and $f_1(x) = 0$ otherwise, $f_2(x) = 3$ for $x \in S$ and $f_2(x) = 0$ otherwise and $f_3(x) = 3$ for $x \in N_G(w) \setminus S$ and $f_3(x) = 0$ otherwise. Since w is adjacent to all vertices of S and to all vertices of $N_G(w) \setminus S$, we conclude that $\{f_1, f_2, f_3\}$ is a restrained double Roman dominating family on G and thus $d_{rdR}(G) \geq 3$. This implies $\gamma_{rdR}(G) + d_{rdR}(G) \geq 6$ in this case.
If $\gamma_{rdR}(G) \geq 4$, then Theorem 1 leads to $\gamma_{rdR}(G) + d_{rdR}(G) \geq 6$, and the lower bound is proved.
Now we prove the upper bound. Theorem 2 implies

$$\gamma_{rdR}(G) + d_{rdR}(G) \leq \frac{3n}{d_{rdR}(G)} + d_{rdR}(G).$$

According to Corollary 1 and Theorem 1, we have $2 \leq d_{rdR}(G) \leq n$. Using these bounds and the fact that the function $g(x) = x + \frac{3n}{x}$ is decreasing for $2 \leq x \leq \sqrt{3n}$ and increasing for $\sqrt{3n} \leq x \leq n$, we obtain

$$\gamma_{rdR}(G) + d_{rdR}(G) \leq \frac{3n}{d_{rdR}(G)} + d_{rdR}(G) \leq \max\left\{\frac{3n}{2} + 2, 3 + n\right\} = \frac{3n}{2} + 2,$$

and the upper bound is proved. □

Example 2. Let $H = pK_2$ with an integer $p \geq 2$. Then $n(H) = n = 2p$, $\gamma_{rdR}(H) = 3p = \frac{3n}{2}$ and $d_{rdR}(H) = 2$. Thus $\gamma_{rdR}(H) + d_{rdR}(H) = \frac{3n}{2} + 2$.

This example shows that the upper bound in Theorem 4 is sharp.

Example 3. Let $Wd(2,p)$ be the *windmill graph* consiting of a center vertex z which is adjacent to the vertices of $p \geq 1$ copies of the complete graph K_2. Then we observe that $\gamma_{rdR}(Wd(2,p)) = 3$, $d_{rdR}(Wd(2,p)) = 3$ and so $\gamma_{rdR}(Wd(2,p)) + d_{rdR}(Wd(2,p)) = 6$. Now let W be the the graph obtained form $Wd(2,p)$ by attaching a leaf. Then we note that $\gamma_{rdR}(W) = 4$, $d_{rdR}(W) = 2$ and so $\gamma_{rdR}(W) + d_{rdR}(W) = 6$.

The graphs in Example 3 show that the lower bound in Theorem 4 is sharp.

3. Complete p-partite graphs

Theorem 5. *If $q \geq p \geq 2$ are integers, then $d_{rdR}(K_{p,q}) = p$.*

Proof. Let $X = \{x_1, x_2, \dots, x_p\}$ and $Y = \{y_1, y_2, \dots, y_q\}$ be a bipartition of $K_{p,q}$. First let $|X| \geq 3$. If f is an RDRDF on $K_{p,q}$, then we show that $f(X) = \sum_{x\in X} f(x) \geq 3$. Suppose on the contrary, that $f(X) \leq 2$. Then, since $|X| \geq 3$, there exists a vertex $v \in X$ with $f(v) = 0$ and therefore a vertex $w \in Y$ with $f(w) = 0$. However, now the definition leads to the contradiction $f(X) = f(N(w)) \geq 3$. If $\{f_1, f_2, \dots, f_d\}$ is a restrained double Roman dominating family on $K_{p,q}$ with $d = d_{rdR}(K_{p,q})$, then it follows that

$$3d \leq \sum_{i=1}^{d} \sum_{x\in X} f_i(x) = \sum_{x\in X} \sum_{i=1}^{d} f_i(x) \leq \sum_{x\in X} 3 = 3|X| = 3p$$

and thus $d_{rdR}(K_{p,q}) \leq p$.
Let now $|X| = 2$. Then $d_{rdR}(K_{p,q}) \leq 3$ by Corollary 1. Suppose that $d = d_{rdR}(K_{p,q}) = 3$, and let $\{f_1, f_2, f_3\}$ be a restrained double Roman dominating family on $K_{p,q}$. If $f_i(x_1) = 0$ or $f_i(x_2) = 0$ for an index $i \in \{1, 2, 3\}$ or $f_i(X) \geq 3$ for all $1 \leq i \leq 3$, then we obtain the contradiction $d \leq p = 2$ as above. Therefore assume, without less of generality, that $f_1(x_1) = f_1(x_2) = 1$. This implies $f_1(y) \geq 2$ for $y \in Y$ and thus $f_2(X), f_3(X) \geq 3$. Hence we arrive at the contradiction

$$8 = 3d - 1 \leq \sum_{i=1}^{3} \sum_{x\in X} f_i(x) = \sum_{x\in X} \sum_{i=1}^{3} f_i(x) \leq \sum_{x\in X} 3 = 6.$$

Altogether, we have $d_{rdR}(K_{p,q}) \leq p$.
Conversely, define $f_i(x_i) = f_i(y_i) = 3$ and $f_i(x) = 0$ otherwise for $1 \leq i \leq p$. Then $\{f_1, f_2, \dots, f_p\}$ is a restrained double Roman dominating family on $K_{p,q}$. Hence $d_{rdR}(K_{p,q}) \geq p$ and thus $d_{rdR}(K_{p,q}) = p$. □

If $p \geq 2$ is an integer, then it follows from Proposition 8 and Theorem 5 that $\gamma_{rdR}(K_{p,p}) \cdot d_{rdR}(K_{p,p}) = 6p$. Thus Theorem 2 is sharp.

Theorem 6. *Let $G = K_{n_1,n_2,\dots,n_p}$ be a complete p-partite graph with $p \geq 3$ and $n_1 \leq n_2 \leq \dots \leq n_p$. If $n = n_1 + n_2 + \dots + n_p$, then:*

(i) If $n_{p-1} = 1$, then $d_{rdR}(G) = p$.

(ii) If $n_1 \geq 2$, then

$$d_{rdR}(G) = \min\left\{n - n_p, \left\lfloor \frac{n}{2} \right\rfloor\right\} = \min\left\{\sum_{i=1}^{p-1} n_i, \left\lfloor \frac{1}{2}\sum_{i=1}^{p} n_i \right\rfloor\right\}.$$

(iii) If $n_t = 1$ and $n_{t+1} \geq 2$ for $1 \leq t \leq p-2$, then

$$d_{rdR}(G) = t + \min\left\{\sum_{i=t+1}^{p-1} n_i, \left\lfloor \frac{1}{2}\sum_{i=t+1}^{p} n_i \right\rfloor\right\}.$$

Proof. Let $S_1, S_2, \dots, S_p$ be the partite sets of G with $|S_i| = n_i$ for $1 \le i \le p$.
(i) Let $n_{p-1} = 1$, and let $S_i = \{s_i\}$ for $1 \le i \le p-1$. Define $f_i(s_i) = 3$ and $f_i(x) = 0$ otherwise for $1 \le i \le p-1$ and $f_p(y) = 3$ for $y \in S_p$ and $f_p(x) = 0$ for $x \in V(G) \setminus S_p$. Then $\{f_1, f_2, \dots, f_p\}$ is a restrained double Roman dominating family on G and therefore $d_{rdR}(G) \ge p$. Since $\delta(G) = p-1$, it follows from Corollary 1 that $d_{rdR}(G) \le p$ and thus $d_{rdR}(G) = p$ in this case.
(ii) Let $n_1 \ge 2$. Then $\Delta(G) \le n-2$ and thus $d_{rdR}(G) \le \frac{n}{2}$ by Corollary 5. Let now $M = \{u_1v_1, u_2v_2, \dots, u_mv_m\}$ be a maximum matching of G.
Define f_i by $f_i(u_i) = f_i(v_i) = 3$ and $f_i(x) = 0$ otherwise for $1 \le i \le m = |M|$. Then $\{f_1, f_2, \dots, f_m\}$ is a restrained double Roman dominating family on G, and therefore we deduce from Proposition 9 that

$$d_{rdR}(G) \ge |M| = \min\left\{n - n_p, \left\lfloor \frac{n}{2} \right\rfloor\right\}. \tag{3.1}$$

If $n - n_p \ge n_p$, then $\min\left\{n - n_p, \left\lfloor \frac{n}{2} \right\rfloor\right\} = \left\lfloor \frac{n}{2} \right\rfloor$ and hence (3.1) and the bound $d_{rdR}(G) \le \frac{n}{2}$ lead to the desired result.
Next assume that $n_p > n - n_p$. Then $\min\left\{n - n_p, \left\lfloor \frac{n}{2} \right\rfloor\right\} = n - n_p$ and (3.1) implies $d_{rdR}(G) \ge n - n_p$. Let now $\{f_1, f_2, \dots, f_d\}$ be a restrained double Roman dominating family on G with $d = d_{rdR}(G)$, and let $X = S_1 \cup S_2 \cup \ldots \cup S_{p-1}$.
Assume first that there exists in index i, say $i = 1$, such that $f_1(X) = 0$. Then $f_1(y) \ge 2$ for $y \in S_p$. Since $n_i \ge 2$, we observe in this case that $f_i(X) \ge 4$ for $2 \le i \le d$. Therefore

$$4(d-1) \le \sum_{i=1}^{d} \sum_{x \in X} f_i(x) = \sum_{x \in X} \sum_{i=1}^{d} f_i(x) \le \sum_{x \in X} 3 = 3|X| = 3(n - n_p).$$

Since $p \ge 3$ and $n_i \ge 2$, this leads to $d_{rdR}(G) = d \le n - n_p$.
Assume next that $f_i(X) \ge 1$ for $1 \le i \le p$ and, without loss of generality, that $f_1(X) = 1$. Then $f_1(y) \ge 2$ for $y \in S_p$, and as in the last case, we obtain $d_{rdR}(G) \le n - n_p$.
Now assume that $f_i(X) \ge 2$ for $1 \le i \le p$. We observe that $f_i(X) = 2$ is possible for at most two indices. It follows that

$$3d - 2 \le \sum_{i=1}^{d} \sum_{x \in X} f_i(x) = \sum_{x \in X} \sum_{i=1}^{d} f_i(x) \le \sum_{x \in X} 3 = 3|X| = 3(n - n_p)$$

and so again $d_{rdR}(G) = d \le n - n_p$. As $d_{rdR}(G) \ge n - n_p$, we conclude that $d_{rdR}(G) = n - n_p$ in this case.
(iii) Finally, let $n_t = 1$ and $n_{t+1} \ge 2$ for $1 \le t \le p-2$. Let $S_i = \{s_i\}$ for $1 \le i \le t$. Clearly, $f_i(s_i) = 3$ and $f_i(x) = 0$ for $1 \le i \le t$ are restrained double Roman dominating functions on G. Applying Theorem 5 when $p - t = 2$ and Part (ii) when $p - t \ge 3$ to the complete $(p-t)$-partite graph $G[S_{t+1} \cup S_{t+2} \cup \ldots \cup S_p]$, we obtain the desired result. □

If $n_1 \geq 2$ and $\min\left\{n - n_p, \left\lfloor \frac{n}{2} \right\rfloor\right\} = \left\lfloor \frac{n}{2} \right\rfloor$ in Theorem 6, then $d_{rdR}(G) = \left\lfloor \frac{n}{2} \right\rfloor$. Thus Corollary 5 is sharp.

Conflict of interest. The authors declare that they have no conflict of interest.

Data Availability. Data sharing is not applicable to this article as no datasets were generated or analyzed during the current study.

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