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Erratum

Erratum to the paper "A study on graph topology"

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Abstract: In this paper, we will point out errors in Theorem 2, Theorem 4, Theorem 5, Proposition 2, Proposition 3, Theorem 8, and Theorem 9 by giving suitable counterexamples. The statements of Theorem 2, Theorem 5, Proposition 2 and Proposition 3 of this paper have been reformulated and proofs are given.

Keywords: graph topology, graph topological space, n-closed, d-closed.

AMS Subject classification: 05C62, 05C75, 54A05

1. A correction of Theorem 2 of [1]

The following example shows that the statement of Theorem 2 of [1] is not correct. Consider graph G' with $V(G') = \{a, b, c\}$ and $E(G') = \{e_1, e_2\}$ where $e_1 = \{a, b\}$ and $e_2 = \{b, c\}$. Let G_1, G_2 be subgraphs of G' as shown in the Figure 1 and let

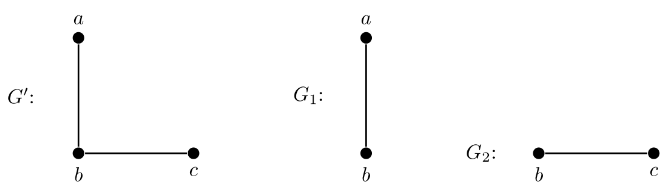

Figure 1.

$\beta_1 = \{G_1, G_2\}$. We see that both condition (i) and (ii) in Theorem 2 are satisfied.

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However, β_1 is not a base for any topology on G'. For if β_1 is a base for some graph topology τ then clearly both G_1 and G_2 belong to τ. So, $G_1 \bigcap G_2 \in \tau$, but $G_1 \bigcap G_2$ is a subgraph containing only the single vertex b and has no edges, so it cannot be written as a union of members of β_1. Hence β_1 is not a base any graph topology on G'.

We restate **Theorem 2** of [1] as follows:

Theorem 1. *Let (G,τ) be a graph topological space and let $\beta \subseteq \tau$. Then, β is a base for the topological space τ if and only if*

i) for each $v \in V(G)$, and for each $H \in \tau$ such that $v \in V(H)$, $\exists$ $G_i \in \beta$ such that $v \in V(G_i) \subseteq V(H)$.

ii) for each $e \in E(G)$, and for each $H \in \tau$ such that $e \in E(H)$, $\exists$ $G_i \in \beta$ such that $e \in E(G_i) \subseteq E(H)$.

Proof. Suppose β is a base of τ.
(i) Let $v \in V(H)$ for some $H \in \tau$. Since β is a base for τ, therefore $H = \bigcup_{i\in I} G_i$, where $G_i \in \beta$ for each $i \in I$. So $v \in V(G_i)$ for some $i \in I$. Thus, $v \in V(G_i) \subseteq V(H)$ for some $G_i \in \beta$ holds.
(ii) Let $e \in E(H)$ for some $H \in \tau$. Again since β is a base for τ we have $H = \bigcup_{i\in I} G_i$ for some index set I. So $e \in E(G_i)$ for some $i \in I$. Thus, $e \in E(G_i) \subseteq E(H)$ for some $G_i \in \beta$ holds.
Conversely, let $H \in \tau$ and $v \in V(H)$. Then by hypothesis, we have $v \in V(G_i) \subseteq V(H)$ for some $G_i \in \beta$. So, we have $V(H) = \bigcup_{i\in I} V(G_i)$ for some indexing set I.
Similarly, for each $e \in E(H)$, we get, $e \in E(G_i) \subseteq E(H)$ for some $G_i \in \beta$. So, we have $E(H) = \bigcup_{i\in J} E(G_i)$ for some indexing set J. Hence, we have $H = \bigcup_{i\in I\cup J} G_i$ which shows that H can be expressed as the union of members of β. Since H is arbitrary, we conclude that β is a base for τ. □

2. An error in Theorem 4 of [1]

In this section, we point out an error in Theorem 4 of [1]. First we give two examples to show that the statement is not correct. Consider the graph G in Figure 2.
Let us consider the subgraphs G_1, G_2, G_3 of G as shown in the Figure 2 and let $\mathcal{K} = \{G_1, G_2, G_3\}$. Then, clearly, we can see that $E(G) = \bigcup_{G_i\in\mathcal{K}} E(G_i)$, $i \in \{1, 2, 3\}$. So, $\mathcal{K}$ is subgraph cover. But $\mathcal{K}$ is not a base for the graph topology τ since no member of $\mathcal{K}$ contains the vertex d.
Next, we consider the connected graph $G = G'$ in the previous section. Here $\beta_1 = \{G_1, G_2\}$ form subgraph cover of G' but we have seen that β_1 is not a base for any graph topology on G'.

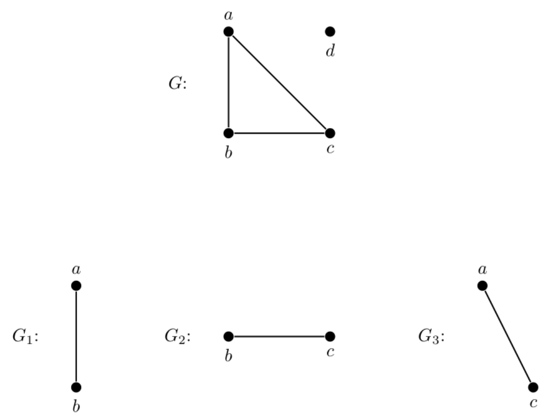

Figure 2.

3. Correction of Theorem 5 of [1]

In this section, we point out an error in the statement of Theorem 5 of [1] by giving some examples and we will restate Theorem 5. Consider the following graph G:

Figure 3.

Consider the subgraph G_1 of G as shown in the figure above. Clearly, $\tau = \{K_0, G, G_1\}$ is graph topology. By definition of τ-neighbourhood of a vertex [1], we see that G_1 is a τ-neighbourhood of the vertex b. But b is not $\tau-$isolated vertex of G. Similarly, if we consider the edge $e = \{a, b\}$ in G, then G_1 is τ-neighbourhood of edge $e = \{a, b\}$, but e is not $\tau-$isolated edge of G.
We now restate **Theorem 5** of [1] as follows:

Theorem 2. *Let v be a vertex and e be an edge in a graph G . Then*

i) Every subgraph H of G containing v is a τ-neighbourhood of v if and only if v is a τ-isolated vertex of G.

ii) Every subgraph H of G containing e is a τ-neighbourhood of e if and only if e is a τ-isolated edge of G.

Proof. (i) Suppose every subgraph H of G containing vertex v is a τ-neighbourhood of v. Therefore, $G[v]$ is a τ-neighbourhood of v which means that v is a τ-isolated vertex of G. Conversely, let v be a τ-isolated vertex of G and a subgraph H of G contains v. Then $v \in G[v] \subset H$ which shows that H is a τ-neighbourhood of v.
The proof of (ii) is similar. □

4. Error in Proposition 2 and Proposition 3 of [1]

In this section, we point out the errors in Proposition 2 and Proposition 3 of [1] by giving some examples:

In Proposition 2, the graph G is d-closed as stated, however for K_0 to be d-closed, the graph under consideration must be without any isolated vertex. This can be illustrated by taking $G = K_2 \cup K_1$ where $V(K_2) = \{a, b\}$ and $V(K_1) = \{c\}$.
If we take $\tau = \{K_0, G, G[b]\}$ then clearly τ is graph topology. But $K_0^* =< E(G) >$ is a subgraph induced by E(G) which has only one edge and only two vertex a and b. Clearly, $K_0^* \notin \tau$ that is K_0^* is not open which show that K_0 is not d-closed.

Similarly in **Proposition 3 of [1]** , the graph under consideration must be a graph without any isolated vertex. For a graph having isolated vertex, the statement of the proposition need not be true. This can be seen by taking G to be the graph shown in Figure 4. Consider the subgraph N_3 of G as illustrated.

Figure 4.

If we take $\tau = \{K_0, G, G[b]\}$ then clearly τ is graph topology. Also, we see that N_3 has no edges, so N_3 is an empty graph that is also not open in τ. But, $N_3^* =< E(G) >$ is the path abc and clearly it is not open. So, N_3 is not d-closed.

Proposition 2 and Proposition 3 of [1] may be reformulated as follows. The proofs of both the two statements are straightforward.

Correction of Proposition 2 of [1] For graph G without isolated vertex, the null graph K_0 and the graph G in a graph topological space is d-closed.

Correction of Proposition 3 of [1] Let G be a graph without any isolated vertex. If N is an empty subgraph of G which is not open in τ then N is d-closed.

5. Error in Theorem 8 of [1]

In this section, we show that Theorem 8 of [1] is not true by giving some examples. As demonstrated in the previous section, K_0 is not d-closed in general. We will illustrate that union of d-closed subgraphs need not be d-closed. Consider the connected graph G illustrated in Figure 5. If G_1, G_2, G_3 are the subgraphs as in the figure below and if we take $\tau = \{K_0, G, G_1, G_2, G_3\}$, then clearly τ is a graph topology on G.

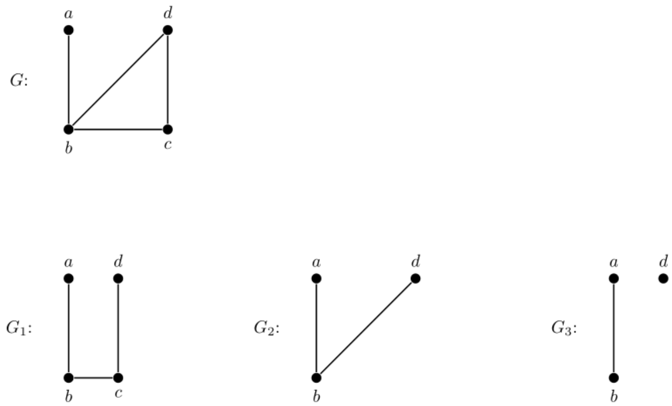

Figure 5.

Here we consider the subgraphs H_1, H_2 of G as indicated in Figure 6. Clearly, we see that $E(H_1^*) = \{\{a,b\},\{b,c\},\{c,d\}\}$ and $E(H_2^*) = \{\{a,b\},\{b,d\}\}$. We observe that

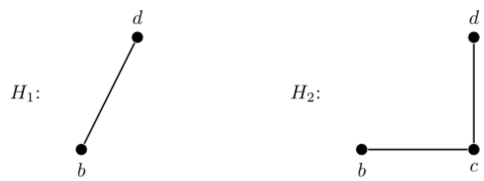

Figure 6.

$H_1^* = G_1$ and $H_2^* = G_2$ which are open in τ. So by definition of d-closed we have both H_1 and H_2 are d-closed. Now the subgraph $H_1 \bigcup H_2$ is a 3-cycle $bcdb$.
So, $E((H_1 \bigcup H_2)^*) = \{\{a,b\}\}$ and so $(H_1 \bigcup H_2)^* = < E(G) - E(H_1 \bigcup H_2) >$ is $K_2 = ab$ and it is not open in τ. This shows that $H_1 \bigcup H_2$ is not d-closed.

6. Error in Theorem 9 of [1]

In this section, we point out an error in Theorem 9 of [1] by giving some examples to show that the statement is not true. We will consider the following example to show

that condition (iii) of Theorem 9 of [1] is false.
Let G be the graph shown in Figure 7 and H_1 a subgraph of G. If we take $\tau = \{K_0, G, H_1, G[e], G[d]\}$ then we see that τ is a graph topology. Now consider two subgraph H_2, H_3 of G illustrated in Figure 8.

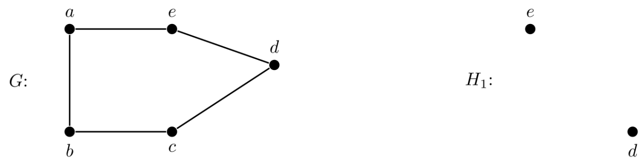

Figure 7.

Figure 8.

Here we have $V(H_2) = \{a, b\}$, $V(H_3) = \{b, c\}$, $N(V(H_2)) = \{e, c\}$ and $N(V(H_3)) = \{a, d\}$. So, $(V(H_2))^{\varsigma} = (V(H_2) \bigcup N(V(H_2)))^{\varsigma} = \{d\}$ and $(V(H_3))^{\varsigma} = (V(H_3) \bigcup N(V(H_3)))^{\varsigma} = \{e\}$. The subgraphs induced by $\{d\}$ and $\{e\}$ are $G[d]$ and $G[e]$ which are open in τ. So, by definition of n-closed we have both H_2 and H_3 are n-closed. However, $H_2 \bigcap H_3 = \{b\}$ and $N(V(H_2 \bigcap H_3)) = \{a, c\}$. So, $(V(H_2 \bigcap H_3))^{\varsigma} = (V(H_2 \bigcap H_3) \bigcup N(V(H_2 \bigcap H_3)))^{\varsigma} = \{d, e\}$. But the subgraph induced by $(V(H_2 \bigcap H_3))^{\varsigma}$ is a subgraph containing two vertex d and e and one edge joining d and e, which is not open. Thus, $H_2 \bigcap H_3$ is not n-closed.

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Conflict of interest. The authors declare that they have no conflict of interest.

Data Availability. Data sharing is not applicable to this article as no datasets were generated or analyzed during the current study.

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[1] A. Aniyan and S. Naduvath, *A study on graph topology*, Commun. Comb. Optim. **8** (2023), no. 2, 397– 409.
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