Communications in Combinatorics and Optimization
Vol. xx, No. x (xxxx), pp. 1-15
https://doi.org/10.22049/cco.2024.28396.1528

CCO
Commun. Comb. Optim.

Research Article

On co-maximal subgroup graph of D_n

Angsuman Das[1], and Manideepa Saha[2,*]

[1]Department of Mathematics, Presidency University, Kolkata, India
angsuman.maths@presiuniv.ac.in

[2]School of Mathematical Sciences, NISER, Bhubaneshwar, India
*manideepasaha1991@gmail.com

Received: 16 March 2023; Accepted: 14 January 2024
Published Online: 31 January 2024

Abstract: Let G be a group and S be the collection of all non-trivial proper subgroups of G. The co-maximal subgroup graph $\Gamma(G)$ of a group G is defined to be a graph with S as the set of vertices and two distinct vertices H and K are adjacent if and only if $HK = G$. In this paper, we study the comaximal subgroup graph on finite dihedral groups. In particular, we study order, maximum and minimum degree, diameter, girth, domination number, chromatic number and perfectness of comaximal subgroup graph of dihedral groups. Moreover, we prove some isomorphism results on comaximal subgroup graph of dihedral groups.

Keywords: dihedral group, graph isomorphism, perfect graph

AMS Subject classification: 05C25, 05C17, 05C45.

1. Introduction

Since the inception of Cayley graphs, many graphs have been defined on groups to study the interplay between groups and graphs. For an extensive survey on different graphs defined on groups, one can refer to [2]. One such graph, namely, the *co-maximal subgroup graph* of a group G was introduced by Akbari *et al.* [1] in 2017. More results on comaximal subgroup graph can be found in [4–7]. In this work, we study the comaximal subgroup graph on finite dihedral groups. We first recall some definitions and results on graph theory and elementary number theory.

* *Corresponding Author*

1.1. Definitions and Preliminaries

Let Γ be a graph. The *diameter* of a connected graph Γ is the maximum distance between any two vertices in Γ. The maximum and minimum degree of a vertex in Γ is denoted by $\Delta(\Gamma)$ and $\delta(\Gamma)$ respectively and $girth(\Gamma)$ denotes the length of a smallest cycle in Γ. A subset S of vertices of a graph Γ is called a dominating set if any vertex of Γ is either in S or adjacent to some vertex in S. The cardinality of a minimum dominating set is called the *domination number* γ of Γ. The *chromatic number* χ of Γ is the minimum number of colours used to colour the vertices of Γ such that no two adjacent vertices get the same colour. The *clique number* ω of Γ is the maximum order of a complete subgraph of Γ. It is known that $\chi \geq \omega$ for any graph. A graph Γ is called *weakly perfect* if $\chi(\Gamma) = \omega(\Gamma)$. A graph Γ is called *perfect* if $\chi(H) = \omega(H)$ for all induced subgraph H of Γ. Two vertices u and v are said to be *twins* if $N(u) = N(v)$, i.e., they have the same set of neighbours.
Let n be a positive integer. $\tau(n)$ denote the number of positive divisors of n, $\sigma(n)$ denote the sum of positive divisors of n and $\pi(n)$ denote the number of distinct prime factors of n. A prime factor p of n is said to be of *even exponent* if the highest power of p dividing n is even.
We first recall a few definitions and results from [1] and [5].

Definition 1. Let G be a group and S be the collection of all non-trivial proper subgroups of G. The co-maximal subgroup graph $\Gamma(G)$ of a group G is defined to be a graph with S as the set of vertices and two distinct vertices H and K are adjacent if and only if $HK = G$. The *deleted co-maximal subgroup graph* of G, denoted by $\Gamma^*(G)$, is defined as the graph obtained by removing the isolated vertices from $\Gamma(G)$.

In [5], authors proved various results on $\Gamma(G)$ and $\Gamma^*(G)$. We recall a result, which will be used in this paper.

Theorem 1 ([5], Corollary 2.3). *Let G be a finite solvable group. Then $\Gamma^*(G)$ is connected and $diam(\Gamma^*(G)) \leq 4$.*

1.2. Our Contribution

In this paper, we compute the maximum and minimum degree, girth, diameter, domination number and chromatic number of $\Gamma(D_n)$. Moreover $\Gamma(D_n)$ was shown to be weakly perfect, and perfect $\Gamma(D_n)$'s are characterized. Finally we prove some isomorphism results on $\Gamma(D_n)$.

2. Structural Properties of $\Gamma(D_n)$ and $\Gamma^*(D_n)$

In this section, we study various structural properties of $\Gamma(D_n)$ and $\Gamma^*(D_n)$ like order, maximum and minimum degree, girth, diameter and when they are Eulerian. We start

by describing the complete list of subgroups of D_n, which constitute the vertex set of the graph to be studied.
The dihedral group D_n has two generators r and s with orders n and 2 such that $srs^{-1} = r^{-1}$. $D_n = \langle r, s : r^n = s^2 = 1, srs = r^{n-1}\rangle$ consists of $2n$ elements. We recall a result on the complete list of subgroups of D_n. For a proof of this listing, please refer to [3].

Proposition 1. *Every subgroup of D_n is either cyclic or dihedral. A complete listing of the subgroups is as follows:*

1. $\langle r^d\rangle$, *where* $d|n$, *with index* $2d$,
2. $\langle r^d, r^i s\rangle$, *where* $d|n$ *and* $0 \leq i \leq d-1$, *with index* d.

Moreover, every subgroup of D_n occurs exactly once in this listing.

Proposition 2. $\Gamma(D_n)$ *has* $\sigma(n) + \tau(n) - 2$ *vertices.*

Proof. $\Gamma(D_n)$ contains all subgroups of the form $\langle r^d\rangle$, where $d|n$ and $d \neq n$. We call this vertices of Type-I, and so number of Type-I vertices is $\tau(n) - 1$. Similarly, $\Gamma(D_n)$ contains all subgroups of the form $\langle r^d, r^i s\rangle$, where $d|n$ and $0 \leq i \leq d-1$ except $d = 1$. We call this vertices of Type-II, and so number of Type-II vertices is $\sigma(n) - 1$. □

Now, we investigate the adjacency between vertices of $\Gamma(D_n)$. It is clear that no two vertices of Type-I are adjacent. Thus, any edge of $\Gamma(D_n)$ occurs either between two vertices of Type-II or one of Type-I and one of Type-II. The edges in $\Gamma(D_n)$ are completely classified in the next theorem.

Theorem 2. *The following are the edges of* $\Gamma(D_n)$*:*

- *A vertex $\langle r^{d_1}\rangle$ of Type-I is adjacent to a vertex $\langle r^{d_2}, r^i s\rangle$ of Type-II if and only if $gcd(d_1, d_2) = 1$.*
- *Two vertices $\langle r^{d_1}, r^i s\rangle$ and $\langle r^{d_2}, r^j s\rangle$ of Type-II are adjacent if and only if one of the two conditions hold:*
  1. $gcd(d_1, d_2) = 1$.
  2. $gcd(d_1, d_2) = 2$ *and* $i - j$ *is odd.*

Proof.
- Let $H = \langle r^{d_1}\rangle$ and $K = \langle r^{d_2}, r^i s\rangle$. We start by noting that $HK = D_n$ if and only if $r \in HK$. If $gcd(d_1, d_2) = 1$, then there exist integers u, v such that $ud_1 + vd_2 = 1$. Thus, $r = (r^{d_1})^u \cdot (r^{d_2})^v \in HK$. Conversely, as $r \notin H, K$, but $r \in HK$, we must get r as product of powers of r^{d_1} and r^{d_2}, i.e., $gcd(d_1, d_2) = 1$.
- Let $H = \langle r^{d_1}, r^i s\rangle$, $K = \langle r^{d_2}, r^j s\rangle$ and $H \sim K$. Then $HK = D_n$. If $d = gcd(d_1, d_2)$, then there exist integers x, y such that $d_1 x + d_2 y = d$, i.e., $r^d = (r^{d_1})^x (r^{d_2})^y \in HK = D_n$. Thus $\langle r^d\rangle \subseteq HK$. Note that r^d is the smallest power

of r that can expressed as product of powers of r^{d_1} and r^{d_2}. If $d \geq 3$, then r and r^2 must be expressible as products of powers of $r^{d_1}, r^i s, r^{d_2}$ and $r^j s$, i.e., there exist integers x_1, x_2, y_1, y_2 such that

$$d_1x_1 + d_2x_2 + (i-j) \equiv 1 \ (mod\ n) \ \text{ and } d_1y_1 + d_2y_2 + (i-j) \equiv 2 \ (mod\ n).$$

Subtracting, we get $d_1u + d_2v \equiv 1 \ (mod\ n)$, i.e., d divides $d_1u + d_2v - 1$, i.e., $d|1$, a contradiction. Thus $d = 1$ or 2. If $d = 1$, we are done. Suppose $d = 2$ and $i-j$ is even. Note that $d = 2$ implies n is even. Now, as $r \in HK$, there exist integers x and y such that $d_1x + d_2y + (i-j) \equiv 1 \ (mod\ n)$. But, $d_1x + d_2y + (i-j)$ is even and it can not be congruent to 1 modulo an even number n. Thus $i-j$ must be odd.

Conversely, let one of the conditions hold. If $d = 1$, then any integer can be expressed as integer linear combination of d_1 and d_2. Thus for any integer l, we have $r^l, r^l s \in HK$, i.e., $HK = D_n$. If $d = 2$ and $i-j$ is odd, then n is even. As $d = 2$, r^2 and all even powers of r can expressed as product of powers of r^{d_1} and r^{d_2} and they belong to HK. For odd powers of r to be in HK, we must have integers x, y such that

$$r^{d_1x+d_2y+(i-j)} = r^{2t+1}, \text{ i.e., } d_1x + d_2y + (i-j) \equiv 2t+1 \ (mod\ n)$$

$$2u = 2t + 1 + j - i \ (mod\ n)$$

Note that as $gcd(d_1, d_2) = d = 2$, for any integer u, we can find x and y such that $d_1x + d_2y = 2u$. Also, $2t + 1 + j - i$ is even. Thus, we have

$$u = \frac{2t+1+j-i}{2} \ (mod\ n)$$

Hence for all values of t, u has a solution and all odd powers of r lies in HK, i.e., $\langle r \rangle \subseteq HK$.

Again, note that $r^{d_1x+d_2y+i}s, r^{d_1x+d_2y+j}s \in HK$ for all values of x, y, i.e., $r^{2l+i}s, r^{2l+j}s \in HK$ for all value of l. As $i-j$ is odd, i and j has different parity, and hence by varying l suitably, all the elements of the form $r^k s \in HK$. Thus $HK = D_n$, i.e., $H \sim K$.

□

In the next few theorems, we find the maximum and minimum degree of $\Gamma(D_n)$, and its number of isolated and pendant vertices.

Theorem 3. *The maximum degree of $\Gamma(D_n)$ is $\sigma(n) - 1$ and is attained by $\langle r \rangle$.*

Proof. Among Type-I vertices, $\langle r\rangle$ has the maximum degree and its degree is

$$\left(\sum_{d|n, d\neq 1} d\right) - 1 = \sigma(n) - 1.$$

We claim that the degree of any Type-II vertex is less than $\sigma(n)-1$.
Case 1: (n is odd, say $n = p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_k^{\alpha_k}$, where p_i's are odd primes). Let $H = \langle r^d, r^i s\rangle$ be a Type-II vertex with $d|n, d\neq 1$. Without loss of generality, let p_1 be a prime divisor of d. Set $K = \langle r^d, s\rangle$ and $L = \langle r^{p_1}, s\rangle$. Clearly $K \subseteq L$. As n is odd, d is also odd. Thus we have

set of neighbours of H = set of neighbours of $K \subseteq$ set of neighbours of L.

Thus $deg(H) = deg(K) \leq deg(L)$. Consider the following two set of vertices

$$A = \{\langle r^{d_1}, s\rangle : p_1|d_1, d_1|n\} \text{ and } B = \{\langle r^{d_1}\rangle : p_1 \nmid d_1\}.$$

It is easy to check that all vertices in A are non-adjacent with L and B is the exactly the set of vertices of Type-I which are adjacent to L. Note that $|A| = \alpha_1(\alpha_2+1)\cdots(\alpha_k+1)$ and $|B| = (\alpha_2+1)\cdots(\alpha_k+1)$. As there are total $(\sigma(n)-1)$ many Type-II vertices and $|A|\geq|B|$, we have

$$deg(H) \leq deg(L) \leq (\sigma(n)-2) - |A| + |B| \leq \sigma(n) - 2 < \sigma(n) - 1.$$

Case 2: (n is even, say $n = p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_k^{\alpha_k}$, where $p_1 = 2$ and other p_i's are odd primes). Let $H = \langle r^d, r^i s\rangle$ be a Type-II vertex with $d|n, d\neq 1$ and p_j be a prime divisor of n. According as i is even or odd, set $K = \langle r^d, s\rangle$ or $\langle r^d, rs\rangle$ respectively, and $L = \langle r^{p_j}, s\rangle$ or $\langle r^{p_j}, rs\rangle$, respectively. As in Case 1, we have $deg(H) = deg(K) \leq deg(L)$. Again, as in Case 1, construct the sets A and B. The rest follows similarly and $deg(H) < \sigma(n) - 1$.
Thus the theorem follows. □

Theorem 4. *Let $n = p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_k^{\alpha_k}$. The number of isolated vertices in $\Gamma(D_n)$ is $\alpha_1\alpha_2\ldots\alpha_k - 1$. Moreover, $\Gamma(D_n)$ is connected if and only if n is square-free.*

Proof. Note that Type-II vertices are never isolated as they are always adjacent to $\langle r\rangle$. A Type-I vertex $\langle r^d\rangle$ is isolated if and only if $p|d$, for all primes $p|n$, i.e., if $n = p_1^{\alpha_1}p_2^{\alpha_2}\ldots p_k^{\alpha_k}$, then the number of isolated vertices are $\alpha_1\alpha_2\ldots\alpha_k - 1$.
As D_n is solvable, it is connected if and only if it has no isolated vertex if and only if $\alpha_1\alpha_2\ldots\alpha_k - 1 = 0$ if and only if n is square-free. □

Theorem 5. *The minimum degree of $\Gamma^*(D_n)$ is given by*

$$\delta(\Gamma^*(D_n)) = \begin{cases} 1, & \text{if } n \text{ is odd,} \\ 2, & \text{if } n \text{ is even.} \end{cases}$$

Proof. If n is odd, then $\langle s\rangle$ is adjacent only to $\langle r\rangle$, and hence $\delta = 1$. If n is even, then $\langle s\rangle$ is adjacent only to $\langle r\rangle$ and $\langle r^2, rs\rangle$. Thus degree of $\langle s\rangle$ is 2. We need to show that no vertex have degree 1. Note that every Type-II vertex is adjacent to $\langle r\rangle$ and exactly one of $\langle r^2, s\rangle$ and $\langle r^2, rs\rangle$, i.e., degree of a Type-II vertex is ≥ 2. Let $\langle r^d\rangle$ be a non-isolated Type-I vertex. Then d misses atleast one prime factor of n, say p. Then $\langle r^d\rangle$ is adjacent to $\langle r^p, s\rangle$ and $\langle r^p, rs\rangle$, i.e., its degree is ≥ 2. □

Corollary 1. *Let $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$ be odd. The number of pendant vertices in $\Gamma(D_n)$ is*

$$p_1 p_2 \cdots p_k \prod_{i=1}^{k} \frac{({p_i}^{\alpha_i} - 1)}{(p_i - 1)}$$

Proof. If n is even, by Theorem 5, the minimum degree is 2 and hence $\Gamma(D_n)$ has no pendant vertex. So, we assume that n is odd.
We start by observing that Type-I vertices of the form $\langle r^d\rangle$ are never pendant, as if $\langle r^d\rangle \sim \langle r^x, r^i s\rangle$, then $\langle r^d\rangle \sim \langle r^x, r^j s\rangle$ for $j \neq i$. Thus Type-II vertices are the only possible choices for pendant vertices.
Let $\langle r^d, r^i s\rangle$ be a pendant vertex. If $p_i \nmid d$ for some i, then $\langle r^d, r^i s\rangle$ is adjacent to at least two vertices, namely $\langle r\rangle$ and $\langle r^{p_i}\rangle$. Thus $p_i | d$ for all i.
Finally, if $p_i | d$ for all i, then it is easy to observe that $\langle r^d, r^i s\rangle$ is adjacent only to $\langle r\rangle$. Now, the corollary follows by counting the number of such vertices. □

Proposition 3. *The girth of $\Gamma(D_n)$ is 3 for $n \geq 3$ and n is not an odd prime power.*

Proof. If n is even, then $\langle r\rangle, \langle r^2, s\rangle$ and $\langle r^2, rs\rangle$ forms a triangle. If n is odd, but not a prime power, then there exist two distinct prime factors, say p, q of n. Then $\langle r\rangle, \langle r^p, s\rangle$ and $\langle r^q, s\rangle$ forms a triangle. □

Proposition 4. *$\Gamma^*(D_n)$ is a star if and only if n is an odd prime power.*

Proof. Let $n = p^k$ where p is an odd prime. Then all Type-I vertices except $\langle r\rangle$ are isolated in $\Gamma(D_n)$ and $\langle r\rangle$ is an universal vertex in $\Gamma^*(D_n)$. Now, as any Type-II vertex is of the form $\langle r^{p^l}, r^i s\rangle$, no two of them are adjacent and hence $\Gamma^*(D_n)$ is a star.
Conversely, if $\Gamma^*(D_n)$ is a star and n is not an odd prime power, by above Proposition, $\Gamma(D_n)$ has a triangle, a contradiction. □

As D_n is a finite solvable group, by Theorem 1, $\Gamma^*(D_n)$ is connected and its diameter is less than or equal to 4. In the next theorem, we compute the diameter of $\Gamma^*(D_n)$ and show that it is either 2 or 3.

Theorem 6.
$$Diam(\Gamma^*(D_n)) = \begin{cases} 2, & n = p^k \\ 3, & else \end{cases}$$

Proof. If n is an odd prime power, by Proposition 4, $\Gamma^*(D_n)$ is a star and hence $Diam\ (\Gamma^*(D_n)) = 2$. If $n = 2^k$, then by Theorem 3.6 [5], $\Gamma^*(D_n)$ has an universal vertex and hence $Diam(\Gamma^*(D_n)) = 2$.
If n is not a prime power, then n has at least two distinct prime factors. Let $n = p^\alpha q^\beta m$, where m is coprime to p and q. Then consider the vertices $A = \langle r^{p^a} \rangle$ and $B = \langle r^{n/p^a} \rangle$. Clearly they are non-adjacent. As both are Type-I vertices, if they have a common neighbour, it must be a Type-II vertex, say $\langle r^d, r^i s\rangle$. But that means $d|n, d \neq 1$ and d is coprime to both p^a and n/p^a, a contradiction. Thus A and B have no common neighbour, i.e., $d(A, B) > 2$. Consider the path $\langle r^{p^a} \rangle \sim \langle r^q, s\rangle \sim \langle r^p, s\rangle \sim \langle r^{n/p^a} \rangle$ and hence $d(A, B) = 3$.
We claim that any two vertices are atmost at distance 3 from the other. If both the vertices are of Type-II, then they always have a common neighbour $\langle r \rangle$ and hence their distance is atmost 2. If both are of Type-I and are not isolated, say $\langle r^{d_1} \rangle$ and $\langle r^{d_2} \rangle$, then both d_1 and d_2 miss at least one prime factor of n, say p and q. If $p \neq q$, then $\langle r^{d_1} \rangle \sim \langle r^p, s\rangle \sim \langle r^q, s\rangle \sim \langle r^{d_2} \rangle$, i.e., their distance is atmost 3. If $p = q$, then $\langle r^{d_1} \rangle \sim \langle r^p, s\rangle \sim \langle r^{d_2} \rangle$, i.e., their distance is at most 2. Thus we are left with the case where one of the vertex is of Type-I and other is of Type-II, say $\langle r^{d_1} \rangle$ and $\langle r^{d_2}, r^i s\rangle$. As $\langle r^{d_1} \rangle$ is not isolated, d_1 misses at least one prime factor of n, say p. Thus $\langle r^{d_1} \rangle \sim \langle r^p, s\rangle \sim \langle r \rangle \sim \langle r^{d_2}, r^i s\rangle$, i.e., their distance is at most 3. Hence the theorem follows. □

In the next theorem, we check when $\Gamma^*(D_n)$ is Eulerian.

Theorem 7. *$\Gamma^*(D_n)$ is Eulerian if and only if n is even and all odd prime factors of n are of even exponent.*

Proof. Let $\Gamma^*(D_n)$ be Eulerian. If n is odd, by Theorem 5, minimum degree is 1, i.e., odd, a contradiction. So n must be even. Let n has an odd prime factor p of odd exponent α, i.e., $n = p^\alpha m$, where m is even and $p \nmid m$. Consider the vertex $\langle r^m \rangle$. Observe that its only neighbours are of the form $\langle r^{p^*}, r^i s\rangle$. Thus degree of $\langle r^m \rangle$ is $p + p^2 + \cdots + p^\alpha$, i.e., odd, a contradiction. Hence all odd prime factors of n are of even exponent.
Conversely, let n be even and all odd prime factors of n are of even exponent. Let $n = 2^\alpha {p_1}^{\alpha_1} {p_2}^{\alpha_2} \cdots {p_k}^{\alpha_k}$, where α_i's are even. We will show that all non-isolated vertices have even degree.

Let us first consider the Type-I vertices of the form $\langle r^d\rangle$. If d is divisible by all the prime factors of n, then $\langle r^d\rangle$ is an isolated vertex. So, we assume that d is not divisible by some prime factors of n. Suppose $p_{i_1}, p_{i_2}, \ldots, p_{i_t}$ are the prime factors of n not dividing d. Then the neighbours of $\langle r^d\rangle$ are of the form $\langle r^{{p_{i_1}}^{\beta_1}{p_{i_2}}^{\beta_2}\cdots{p_{i_t}}^{\beta_t}}, r^js\rangle$, where not all β_i's are zero simultaneously. Thus degree of $\langle r^d\rangle$ is $\sigma({p_{i_1}}^{\alpha_{i_1}}{p_{i_2}}^{\alpha_{i_2}}\cdots{p_{i_t}}^{\alpha_{i_t}})-1$, which is even, as each α_i is even. Thus Type-I vertices are of even degree.
Now, we consider the Type-II vertices of the form $\langle r^d, r^is\rangle$. If d is divisible by all the prime factors of n, then $\langle r^d, r^is\rangle$ has precisely two neighbours, $\langle r\rangle$ and exactly one of $\langle r^2, s\rangle$ and $\langle r^2, rs\rangle$. So, we assume that d is not divisible by some prime factors of n. Suppose $p_{i_1}, p_{i_2}, \ldots, p_{i_t}$ are the prime factors of n not dividing d.
Case 1: ($2 \nmid d$) In this case, the neighbours of $\langle r^d, r^is\rangle$ are of the form $\langle r^{{p_{i_1}}^{\beta_1}{p_{i_2}}^{\beta_2}\cdots{p_{i_t}}^{\beta_t}}\rangle$ and $\langle r^{{p_{i_1}}^{\beta_1}{p_{i_2}}^{\beta_2}\cdots{p_{i_t}}^{\beta_t}}, r^js\rangle$ where not all β_i's are zero. Thus the degree of $\langle r^d, r^is\rangle$ is

$$\tau({p_{i_1}}^{\alpha_{i_1}}{p_{i_2}}^{\alpha_{i_2}}\cdots{p_{i_t}}^{\alpha_{i_t}})+\sigma({p_{i_1}}^{\alpha_{i_1}}{p_{i_2}}^{\alpha_{i_2}}\cdots{p_{i_t}}^{\alpha_{i_t}})-2,$$

which is even, as explained earlier.
Case 2: ($2|d$) In this case, apart from the neighbours mentioned in Case 1, $\langle r^d, r^is\rangle$ has neighbours of the form $\langle r^{2^{\beta}{p_{i_1}}^{\beta_1}{p_{i_2}}^{\beta_2}\cdots{p_{i_t}}^{\beta_t}}, r^js\rangle$, where $i-j$ is odd. However, proceeding similarly as above, it can be shown that the number of such neighbours is also even. As a result the degree of Type-II vertices are also even. This proves the theorem. □

3. Domination number, Chromatic Number and Perfectness of $\Gamma(D_n)$

In this section, we study the domination number, chromatic number of $\Gamma(D_n)$ and characterize when $\Gamma(D_n)$ is perfect.

Theorem 8. *The domination number of $\Gamma^*(D_n)$ is given by*

$$\gamma(\Gamma^*(D_n)) = \begin{cases} 1, & \text{if } n \text{ is a prime power,} \\ \pi(n)+1, & \text{otherwise.} \end{cases}$$

Proof. If n is a prime power, by Proposition 4, $\Gamma^*(D_n)$ is a star and hence the theorem follows. Let $n = p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_k^{\alpha_k}$. Clearly $\{\langle r\rangle, \langle r^{p_1}, s\rangle, \langle r^{p_2}, s\rangle, \ldots, \langle r^{p_k}, s\rangle\}$ is a dominating set of $\Gamma^*(D_n)$ and hence $\gamma(\Gamma^*(D_n)) \le k+1$.
If possible, let $S = \{x_1, x_2, \ldots, x_k\}$ be a dominating set of size k. Set $m = p_1p_2\cdots p_k$ and consider the set of $k+1$ vertices $A = \{\langle r^{m/p_1}\rangle, \langle r^{m/p_2}\rangle, \ldots, \langle r^{m/p_k}\rangle, \langle r^m, s\rangle\}$. Among these $k+1$ vertices, at least one of them is not in S. Without loss of generality, let $\langle r^{m/p_1}\rangle \notin S$ and $\langle r^{m/p_1}\rangle \sim x_1$. Then x_1 is of the form $\langle r^{p_1^{\beta_1}}, r^{i_1}s\rangle$. Note that x_1 is not adjacent to any one of k vertices in the set $A' = \{\langle r^{m/p_2}\rangle, \ldots, \langle r^{m/p_k}\rangle, \langle r^m, s\rangle\}$. By similar argument, not all of these k vertices in A' belong to S. Without loss of

generality, let $\langle r^{m/p_2}\rangle \notin S$ and $\langle r^{m/p_1}\rangle \sim x_2$. Proceeding similarly, we get $x_2 = \langle r^{p_2^{\beta_2}}, r^{i_2}s\rangle, \ldots, x_k = \langle r^{p_k^{\beta_k}}, r^{i_k}s\rangle$.
If n is odd, then $\langle r^m, s\rangle$ is not adjacent to any x_i, a contradiction. If n is even, then either $\langle r^m, s\rangle$ or $\langle r^m, rs\rangle$ is not dominated by any x_i, a contradiction. Hence, $\gamma(\Gamma^*(D_n)) = k + 1$. □

Theorem 9. *$\Gamma(D_n)$ is weakly perfect, i.e., the clique number and chromatic number of $\Gamma(D_n)$ are given by*

$$\chi(\Gamma(D_n)) = \omega(\Gamma(D_n)) = \begin{cases} \pi(n) + 1, & \text{if } n \text{ is odd} \\ \pi(n) + 2, & \text{if } n \text{ is even.} \end{cases}$$

Proof. We first deal with the case when n is odd, say $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$, where p_i's are distinct odd primes. Consider the set $A = \{\langle r\rangle, \langle r^{p_1}, s\rangle, \langle r^{p_2}, s\rangle, \ldots, \langle r^{p_k}, s\rangle\}$. Clearly A forms a clique of size $k+1 = \pi(n)+1$, i.e., $\omega(\Gamma(D_n)) \geq \pi(n)+1$. Let M be a maximum clique of $\Gamma(D_n)$ of size $t \geq k+2$. If M contains only vertices of Type-II, then $M \cup \langle r\rangle$ is a clique properly containing M, a contradiction. Thus M always contains a vertex of Type-I. As no two vertices of Type-I are adjacent, M can have exactly one vertex of Type-I. Without loss of generality, we can assume the Type-I vertex in M to be $\langle r\rangle$. Let $M = \{\langle r\rangle, \langle r^{a_1}, r^{b_1}s\rangle, \langle r^{a_2}, r^{b_2}s\rangle, \ldots, \langle r^{a_{t-1}}, r^{b_{t-1}}s\rangle\}$. Thus a_1, a_2, a_{t-1} are mutually coprime factors of n and $a_i \neq 1$. But as n has $\pi(n)$ distinct prime factors, it can have atmost $\pi(n) = k < t-1$ mutually coprime factors. Thus $\omega(\Gamma(D_n)) = \pi(n) + 1$.
Similarly, if n is even, i.e., $n = 2^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$, it can be easily checked that $B = \{\langle r\rangle, \langle r^2, s\rangle, \langle r^2, rs\rangle, \langle r^{p_2}, s\rangle, \ldots, \langle r^{p_k}, s\rangle\}$ is a clique of size $k+2 = \pi(n)+2$. Thus $\omega(\Gamma(D_n)) \geq \pi(n) + 2$. Let M be a maximum clique of $\Gamma(D_n)$ of size t. As in the previous case, M have exactly one vertex of Type-I. Let $M = \{\langle r\rangle, \langle r^{a_1}, r^{b_1}s\rangle, \langle r^{a_2}, r^{b_2}s\rangle, \ldots, \langle r^{a_{t-1}}, r^{b_{t-1}}s\rangle\}$. Arguing as in the previous case, the number of odd divisors of n among a_1, a_2, a_{t-1} is atmost $k-1$. Again due to the adjacency condition of Type-II vertices, the number of odd divisors of n among a_1, a_2, a_{t-1} is atmost 2. Thus M can have atmost $1 + 2 + (k-1) = k+2$ vertices, i.e., $\omega(\Gamma(D_n)) = \pi(n) + 2$.
As $\chi \geq \omega$, it suffices to produce a proper colouring using ω colours. If $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$ is odd, define

$$A_1 = \{\langle r^d\rangle, \langle r^d, r^i s\rangle : p_1 | d\},\ A_2 = \{\langle r^d\rangle, \langle r^d, r^i s\rangle : p_2 | d\} \setminus A_1, \cdots,$$

$$A_j = \{\langle r^d\rangle, \langle r^d, r^i s\rangle : p_j | d\} \setminus \bigcup_{l=1}^{j-1} A_l, \text{ where } j = 1, 2, \ldots, k.$$

Observe that $A_1, A_2, \ldots, A_k$ are independent sets in $\Gamma(D_n)$. We assign the colour j to all the vertices in A_j and the $k+1$ the colour to $\langle r\rangle$. It can be easily checked that this is a proper colouring of $\Gamma(D_n)$ using $k+1 = \pi(n)+1$ colours.

Similarly, if n is even, we construct similar independent sets for each prime as above, with the following exception for the prime 2. For the prime 2, we construct two sets $X = \{\langle r^d\rangle, \langle r^d, r^i s\rangle : 2|d, i \text{ is odd}\}$ and $Y = \{\langle r^d\rangle, \langle r^d, r^i s\rangle : 2|d, i \text{ is even}\}$. One can easily check that this gives a proper colouring $\Gamma(D_n)$ using $\pi(n) + 2$ colours. □

Theorem 10. *$\Gamma(D_n)$ is perfect if and only if one of the two conditions hold:*

- *n is odd and $\pi(n) \leq 4$.*
- *n is even and either $\pi(n) \leq 2$ or $\pi(n) = 3$ and $4 \nmid n$.*

Proof. If n is odd and $\pi(n) \geq 5$, let $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_5^{\alpha_5} m$, where p_i's are odd primes which are coprime to m. Then $\langle r^{p_1p_2}, s\rangle \sim \langle r^{p_3p_4}, s\rangle \sim \langle r^{p_2p_5}, s\rangle \sim \langle r^{p_1p_4}, s\rangle \sim \langle r^{p_3p_5}, s\rangle \sim \langle r^{p_1p_2}, s\rangle$ is an induced 5-cycle in $\Gamma(D_n)$ and hence $\Gamma(D_n)$ is not perfect.
Let n be odd and $\pi(n) \leq 4$. Let $C : x_1 \sim x_2 \sim \cdots \sim x_{2t+1} \sim x_1$ be an induced odd cycle in $\Gamma(D_n)$. As n is odd and any subgroup of D_n is of the form $\langle r^d\rangle$ or $\langle r^d, r^i s\rangle$, it follows from the adjacency condition that $\langle r^{d_1}\rangle \sim \langle r^{d_2}, r^i s\rangle$ or $\langle r^{d_1}, r^i s\rangle \sim \langle r^{d_2}, r^j s\rangle$ if and only if $gcd(d_1, d_2) = 1$. Thus for each vertex x_i in C we can associate a factor d_i of n such that $x_i \sim x_j$ if and only if $gcd(d_i, d_j) = 1$. Now, by following the steps in the proof of Theorem 3.2 in [7], one can show that $\Gamma(D_n)$ is perfect.
If n is even and $\pi(n) \geq 4$, let $n = 2^{\alpha_1} p_2^{\alpha_2} \cdots p_4^{\alpha_4} m$, where p_i's are odd primes which are coprime to m. Then $\langle r^{p_2}\rangle \sim \langle r^{2p_2p_3}, rs\rangle \sim \langle r^{p_3p_4}\rangle \sim \langle r^{2p_4}, r^2 s\rangle \sim \langle r^{2p_2}, s\rangle \sim \langle r^{p_2}\rangle$ is an induced 5-cycle in the complement of $\Gamma(D_n)$ and hence $\Gamma(D_n)$ is not perfect.
If $\pi(n) = 3$ and $4|n$, let $n = 2^{\alpha} p_2^{\alpha_2} p_3^{\alpha_3}$ where p_i's are odd primes. Then $\langle r^{p_1}\rangle \sim \langle r^4, s\rangle \sim \langle r^{p_2}\rangle \sim \langle r^{2p_1}, s\rangle \sim \langle r^{4p_2}, rs\rangle \sim \langle r^{p_1}\rangle$ is an induced 5-cycle in the complement of $\Gamma(D_n)$ and hence $\Gamma(D_n)$ is not perfect.
Thus, if n is even, we are left with two cases, either $n = 2^{\alpha} p_2^{\alpha_2}$ or $n = 2p_2^{\alpha_2} p_3^{\alpha_3}$. These two cases are dealt with in the following two lemmas. □

Lemma 1. *If $n = 2^{\alpha} p_2^{\alpha_2}$, then $\Gamma(D_n)$ is perfect.*

Proof. Note that any vertex of the form $\langle r^d\rangle$ or $\langle r^d, r^i s\rangle$ where $2p_2|d$ are of degree 0 or 2 respectively in $\Gamma(D_n)$. In fact, $\langle r^d, r^i s\rangle$ is adjacent to exactly two vertices, namely $\langle r\rangle$ and exactly one of $\langle r^2, s\rangle$ and $\langle r^2, rs\rangle$. If possible, let $C : x_1 \sim x_2 \sim \cdots \sim x_{2t+1} \sim x_1$ be an induced odd cycle of length atleast 5 in $\Gamma(D_n)$. Clearly C must have atleast one Type-II vertex. As $\langle r\rangle$ does not lie on C, any vertex of the form $\langle r^d, r^i s\rangle$ where $2p_2|d$ does not lie on C.
Claim A: If $x_1 = \langle r^{d_1}, r^i s\rangle$ is a Type-II vertex on C, then d_1 is even.
Proof of Claim A: If d_1 is odd, then $d_1 = p_2^{\beta}$. As $x_1 \not\sim x_3, x_4$, we have $x_3 = \langle r^{p_2^a}\rangle$ or $\langle r^{p_2^a}, r^j s\rangle$ and $x_4 = \langle r^{p_2^b}\rangle$ or $\langle r^{p_2^b}, r^k s\rangle$. In any case, we have $x_3 \not\sim x_4$, a contradiction.
Claim B: There exists no Type-I vertex on C.
Proof of Claim B: If there exists two vertices, say x_1, x_k of Type-I on C. Clearly they must be non-adjacent. Using Claim A, $x_1 = \langle r^{p_2^{\beta}}\rangle$ and $x_k = \langle r^{p_2^{\beta'}}\rangle$. But as $x_2, x_{2t+1} \sim x_1$, we must have $x_k \sim x_2, x_{2t+1}$, a contradiction. So atmost one Type-I

vertex can be on C, say $x_1 = \langle r^{p_2^{\beta}} \rangle$. As $x_1 \not\sim x_3, x_4$ and both are Type-II vertices, by Claim A, we must have $x_3 = \langle r^{d_3}, r^i s\rangle$ and $x_4 = \langle r^{d_4}, r^j s\rangle$ where $2p_2$ divides d_3 and d_4. However such vertices do not lie on C.
Thus all the vertices on C are of Type-II, i.e., $x_l = \langle r^{d_l}, r^{i_l} s\rangle$ for $l = 1, 2, \ldots, 2t+1$ where d_l's are even. Again from the adjacency condition, we have all of $i_1 - i_2, i_2 - i_3, \ldots, i_{2t+1} - i_1$ to be odd. Adding all of them, we get the sum of odd number of odd integers to be zero, a contradiction. Thus $\Gamma(D_n)$ has no induced odd cycle of length atleast 5. Similarly, it can be shown that $\Gamma(D_n)^c$ has no induced odd cycle of length atleast 5. Hence $\Gamma(D_n)$ is perfect. □

Lemma 2. *If $n = 2^{\alpha} p_2^{\alpha_2} p_3^{\alpha_3}$, then $\Gamma(D_n)$ is perfect.*

Proof. If possible, let $C : x_1 \sim x_2 \sim \cdots \sim x_{2t+1} \sim x_1$ be an induced odd cycle of length atleast 5 in $\Gamma(D_n)$. As no two Type-I vertices are adjacent, thus we must have atleast $t + 1 \geq 3$ Type-II vertices in C.

Claim 1: $\langle r^d, r^i s\rangle$, where $2p_1p_2|d$ does not lie in C.
Proof of Claim 1: Its only neighbours are $\langle r\rangle$ and exactly one of $\langle r^2, s\rangle$ and $\langle r^2, rs\rangle$. As $\langle r\rangle$ is adjacent to all Type-II vertices and there are atleast 3 Type-II vertices in C, $\langle r\rangle$ does not lie on C. Thus $\langle r^d, r^i s\rangle$ can have atmost one neighbour in C, which is a contradiction as C is a cycle.

Claim 2: None of $\langle r^2, s\rangle$ and $\langle r^2, rs\rangle$ lie in C.
Proof of Claim 2: If $x_1 = \langle r^2, s\rangle$ lies in C, then as $\langle r^2, s\rangle$ is a maximal subgroup of index 2 in D_n, all of $x_3, x_4, \ldots, x_{2t}$ are contained in x_1. Thus, using Claim 1, without of loss of generality, we can assume that $x_3 = \langle r^{2p_1^{\beta_1}}, r^i s\rangle$ and $x_3 = \langle r^{2p_2^{\beta_2}}, r^j s\rangle$. As $x_3 \sim x_4$, we have $i - j$ is odd. On the other hand, as $x_1 \not\sim x_3, x_4$, we must have i and j to be both even. This contradicts the parity of $i - j$.

Claim 3: Vertices of the form $\langle r^{p_1^{\beta_1} p_2^{\beta_2}}\rangle$ and $\langle r^{p_1^{\beta_1} p_2^{\beta_2}}, r^i s\rangle$ do not lie in C.
Proof of Claim 3: As $\langle r^{p_1^{\beta_1} p_2^{\beta_2}}\rangle$ is adjacent only with $\langle r^2, s\rangle$ and $\langle r^2, rs\rangle$, the claim follows from Claim 2. Similarly, only neighbours of $\langle r^{p_1^{\beta_1} p_2^{\beta_2}}, r^i s\rangle$ in $\Gamma(D_n)$ are $\langle r\rangle$, $\langle r^2\rangle$ and exactly one of $\langle r^2, s\rangle$ and $\langle r^2, rs\rangle$. However, from Claim 2, its only possible neighbour in C is $\langle r^2\rangle$, a contradiction. Hence Claim 3 holds.

Claim 4: $\langle r^2\rangle$ lies in C.
Proof of Claim 4: Suppose $\langle r^2\rangle$ does not in C. Then from Claims 1,2 and 3, it follows that for any vertex $\langle r^{d_i}\rangle$ or $\langle r^{d_i}, r^i s\rangle$ in C, d_i must be of the form $p_1^{\beta_1}, p_2^{\beta_2}, 2p_1^{\beta_1}$ or $2p_2^{\beta_2}$. Again, as C is cycle, d_i must be alternately divisible by p_1 and p_2. But this contradicts that C is an odd cycle. Thus the claim follows.
Let $x_1 = \langle r^2\rangle$ be a vertex on C. As x_1 is a Type-I vertex, from the adjacency condition and previous claims, without loss of generality, we have $x_2 = \langle r^{p_1^{\beta}}, r^i s\rangle$ and $x_{2t+1} = \langle r^{p_1^{\beta'}}, r^j s\rangle$. Then x_3 must be of one of the 4 forms, namely $\langle r^{p_2^{\beta_2}}\rangle, \langle r^{p_2^{\beta_2}}, r^j s\rangle, \langle r^{2p_2^{\beta_2}}\rangle$ and $\langle r^{2p_2^{\beta_2}}, r^j s\rangle$. However, in any case, we have $x_3 \sim x_{2t+1}$, a contradiction. Thus $\Gamma(D_n)$ has no induced odd cycle of length atleast 5. □

4. Isomorphisms of $\Gamma(D_n)$

In this section, we discuss some isomorphism results of $\Gamma(D_n)$. The first result (Theorem 11) shows that co-maximal graph of D_n uniquely determines n. The second result (Theorem 12) is more general in nature. It shows that nilpotent dihedral groups are uniquely determined by their comaximal subgroup graphs.

Lemma 3. *Let n and m be two positive integers such that $\Gamma(D_n) \cong \Gamma(D_m)$. Then n and m are of same factorization type.*

Proof. As $\Gamma(D_n) \cong \Gamma(D_m)$, from Theorem 5, it follows that n and m have same parity. Thus, by Theorem 9, $\pi(n) = \pi(m)$, i.e., m and n have same number of distinct prime factors. So we assume that $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$ and $m = q_1^{\beta_1} q_2^{\beta_2} \cdots q_k^{\beta_k}$.
Consider the Type-I vertices other than $\langle r \rangle$ in $\Gamma(D_n)$. Note that $\{\langle r^{p_1} \rangle, \langle r^{p_1^2} \rangle, \cdots, \langle r^{p_1^{\alpha_1}} \rangle\}$ is one of the twin class of size α_1. Similarly, we get twin classes of size $\alpha_2, \alpha_3, \ldots, \alpha_k$. Again, note $\{\langle r^{p_1 p_2} \rangle, \langle r^{p_1^2 p_2} \rangle, \cdots, \langle r^{p_1^{\alpha_1} p_2^{\alpha_2}} \rangle\}$ is a twin class of size $\alpha_1\alpha_2$. Proceeding this way, Type-I vertices other than $\langle r \rangle$, can be partitioned into twin classes of size

$$\mathcal{P}_n = \{\alpha_1, \alpha_2, \ldots, \alpha_k, \alpha_1\alpha_2, \alpha_2\alpha_3, \ldots, \alpha_1\alpha_2 \cdots \alpha_k\}.$$

Similarly for $\Gamma(D_m)$, we get

$$\mathcal{P}_m = \{\beta_1, \beta_2, \ldots, \beta_k, \beta_1\beta_2, \beta_2\beta_3, \ldots, \beta_1\beta_2 \cdots \beta_k\}.$$

As $\Gamma(D_n) \cong \Gamma(D_m)$, we have $\mathcal{P}_n = \mathcal{P}_m$. If $\alpha_i = \beta_{\sigma(i)}$ for some $\sigma \in S_k$, we are done. If no α_i is equal to any β_j, then without loss of generality, let $\alpha_1 = \min\{\alpha_1, \alpha_2, \ldots, \alpha_k, \beta_1, \beta_2, \ldots, \beta_k\}$. Therefore, $\alpha_1 < \beta_i$ for all i. Thus $\alpha_1 \in \mathcal{P}_n$, but $\alpha_1 \in \mathcal{P}_m$, as $\beta_i > \alpha_1$. This contradicts the fact $\mathcal{P}_n = \mathcal{P}_m$. Thus some α_i's are equal to some β_j. By suitable renaming, let $\alpha_1 = \beta_1, \alpha_2 = \beta_2, \ldots, \alpha_i = \beta_i$ and none of $\alpha_{i+1}, \ldots, \alpha_k$ is not equal to any of $\beta_{i+1}, \ldots, \beta_k$. Therefore each of $\alpha_{i+1}, \ldots, \alpha_k$ is product of atleast two β_j's. Similarly, each of $\beta_{i+1}, \ldots, \beta_k$ is product of atleast two α_j's.
We remove all the terms involving $\alpha_1, \alpha_2, \ldots, \alpha_i$ from $\mathcal{P}_n$ to get a new set $\mathcal{P}'_n$. Similarly, we remove all the terms involving $\beta_1, \beta_2, \ldots, \beta_i$ from $\mathcal{P}_m$ to get a new set $\mathcal{P}'_m$. Hence we have $\mathcal{P}'_n = \mathcal{P}'_m$.
Let $\alpha_{i_1}\alpha_{i_2} \cdots \alpha_{i_t}$ be the smallest element of $\mathcal{P}'_n$. Then at least one of $\alpha_{i_1}, \alpha_{i_2}, \ldots, \alpha_{i_t}$ does not belong to $\{\alpha_1, \alpha_2, \ldots, \alpha_i\}$. Let $\alpha_{i_1} \notin \{\alpha_1, \alpha_2, \ldots, \alpha_i\}$. Then $\alpha_{i_1} \in \mathcal{P}'_n$ and $\alpha_{i_1} \leq \alpha_{i_1}\alpha_{i_2} \cdots \alpha_{i_t}$. Thus α_{i_1} is also smallest in $\mathcal{P}'_n = \mathcal{P}'_m$.
Therefore $\alpha_{i_1} = \beta_{j_1}\beta_{j_2} \cdots \beta_{j_t} \in \mathcal{P}'_m$. Arguing similarly, without loss of generality, β_{j_1} is the smallest element in $\mathcal{P}'_m$. Thus $\alpha_{i_1} = \beta_{j_1}$, a contradiction. Hence, $\alpha_i = \beta_{\sigma(i)}$ for some $\sigma \in S_k$ and the theorem follows. $\square$

Theorem 11. *Let n and m be two positive integers such that $\Gamma(D_n) \cong \Gamma(D_m)$. Then $n = m$.*

Proof. From Lemma 3, we get that $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$ and $m = q_1^{\alpha_1} q_2^{\alpha_2} \cdots q_k^{\alpha_k}$. Thus, it suffices to show that $p_i = q_i$ for all i. We consider the case when both m and n are odd. The case when both m and n are even can be handled similarly.
Consider the maximum clique $A = \{\langle r\rangle, \langle r^{p_1}, s\rangle, \langle r^{p_2}, s\rangle, \ldots, \langle r^{p_k}, s\rangle\}$ of $\Gamma^*(D_n)$ as defined in the proof of Theorem 9. Note that it contains exactly one vertex of Type-I and k-vertices of Type-II. As $\Gamma(D_n) \cong \Gamma(D_m)$, under any isomorphism, A is mapped to a maximum clique B of $\Gamma^*(D_m)$. Without loss of generality,

$$B = \{\langle r\rangle, \langle r^{q_1}, r^{i_1}s\rangle, \langle r^{q_2}, r^{i_2}s\rangle, \ldots, \langle r^{q_k}, r^{i_k}s\rangle\}.$$

Now, consider the number of Type-I and Type-II neighbours of Type-II vertices in A. For example, $\langle r^{p_i}, s\rangle$ has $(\tau(n/{p_i}^{\alpha_i}) - 1)$ many Type-I neighbours and $(\sigma(n/{p_i}^{\alpha_i}) - 1)$ many Type-II neighbours in $\Gamma^*(D_n)$. Similarly, we can compute the number of Type-I and Type-II neighbours of Type-II vertices in B. As $\Gamma(D_n) \cong \Gamma(D_m)$, the following two sets consisting of ordered pairs are equal.

$$\{(\tau(n/{p_1}^{\alpha_1}), \sigma(n/{p_1}^{\alpha_1})), (\tau(n/{p_2}^{\alpha_2}), \sigma(n/{p_2}^{\alpha_2})), \ldots, (\tau(n/{p_k}^{\alpha_k}), \sigma(n/{p_k}^{\alpha_k}))\}$$

$$= \{(\tau(m/{q_1}^{\alpha_1}), \sigma(m/{q_1}^{\alpha_1})), (\tau(m/{q_2}^{\alpha_2}), \sigma(m/{q_2}^{\alpha_2})), \ldots, (\tau(m/{q_k}^{\alpha_k}), \sigma(m/{q_k}^{\alpha_k}))\}$$

Again, as $\tau(m) = \tau(n), \sigma(m) = \sigma(n)$ and τ, σ are multiplicative functions, we have

$$\{(\tau({p_1}^{\alpha_1}), \sigma({p_1}^{\alpha_1})), (\tau({p_2}^{\alpha_2}), \sigma({p_2}^{\alpha_2})), \ldots, (\tau({p_k}^{\alpha_k}), \sigma({p_k}^{\alpha_k}))\}$$

$$= \{(\tau({q_1}^{\alpha_1}), \sigma({q_1}^{\alpha_1})), (\tau({q_2}^{\alpha_2}), \sigma({q_2}^{\alpha_2})), \ldots, (\tau({q_k}^{\alpha_k}), \sigma({q_k}^{\alpha_k}))\}$$

As these two sets are equal, there exists i such that $(\tau({p_1}^{\alpha_1}), \sigma({p_1}^{\alpha_1})) = (\tau({q_i}^{\alpha_i}), \sigma({q_i}^{\alpha_i}))$, i.e., $\alpha_1 = \alpha_i$ and hence $\sigma({p_1}^{\alpha_1}) = \sigma({q_i}^{\alpha_1})$, i.e., $p_1 = q_i$. Similarly, it can be shown that set of prime factors of m and n are same and as a result, $m = n$. □

Theorem 12. *Let G be a finite solvable group such that $\Gamma(G) \cong \Gamma(D_{2^\alpha})$. Then $G \cong D_{2^\alpha}$.*

Proof. As $\Gamma^*(D_{2^\alpha})$ has a unique universal vertex, namely $\langle r\rangle$ and all other Type-I vertices are isolated, we get a subgroup H which is the unique universal vertex in $\Gamma^*(G)$.
Claim 1: H is a maximal subgroup G and $H \lhd G$.
Proof of Claim 1: If there exists a proper subgroup X of G such that $H \subsetneq X$, then $deg(H) \le deg(X)$ in $\Gamma(G)$, a contradiction. Thus H is a maximal subgroup of G. If H is not normal in G, there exists $g \in G$ such that $H' = gHg^{-1} \neq H$. Note that

$K \sim H$ if and only if $gKg^{-1} \sim gHg^{-1}$, i.e., $deg(H) = deg(H')$, a contradiction. Thus $H \lhd G$.

From Claim 1, it follows that G/H is a prime order group, i.e., $[G : H] = p$, for some prime p. Thus $|G| = p^a m$ and $|H| = p^{a-1}m$, where $p \nmid m$.

Claim 2: G is a group of prime power order.

Proof of Claim 2: Let q be a prime factor of m and K be a Sylow q-subgroup of G. If $K \nsubseteq H$, then $KH = G$, i.e.,

$$p^a m = \frac{(q^b)(p^{a-1}m)}{|H \cap K|} = \frac{(q^b)(p^{a-1}m)}{q^t} = q^{b-t}p^{a-1}m, \text{ i.e., } q^{b-t} = p, \text{ a contradiction.}$$

Thus if q is a prime factor of m, then every Sylow q-subgroup K of G is contained in H. Thus K corresponds to a Type-I vertex in $\Gamma(D_{2^\alpha})$ and hence, if $K \neq H$, then K is an isolated vertex in $\Gamma(D_{2^\alpha})$. However, as G is solvable, K has a Hall complement L of order $p^a m/q^b$ in G, i.e., $KL = G$, i.e., $K \sim L$. Thus either m has no prime factor, i.e., $m = 1$ or $K = H$. If $m = 1$, then G is p-group and the claim holds. If $K = H$, then $|H| = q^b$, i.e., $a = 1$ and $|G| = pq^b$.

Again, note that $\Gamma^*(D_{2^\alpha})$ has exactly two Type-II vertices of second highest degree, namely $\langle r^2, s\rangle$ and $\langle r^2, rs\rangle$ and every other Type-II vertices is adjacent to exactly one of $\langle r^2, s\rangle$ and $\langle r^2, rs\rangle$. Let K_1, K_2 be the two vertices in $\Gamma^*(G)$ corresponding to $\langle r^2, s\rangle$ and $\langle r^2, rs\rangle$ respectively. As H is the universal vertex in $\Gamma^*(G)$, we have $H \sim K_1$ and $H \sim K_2$, i.e., $K_1, K_2 \nsubseteq H$. Thus $|K_1| = pq^{t_1}$ and $|K_2| = pq^{t_2}$. Again, as $\langle r^2, s\rangle \sim \langle r^2, rs\rangle$, we have $K_1 \sim K_2$, i.e., $K_1K_2 = G$, i.e.,

$$pq^b = \frac{pq^{t_1} \cdot pq^{t_2}}{|K_1 \cap K_2|}, \text{ i.e., } |K_1 \cap K_2| = pq^{t_1+t_2-b}.$$

If $p \neq q$, $K_1 \cap K_2 \nsubseteq H$, i.e., $H \sim K_1 \cap K_2$ and $K_1 \cap K_2$ corresponds to a Type-II vertex. Hence, $K_1 \cap K_2$ must be adjacent to one of K_1 and K_2. However, $K_1 \cap K_2 \subseteq K_1, K_2$, this is a contradiction. Thus we must have $p = q$ and $|G| = p^{b+1}$. Hence Claim 2 holds.

As G is a group of prime-power order, G is nilpotent and $\Gamma^*(G)$ has a unique universal vertex. Thus by Theorem 3.6 in [5], G must belong to one of the five families of groups, namely $3, 4, 5, 6, 7$. As $\Gamma^*(\mathbb{Z}_{p^{n-1}} \times \mathbb{Z}_p)$ and $\Gamma^*(M_{p^n})$ has p many universal vertices, G is not isomorphic to $\mathbb{Z}_{p^{n-1}} \times \mathbb{Z}_p$ or M_{p^n}. Again, as $\Gamma^*(SD_{2^n})$ a unique vertex of second highest degree, G is not isomorphic to SD_{2^n}. If $G \cong Q_{2^n}$, then number of isolated vertices in $\Gamma(G)$ is $n-2$ and the second highest degree is 2^{n-2}. However, $\Gamma(D_{2^\alpha})$ has $\alpha - 1$ isolated vertices and its second highest degree is 2^α. This is a contradiction and hence $G \ncong Q_{2^n}$. Hence $G \cong D_{2^{n-1}}$. Finally, comparing the number of isolated vertices, we get $G \cong D_{2^\alpha}$. □

Research Funding. The first author acknowledges the financial support from DST-FIST [File No. $SR/FST/MS - I/2019/41$ and DST-MATRICS grant $MTR/2022/000020$.

Acknowledgements. The authors are thankful to Prof. Gutman and the anonymous referees for their careful reading and the suggestions that improved the article.

Conflict of interest. The authors have no relevant financial or non-financial interests to disclose.

Data Availability. Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

References

[1] S. Akbari, B. Miraftab, and R. Nikandish, *Co-maximal graphs of subgroups of groups*, Can. Math. Bull. **60** (2017), no. 1, 12–25.
https://doi.org/10.4153/CMB-2016-026-0.

[2] P.J. Cameron, *Graphs defined on groups*, Int. J. Group Theory **11** (2022), no. 2, 53–107.
https://doi.org/10.22108/ijgt.2021.127679.1681.

[3] K. Conrad, *Notes on dihedral groups*,
https://kconrad.math.uconn.edu/blurbs/grouptheory/dihedral2.pdf.

[4] A. Das and M. Saha, *On co-maximal subgroup graph of a group-II*, Ric. Mat., In press.
https://doi.org/10.1007/s11587-023-00836-3.

[5] A. Das, M. Saha, and S. Al-Kaseasbeh, *On co-maximal subgroup graph of a group*, Ric. Mat., In press.
https://doi.org/10.1007/s11587-022-00718-0.

[6] B. Miraftab and R. Nikandish, *Co-maximal graphs of two generator groups*, J. Algebra Appl. **18** (2019), no. 4, Article ID:1950068.
https://doi.org/10.1142/S0219498819500683.

[7] M. Saha, S. Biswas, and A. Das, *On co-maximal subgroup graph of* $\mathbb{Z}_n$, Int. J. Group Theory **11** (2022), no. 4, 221–228.
https://doi.org/10.22108/ijgt.2021.129788.1732.