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Research Article

On connected bipartite Q-integral graphs

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Abstract: A graph G is said to be H-free if G does not contain H as an induced subgraph. Let $\mathcal{S}_n^2(m)$ be a variation of double star $\mathcal{S}_n^2$ obtained by adding $m(\leq n)$ disjoint edges between the pendant vertices which are at distance 3 in $\mathcal{S}_n^2$. A graph having integer eigenvalues for its signless Laplacian matrix is known as a Q-integral graph. The Q-spectral radius of a graph is the largest eigenvalue of its signless Laplacian. Any connected Q-integral graph G with Q-spectral radius 7 and maximum edge-degree 8 is either $K_{1,4}\Box K_2$ or contains $\mathcal{S}_4^2(0)$ as an induced subgraph or is a bipartite graph having at least one of the induced subgraphs $\mathcal{S}_4^2(m), (m = 1, 2, 3)$. In this article, we improve this result by showing that every connected Q-integral graph G having Q-spectral radius 7, maximum edge-degree 8 is always bipartite and $\mathcal{S}_4^2(3)$-free.

Keywords: edge-degree, H-free graph, signless Laplacian matrix, Q-integral graph

AMS Subject classification: 05C50, 05C07

1. Introduction

All the graphs considered in this article are simple and undirected. Let $G = (V(G), E(G))$ be a graph, where $V(G)$ and $E(G)$ denote the set of vertices and edges, respectively. Let $d(v)$ be the *degree* of a vertex v and $N(v)$ be the *neighborhood* of $v \in V(G)$. The Cartesian product $G_1\Box G_2$ obtained from the graphs G_1 and G_2 is a graph with vertex set $V(G_1) \times V(G_2)$, and two vertices (v_1, v_2) and (u_1, u_2) are adjacent in $G_1\Box G_2$ if and only if either $v_1 = u_1$ and v_2 is adjacent to u_2 in G_2 or $v_2 = u_2$ and v_1 is adjacent to u_1 in G_1. A graph G is said to be *H-free* if H is not an induced subgraph of G.

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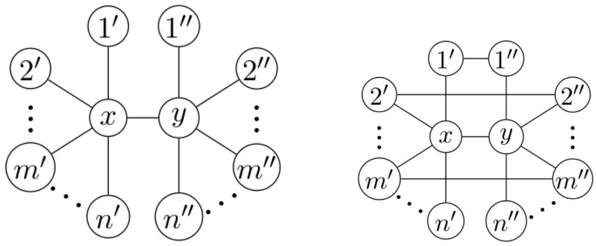


(a) $\mathcal{S}_n^2$ (b) $\mathcal{S}_n^2(m)$

Figure 1.

Let $D(G)$ be the diagonal matrix with $D(G)_{vv} = d(v)$, for $v \in V(G)$. The *signless Laplacian* $Q(G)$ of G is defined by the matrix $D(G) + A(G)$. The matrix $Q(G)$ is positive semidefinite and irreducible. The *Q-eigenvalues* and *Q-spectral radius* $q(G)$ of G are the eigenvalues and largest eigenvalue of $Q(G)$, respectively. A graph G is called *Q-integral* if all the Q-eigenvalues of G are integral. For some more studies related to signless Laplacian and Q-integral graphs, see [2–4, 6, 10–14, 16].

We define a *double star* $\mathcal{S}_n^2$ by taking two disjoint copies of star graph $K_{1,n}$ and adding an edge between the vertices of degree n. Let $\mathcal{S}_n^2(m)$ be a *variation of double star* $\mathcal{S}_n^2$ obtained by adding $m(\leq n)$ disjoint edges between the pendant vertices which are at distance 3 in $\mathcal{S}_n^2$ (see Figure 1).

We use $e' = uv \in E(G)$ to denote an edge having u, v as incident vertices and the *edge-degree* $e\text{-}deg_G(e')$ is given by $|N(u) \cup N(v)| - 2$. The *maximum edge-degree* of G is denoted by $e\text{-}deg_G^{max}$. Simić and Stanić [15] studied connected Q-integral graphs with $e\text{-}deg_G^{max} \leq 4$ and also gave partial results about the spectra of $Q(G)$ for $e\text{-}deg_G = 5$. Park and Sano [8] investigated on connected Q-integral graphs G having $e\text{-}deg_G^{max} \leq 6$ and gave a structural theorem when $q(G) = 6$. An improvement of this result can be found in [7].

In 2022 [9], we studied connected Q-integral graph having $e\text{-}deg_G^{max} \leq 8$ and gave a structural characterization under the restriction $q(G) = 7$. We have shown that $G(\neq K_{1,4} \square K_2)$ must contain one of the four special subgraphs $\mathcal{S}_4^2(m)$ for $m = 0, \ldots, 3$ (as shown in Figure 2) as an induced subgraph.

2. Preliminaries

Let $n \in \mathbb{N}$ be any number. We use $\mathbf{0}$ and $\mathbf{1}$ to denote the matrices of appropriate orders whose entries are all equal to 0 and 1, respectively. The multiset of all the eigenvalues of N together with their multiplicities is called as the *spectrum* $\text{Spec}(N)$ of the matrix $N_{n \times n}$. The *spectral radius* of N is $\rho(N) = \max\{|\beta| \mid \beta \in Spec(N)\}$. The *least, second smallest*, and the *second largest eigenvalues* of a matrix N are denoted by $\lambda_{\min}(N), \lambda_{s2}(N)$, and $\lambda_{l2}(N)$, respectively.

The principal submatrix of $Q(G)$ corresponding to the vertices of $H \subseteq V(G)$ is denoted by $Q_p(H)$. For any two distinct vertices $u, v \in V(G)$, the (u,v)-th entry of $Q_p(H)$ is denoted by a_{uv}. We use $a_{..}$ and $d(\cdot)$ in place of a_{uv} and $d(z)$ when u, v and z are suitable vertices within the context.

Proposition 1 ([1], Proposition 1.3.9). *The number of connected bipartite components of G is equal to the multiplicity of the Q-eigenvalue 0 in G.*

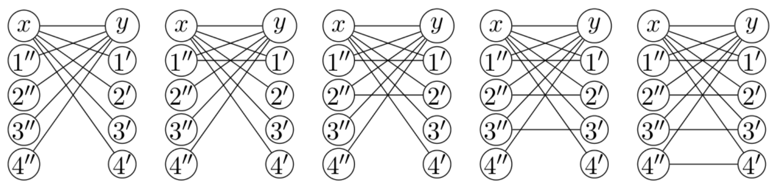
Proposition 2 ([8], Proposition 2.7). *A connected graph G has $d(v) \leq \lceil q(G) - 1 \rceil$ for any $v \in V(G)$, where $q(G)$ is the Q-spectral radius of G. If G has a vertex v having $d(v) = q(G) - 1$ and $q(G) \in \mathbb{Z}^+$, then $G = K_{1,q(G)-1}$.*

The following results give bounds for the maximum edge-degree of a graph G.

Remark 1 ([9], Remark 3.2). For a connected edge-regular graph G, $e\text{-}deg_G = q(G) - 2$.

Lemma 1 ([9], Lemma 3.3). *Let G be a connected edge-non-regular Q-integral graph. Then $q(G) - 1 \leq e\text{-}deg_G^{max} \leq 2q(G) - 6$.*

Remark 2 ([9], Remark 3.4). There does not exist any connected edge-non-regular Q-integral graph with $q(G) \leq 4$. Moreover, if $q(G) = 5$, then e-$deg_G^{max} = 4$.



(a) $\gamma_0 = \mathcal{S}_4^2(0)$ (b) $\mathcal{S}_4^2(1)$ (c) $\mathcal{S}_4^2(2)$ (d) $\gamma_3 = \mathcal{S}_4^2(3)$ (e) $K_{1,4} \square K_2$

Figure 2.

The following remark is a consequence of the results (Remark 1, Lemma 1, Remark 2) stated above.

Remark 3. A connected Q-integral graph G is edge-regular if and only if $q(G) = 4$.

Now, suppose G is any connected Q-integral graph with e-$deg_G^{max} = 2q(G) - 6$.

- $q(G) \leq 3$: There does not exist any such graph G.
- $q(G) = 4$: G must be one of the graphs $C_3, C_4, C_6, K_{1,3}$. Note that $G(\neq C_3)$ is bipartite and $G(\neq C_6)$ is $\mathcal{S}_1^2(0)$-free.

- $q(G) = 5$: G must be one of the graphs $K_{1,2}\square K_2$, $\overline{K_{3,3} - e}$ (the complement of the graph $K_{3,3} - e$, where e is an edge). Note that both of them are $\mathcal{S}_2^2(1)$-free.

- $q(G) = 6$: G is bipartite and $\mathcal{S}_3^2(2)$-free.

Recently in [9], it was proved that Q-integral graph contains $\mathcal{S}_4^2(m)$ $(0 \leq m \leq 3)$ as an induced subgraph when $q(G) = 7$.

Theorem 1 ([9], Theorem 4.1). *Suppose G is a connected Q-integral graph having $q(G) = 7$. If e-$deg_G^{max} = 8$, then one of the following hold.*

(a) $G = K_{1,4}\square K_2$.

(b) *G is bipartite with at least one of $\mathcal{S}_4^2(m)(m = 1, 2, 3)$, given in Figure 2, as induced subgraph, and $1, 6 \in Spec(Q(G))$.*

(c) *G has $\mathcal{S}_4^2(0)$ as induced subgraph, and $1, 6 \in Spec(Q(G))$.*

In this article, we show that a connected Q-integral graph G with $q(G) = 7$ and e-$deg_G^{max} = 8$ is bipartite and $\mathcal{S}_4^2(3)$-free.

3. Main Result

For the rest of the article, G denotes a connected Q-integral graph having $q(G) = 7$ and maximum edge-degree e-$deg_G^{max} = 8$. As a main result, we improve Theorem 1 by showing that G is bipartite, and if $G \neq K_{1,4}\square K_2$ then it contains at least one of $\mathcal{S}_4^2(m)$, for $m = 0, 1, 2$ as an induced subgraph.

Theorem 2. *(Main Result) Suppose G is a connected Q-integral graph having $q(G) = 7$. If e-$deg_G^{max} = 8$, then G is bipartite and $\mathcal{S}_4^2(3)$-free, where $\mathcal{S}_4^2(3)$ is given in Figure 2.*

Before we prove the theorem, we require the following notations.
For any $S_1, S_2 \subseteq V(G)$ and $S_1 \cap S_2 = \phi$, let A_{S_1,S_2} be a matrix of order $|S_1| \times |S_2|$ whose rows and columns corresponds to the vertices of S_1 and S_2, respectively. Let the (u, v)-th element, a_{uv}, for $u \in S_1, v \in S_2$ be such that

$$a_{uv} = \begin{cases} 1, & \text{if } uv \in E(G), \\ 0, & \text{otherwise.} \end{cases}$$

Thus, $A_{S_1,S_2}^t = A_{S_2,S_1}$. For brevity and clarity, we use γ_0 (resp. γ_3) to denote $\mathcal{S}_4^2(0)$ (resp. $\mathcal{S}_4^2(3)$) in the rest of the article. Let $\Gamma_0 = V(\gamma_0) = V(\gamma_3) = \{x, y, 1', 2', 3', 4', 1'', 2'', 3'', 4''\}$ be the subset of $V(G)$. In the proof, we iteratively define $\Gamma_{i+1} = \Gamma_i \cup S_i$ where $S_i \subseteq V(G) \setminus \Gamma_i,\ i \geq 0$. For Γ_{i+1} obtained from the pair (Γ_i, S_i), we have the principal submatrix

$$Q_p(\Gamma_{i+1}) = \begin{pmatrix} Q_p(\Gamma_i) & A_{\Gamma_i,S_i} \\ A_{S_i,\Gamma_i} & Q_p(S_i) \end{pmatrix}. \tag{3.1}$$

We will use these notations repeatedly in rest of the paper with appropriate definitions of Γ_i and S_i, respectively. We use MATLAB to calculate the eigenvalues of matrices.

Proof of Theorem 2

Let G be as stated in the theorem with $e\text{-}deg_G^{max} = 8$. By Proposition 2, the maximum vertex-degree of G must be less than or equal to 5. In order to prove this theorem, we observe that it is sufficient to show that

(a) G is γ_3-free, where γ_3 is given in Figure 2.

(b) G is bipartite whenever it contains the induced subgraph γ_0.

We prove **(a)** by contradiction, that is, suppose γ_3 is an induced subgraph of G. By Theorem 1, G is bipartite. Thus, $0, 7$ are simple Q-eigenvalues of G by Proposition 1 and Perron-Frobenius Theorem [[5], Theorem 8.4.4]. Since $Spec(Q(G)) \subseteq \mathbb{Z}$, we have $\lambda_{s2}(Q(G)) = 1$ and $\lambda_{l2}(Q(G)) = 6$. By Interlacing Theorem [[5], Theorem 4.3.17] on eigenvalues, we have the following remark which will be used repeatedly to prove our main theorem.

Remark 4. If G is a bipartite connected Q-integral graph having e-$deg_G^{max} = 8$ and $q(G) = 7$, then every principal submatrix $Q_p(H)$ of $Q(G)$ corresponding to a set of vertices $H \subseteq V(G)$ have $\lambda_{s2}(Q_p(H)) \geq 1$ and $\lambda_{l2}(Q_p(H)) \leq 6$.

Recall $\Gamma_0 = V(\gamma_3) = \{x, y, 1', 2', 3', 4', 1'', 2'', 3'', 4''\} \subseteq V(G)$. Note that the subgraph γ_3 is given by Figure 2(d) and its principal submatrix $Q_p(\Gamma_0)$ of $Q(G)$ is given by

$$Q_p(\Gamma_0) = \begin{pmatrix} 5 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 5 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & d(1') & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & d(2') & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & d(3') & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & d(4') & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & d(1'') & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & d(2'') & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & d(3'') & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & d(4'') \end{pmatrix} \tag{3.2}$$

where $d(\cdot) \in \{1, \ldots, 5\}$. We have the following 4 non-isomorphic choices for $d(\cdot)$ as G has Q-spectral radius 7:

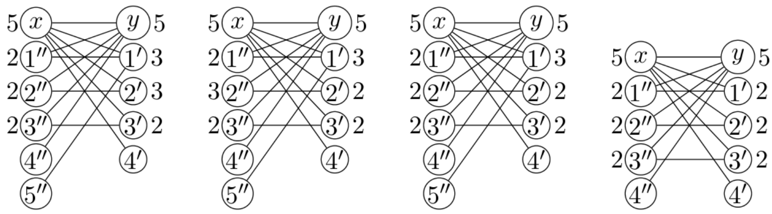
Case 1. $d(i') = 3, d(3') = d(j'') = 2; i = 1, 2; j = 1, 2, 3;$

Case 2. $d(1') = d(2'') = 3, d(2') = d(3') = d(1'') = d(3'') = 2;$

Case 3. $d(1') = 3, d(i') = d(j'') = 2; i = 2, 3; j = 1, 2, 3;$

Case 4. $d(j') = d(j'') = 2; j = 1, 2, 3.$

To complete the proof, we show that all the four cases mentioned above are not possible with the help of various claims. To this end, we begin by showing that the first 3 cases are not possible in Claim 1 and all the remaining claims are used to show that Case 4 can not hold true.



(a) Case 1 (b) Case 2 (c) Case 3 (d) Case 4

Figure 3.

Claim 1. Cases 1,2, and 3 are not possible.

Suppose either of the three cases hold true. Let $N(1') = \{x, 1'', 5''\}$, where $5'' \in V(G) \setminus \Gamma_0$ and $\Gamma_1 = \Gamma_0 \cup \{5''\}$. With respect to each of the considered cases, the subgraph on Γ_1 along with the degrees of its vertices are given in Figures 3(a)-(c). Note that the vertex $5''$ is not adjacent to any other vertex of Γ_0 as the following conditions on G has to hold, that is, G is bipartite, $q(G) = 7$, and $d(2') = 2$ (for Cases 2,3).
However, for all the possible choices of $d(\cdot)$ in $Q_p(\Gamma_1)$, we get either $\lambda_{s2}(Q_p(\Gamma_1)) < 1$ or $\rho(Q_p(\Gamma_1)) > 7$, which contradicts Remark 4 or $q(G) = 7$. Hence, the cases 1, 2, and 3 are not possible which proves our Claim 1.

Claim 2. Case 4 is not possible.

On the contrary, suppose Case 4 is true, that is, $d(j') = d(j'') = 2; j = 1, 2, 3$. The subgraph of G corresponding to the vertices of Γ_0 and their vertex degrees is as given in Figure 3(d). Thus, the principal submatrix $Q_p(\Gamma_0)$ is

$$Q_p(\Gamma_0) = \begin{pmatrix} 5 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 5 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 2 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 2 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & d(4') & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 2 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & d(4'') \end{pmatrix}. \quad (3.3)$$

Claim 2.1. $d(4') = d(4'') = 2$.

The degree of the vertices $4'$ and $4''$ can be at most 3, otherwise $\rho(Q_p(\Gamma_0))$ exceeds 7. Let $d(4') = 3$ such that the distinct vertices $5'', 6'' \in V(G) \setminus \Gamma_0$ are the other neighbors of $4'$. Consider $S_0 = \{5'', 6''\}$, and define $\Gamma_1 = \Gamma_0 \cup S_0$. Here $5'', 6''$ can not be adjacent to any vertices of Γ_1 except $4'$ as G is bipartite and $d(i') = 2 (i = 1, 2, 3)$. Thus, we have $a_{5''j} = a_{6''j} = a_{5''6''} = 0 (j \in \Gamma_0 \setminus \{4'\})$. Now, the matrix $Q_p(\Gamma_1)$ is given by

$$Q_p(\Gamma_1) = \begin{pmatrix} Q_p(\Gamma_0) & A_{\Gamma_0,S_0} \\ A_{S_0,\Gamma_0} & Q_p(S_0) \end{pmatrix} \tag{3.4}$$

where $Q_p(\Gamma_0)$ is given in (3.3), $A_{\Gamma_0,S_0} = A^t_{S_0,\Gamma_0}$,

$$A_{S_0,\Gamma_0} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix},\ Q_p(S_0) = \begin{pmatrix} d(5'') & 0 \\ 0 & d(6'') \end{pmatrix},$$

and $d(4') = 3,\ 1 \le d(4'') \le 3$. Also, $d(4'') = 2$, otherwise we get a contradiction to $\lambda_{s2}(Q(G)) \ge 1$ and $q(G) = 7$ by interlacing theorem. Therefore $(d(4'), d(4'')) = (3, 2)$. Let $4''$ be adjacent to a vertex $5' \in V(G) \setminus \Gamma_1$ and $S_1 = \{5'\}$ and $\Gamma_2 = \Gamma_1 \cup S_1$. In this case, $5'$ is not adjacent to $5'', 6''$, otherwise either $\lambda_{s2}(Q_p(\Gamma_2)) < 1$ or $\lambda_{l2}(Q_p(\Gamma_2)) > 6$ or $\rho(Q_p(\Gamma_2)) > 7$. The principal submatrix corresponding to Γ_2 is

$$Q_p(\Gamma_2) = \begin{pmatrix} Q_p(\Gamma_1) & A_{\Gamma_1,S_1} \\ A_{S_1,\Gamma_1} & Q_p(S_1) \end{pmatrix} \tag{3.5}$$

where $Q_p(\Gamma_1)$ is given in (3.4) with $d(4') = 3$, $d(4'') = 2$, and

$$A_{S_1,\Gamma_1} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix},\ Q_p(S_1) = \begin{pmatrix} d(5') \end{pmatrix}.$$

We have the following three choices for $(d(5''), d(6''), d(5'))$ so that $\lambda_{s2}(Q_p(\Gamma_2)) \ge 1$, and $\lambda_{l2}(Q_p(\Gamma_2)) \le 6, \rho(Q_p(\Gamma_2)) \le 7$: (i) $(3, 2, 4)$, (ii) $(3, 3, 3)$, (iii) $(3, 2, 3)$.
Suppose that $5''$ is adjacent to the vertices of $S_2 = \{6', 7'\} \subseteq V(G) \setminus \Gamma_2$, where $6' \ne 7'$ and $\Gamma_3 = \Gamma_2 \cup S_2$. Therefore, we conclude the following to have $\lambda_{s2}(Q_p(\Gamma_3)) \ge 1$ and $\rho(Q_p(\Gamma_3)) \le 7$:

- $(d(5''), d(6''), d(5')) = (3, 2, 3)$;
- $a_{6'i'} = a_{6'j''} = a_{7'i'} = a_{7'j''} = 0 (i = 1, \ldots, 5; j = 1, \ldots, 4, 6)$ in $Q_p(\Gamma_3)$.

Hence, $d(6'') = 2$ and let $S_3 = \{8'\} \subset V(G) \cap N(6'') \setminus \{\Gamma_3\}$, and define $\Gamma_4 = \Gamma_3 \cup S_3$. Clearly, $8'$ is not adjacent to any vertices of Γ_4 except $6''$ as G is bipartite, $d(j'') = 2 (j = 1, \ldots, 4)$ and $d(5'') = 3$; see Figure 4(a). For every possible choices of $(d(6'), d(7'), d(8'))$, we get either $\lambda_{s2}(Q_p(\Gamma_4)) < 1$ or $\rho(Q_p(\Gamma_4)) > 7$, which is a contradiction by Remark 4. Hence, $d(4') \le 2$ and by symmetric structure given in Figure 3(d), we have $d(4'') \le 2$.
However, if any one of $d(4'), d(4'') < 2$, then $\lambda_{s2}(Q_p(\Gamma_0))$ given in (3.3) becomes less than 1 which contradicts to Remark 4. Therefore, we conclude that $(d(4'), d(4'')) = (2, 2)$ which completes the proof of Claim 2.1.

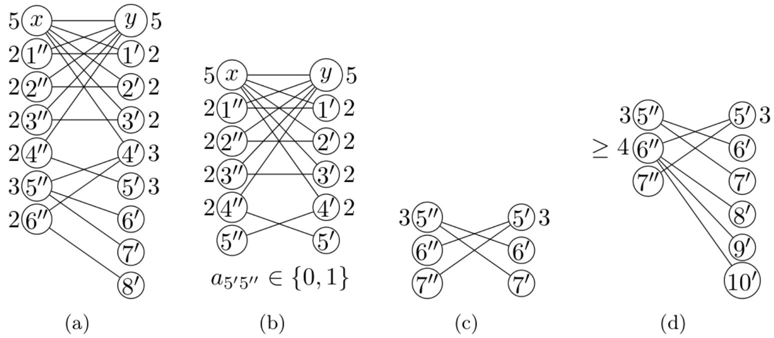


Figure 4.

Next, we look into the neighbors of $4'$ and $4''$ in the subgraph induced by $\Gamma_0 = V(\gamma_3)$. Let $5', 5'' \in V(G) \setminus \Gamma_0$ be the neighbors of $4'', 4'$, respectively. Since G is bipartite, $5' \neq 5''$. Let $\Gamma_1 = \Gamma_0 \cup S_0$, where $S_0 = \{5', 5''\}$. So, we get the subgraph given in Figure 4(b) corresponding to the vertex set Γ_1. The principal submatrix $Q_p(\Gamma_1)$ is given by

$$Q_p(\Gamma_1) = \begin{pmatrix} 5 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 5 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 2 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 2 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & d(5') & a_{5'5''} \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & a_{5'5''} & d(5'') \end{pmatrix}. \tag{3.6}$$

Now, we claim the following about vertices $5'$ and $5''$:

Claim 2.2. $5'$ is not adjacent to $5''$.

Assume on the contrary that $5'$ is adjacent to $5''$. Since G is connected, $d(5') \geq 3$ or $d(5'') \geq 3$. Now, we look into all possible degree pairs for $5'$ and $5''$.

(i) $(d(5'), d(5'')) \in \{(2, 3), (3, 3)\}$: Let $6' \in V(G) \setminus \Gamma_1$ be the remaining neighbor of $5''$. Obviously, $5'$ is not adjacent to $6'$ as G is bipartite. Define $\Gamma_2 = \Gamma_1 \cup S_1$, where $S_1 = \{5''\}$. Thus, the matrix $Q_p(\Gamma_2)$ is

$$Q_p(\Gamma_2) = \begin{pmatrix} Q_p(\Gamma_1) & A_{\Gamma_1, S_1} \\ A_{S_1, \Gamma_1} & Q_p(S_1) \end{pmatrix}, \tag{3.7}$$

where $Q_p(\Gamma_1)$ is given in (3.6) with $d(5') \in \{2, 3\}, d(5'') = 3, a_{5'5''} = 1$, and

$$A_{S_1, \Gamma_1} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},\ Q_p(S_1) = \begin{pmatrix} d(6') \end{pmatrix}.$$

Now $(d(5'), d(5'')) \neq (2, 3)$, otherwise $\lambda_{s2}(Q_p(\Gamma_2)) \leq 0.0669 < 1$. Let $d(5') = 3$ and $6'' \in V(G) \setminus \Gamma_2$ be the remaining neighbor of $5'$. Define $\Gamma_3 = \Gamma_2 \cup S_2$, where $S_2 = \{6''\}$. For each admissible choices of $d(\cdot), a_{\cdot\cdot}$, we get a contradiction to $\lambda_{s2}(Q_p(\Gamma_3)) \geq 1$. Hence, $(d(5'), d(5'')) \neq (3, 3)$. Therefore, (i) is not possible.

(ii) $((d(5'), d(5'')) = (4,4)$: Let $S_1 = \{6', 7', 6'', 7''\} \subseteq V(G) \setminus \Gamma_1$ be a set of distinct vertices in G and define $\Gamma_2 = \Gamma_1 \cup S_1$. Assume that $5''$ (resp. $5'$) is adjacent to $6', 7'$ (resp. $6'', 7''$). The matrix A_{S_1,Γ_1} and $Q_p(S_1)$ are given by

$$A_{S_1,\Gamma_1} = \left(\mathbf{0} \;\middle|\; \begin{smallmatrix} 0 & 1 \\ 0 & 1 \\ 1 & 0 \\ 1 & 0 \end{smallmatrix} \right), \; Q_p(S_1) = \begin{pmatrix} d(6') & 0 & a_{6'6''} & a_{6'7''} \\ 0 & d(7') & a_{7'6''} & a_{7'7''} \\ a_{6'6''} & a_{7'6''} & d(6'') & 0 \\ a_{6'7''} & a_{7'7''} & 0 & d(7'') \end{pmatrix}.$$

Here, $a_{i'j''} = 0, \forall i, j = \{6, 7\}$ and $d(7'') = 3$ in G to have $\lambda_{s2}(Q_p(\Gamma_2)) \geq 1$ or $\lambda_{l2}(Q_p(\Gamma_2)) \leq 6$ or $\rho(Q_p(\Gamma_2)) = 7$. Let $S_2 = \{8', 9'\} \subseteq V(G) \setminus \Gamma_2$ be a subset of $N(7'')$ containing distinct vertices and define $\Gamma_3 = \Gamma_2 \cup S_2$. However, $\lambda_{s2}(Q_p(\Gamma_3)) < 1$, which contradicts to Remark 4. Therefore, $(d(5'), d(5'')) \neq (4, 4)$.

(iii) $(d(5'), d(5'')) \in \{(3,4), (3,5), (4,5), (5,5)\}$: Similar to **(ii)**, it can be verified using Remark 4 and $q(G) = 7$ that this subcase is not possible.

From above, we conclude that Claim 2.2 holds, i.e., $a_{5'5''} = 0$ in $Q_p(\Gamma_1)$ given by (3.6).
From Claim 2.2, the induced subgraph $G[\Gamma_1]$ is as shown in Figure 4(b). Now, we look into the degrees of $d(5')$ and $d(5'')$.

Claim 2.3. $d(5') \geq 3$ and $d(5'') \geq 3$.

The claim holds otherwise $\lambda_{s2}(Q_p(\Gamma_1)) \leq 0.9194 < 1$, where $Q_p(\Gamma_1)$ given in (3.6).

Claim 2.4. $(d(5'), d(5'')) \neq (3, 3)$.

Assume on the contrary that $d(5') = d(5'') = 3$. Let $S_1 = \{6', 7', 6'', 7''\} \subseteq V(G) \setminus \Gamma_1$ be a set of distinct vertices in G such that $5''$ (resp. $5'$) is adjacent to $6', 7'$ (resp. $6'', 7''$), see Figure 4(c). The corresponding submatrix $Q_p(\Gamma_1)$ of the principal submatrix $Q_p(\Gamma_2)$ in (3.1), where $\Gamma_2 = \Gamma_1 \cup S_1$ is given in (3.6) with $a_{5'5''} = 0$, and A_{S_1,Γ_1}, $Q_p(S_1)$ are

$$A_{S_1,\Gamma_1} = \left(\mathbf{0} \;\middle|\; \begin{smallmatrix} 0 & 1 \\ 0 & 1 \\ 1 & 0 \\ 1 & 0 \end{smallmatrix} \right), \; Q_p(S_1) = \begin{pmatrix} d(6') & 0 & a_{6'6''} & a_{6'7''} \\ 0 & d(7') & a_{7'6''} & a_{7'7''} \\ a_{6'6''} & a_{7'6''} & d(6'') & 0 \\ a_{6'7''} & a_{7'7''} & 0 & d(7'') \end{pmatrix}. \tag{3.8}$$

From $q(G) = 7$, Remark 4 on $Q_p(\Gamma_2)$, we have $a_{i'j''} = 0; i, j \in \{6, 7\}$ and $d(6'') \geq 4$. Let $S_2 = \{8', 9', 10'\} \subseteq N(6'') \setminus \Gamma_2$ be set of distinct vertices and $\Gamma_3 = \Gamma_2 \cup S_2$, see Figure 4(d). However, for each choices of $a_{.,.}, d(\cdot)$, we get a contradiction to Remark 4. Hence, $(d(5'), d(5'')) \neq (3, 3)$.

Claim 2.5. $d(i) \neq 5$ for $i = 5', 5''$.

Let $d(5'') = 5$ and $S_1 = \{6', 7', 8', 9'\} \subseteq (V(G) \setminus \Gamma_1) \cap N(5'')$ be a set of distinct vertices in G and define $\Gamma_2 = \Gamma_1 \cup S_1$, see Figure 5(a). We get a contradiction to Remark 4 for $Q_p(\Gamma_2)$. Therefore, neither $d(5') = 5$ nor $d(5'') = 5$.

Claim 2.6. $d(5'), d(5'') \neq 4$.

Assume on the contrary that the claim holds. From Claim 2.3-2.5, we have the non-isomorphic choices for $(d(5'), d(5''))$ as $(3, 4)$, $(4, 4)$.

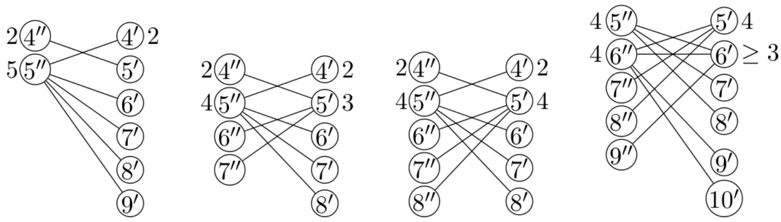


Figure 5.

Case 2.6.1. $(d(5'), d(5'')) = (3, 4)$.

Let $S_1 = \{6', 7', 8', 6'', 7''\} \subseteq V(G) \setminus \Gamma_1$ be a set of distinct vertices such that $\{6', 7', 8'\} \subseteq N(5'')$ and $\{6'', 7''\} \subseteq N(5')$, see Figure 5(b). Define $\Gamma_2 = \Gamma_1 \cup S_1$, and the principal submatrix $Q_p(\Gamma_2)$ given by (3.1) contains $Q_p(\Gamma_1)$ as given in (3.6). Hence, each neighbor of $5'$ can be adjacent to at most one vertex of $\{6', 7', 8'\}$ in S_1, otherwise either $\lambda_{s2}(Q_p(\Gamma_2)) < 1$ or $\lambda_{l2}(Q_p(\Gamma_2)) > 6$. Since G is bipartite, we have the following non-isomorphic cases: (2.6.1.1) $E(S_1) = \{6'6'', 6'7''\}$, (2.6.1.2) $E(S_1) = \{6'6'', 7'7''\}$, (2.6.1.3) $E(S_1) = \{6'6''\}$, (2.6.1.4) $E(S_1) = \phi$.
Now, we analyze these possible cases. Here, (2.6.1.1), (2.6.1.2) are not possible since either $\lambda_{s2}(Q_p(\Gamma_2)) < 1$ or $\lambda_{l2}(Q_p(\Gamma_2)) > 6$.

(2.6.1.3) $E(S_1) = \{6'6''\}$: Degree of $6''$ is at least 4, otherwise $\lambda_{s2}(Q_p(\Gamma_2)) \leq 0.9923 < 1$. Let $6''$ be adjacent to the set of distinct vertices $S_2 = \{9', 10'\} \subseteq V(G) \setminus \Gamma_1$, and define $\Gamma_3 = \Gamma_2 \cup S_2$. However, for all the admissible choices of $d(\cdot)$ and $a_{..}$ in $Q_p(\Gamma_3)$, either $\lambda_{s2}(Q_p(\Gamma_3)) < 1$ or $\lambda_{l2}(Q_p(\Gamma_3)) > 6$. Hence, $E(S_1) \neq \{6'6''\}$.

(2.6.1.4) $E(S_1) = \phi$: Similar to (2.6.1.3), it can be verified that this case is not possible using Remark 4 and $q(G) = 7$.

Therefore, Case 2.6.1 is not valid i.e., $(d(5'), d(5'')) \neq (3, 4)$.

Case 2.6.2. $(d(5'), d(5'')) = (4, 4)$.

Let $S_1 = \{6', 7', 8', 6'', 7'', 8''\} \subseteq V(G) \setminus \Gamma_1$ be a set of distinct vertices such that $\{6', 7', 8'\} \subset N(5'')$ and $\{6'', 7'', 8''\} \subseteq N(5')$, see Figure 5(c). Define $\Gamma_2 = \Gamma_1 \cup S_1$. Now, $i'' \in N(5')(i = 6, 7, 8)$ is adjacent to at most one vertex of $\{6', 7', 8'\}$, otherwise either $\lambda_{s2}(Q_p(\Gamma_2)) < 1$ or $\lambda_{l2}(Q_p(\Gamma_2)) > 6$. Since G is bipartite, we have the following non-isomorphic choices for $E(S_1)$: (2.6.2.1) $E(S_1) = \{6'6'', 7'7'', 8'8''\}$, (2.6.2.2) $E(S_1) = \{6'6'', 7'7''\}$, (2.6.2.3) $E(S_1) = \{6'6''\}$, (2.6.2.4) $E(S_1) = \phi$. Now, we analyze these cases.

(2.6.2.1) $E(S_1) = \{6'6'', 7'7'', 8'8''\}$: This case is not possible, otherwise either we get $\lambda_{s2}(Q_p(\Gamma_2)) < 1$ or $\lambda_{l2}(Q_p(\Gamma_2)) > 6$ or $\rho(Q_p(\Gamma_2)) > 7$.

(2.6.2.2) $E(S_1) = \{6'6'', 7'7''\}$: We have $d(6'') \geq 3$, otherwise we arrive at a contradiction using Remark 4. Suppose that $S_2 = \{9''\} \subseteq N(6'')$ be such that $\Gamma_2 \cap S_2 = \phi$ and define $\Gamma_3 = \Gamma_2 \cup S_2$. Then for each possible submatrix $Q_p(\Gamma_3)$ in (3.1), we get $\lambda_{l2}(Q_p(\Gamma_3)) > 6$, which is a contradiction. Therefore, $E(S_1) \neq \{6'6'', 7'7''\}$.

(2.6.2.3) $E(S_1) = \{6'6''\}$: Here we have $d(6'), d(6'') \geq 3$, otherwise $\lambda_{s2}(Q_p(\Gamma_2)) \leq 0.9893$. Let $S_2 = \{9', 9''\} \subseteq V(G) \setminus \Gamma_2$ be such that $9'$ (resp. $9''$) is adjacent to $6''$ (resp. $6'$). Since G is bipartite, we have $9' \neq 9''$. Define $\Gamma_3 = \Gamma_2 \cup S_2$. We have $N(9') \cap \{7'', 8'', 9''\} = \phi = N(9'') \cap \{7', 8', 9'\}$ and $d(6'') = 4$, otherwise we get a contradiction to Remark 4. Let $S_3 = \{10'\} \subseteq N(6'') \setminus \Gamma_3$ and define $\Gamma_4 = \Gamma_3 \cup S_3$, see Figure 5(d). Now for all the possible choices of $d(\cdot)$, we arrive at a contradiction to Remark 4 for $Q_p(\Gamma_4)$. Hence $E(S_1) \neq \{6'6''\}$.

(2.6.2.4) $E(S_1) = \phi$**:** Similar to (2.6.2.3), this case is not possible due to Remark 4, $q(G) = 7$, and bipartieness of G.

Therefore Claim 2.6 holds i.e., $d(i) \neq 4$ for all $i = 5', 5''$.
Further from Claims 2.1- 2.6, we conclude that Case 4 does not hold. Finally from Claim 1 and 2, we obtain that G is γ_3-free.

Next, we prove the second part of the theorem, namely **(b)**, by contradiction. Let us assume that G is non-bipartite and has an induced subgraph γ_0. Thus $\lambda_{\min}(Q(G)) \geq 1$ by Proposition 1, and hence $\lambda_{\min}(Q_p(H)) \geq 1$ for every $H \subseteq V(G)$. The principal submatrix corresponding to Γ_0 is

$$Q_p(\Gamma_0) = \begin{pmatrix} 5 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 5 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & d(1') & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & d(2') & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & d(3') & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & d(4') & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & d(1'') & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & d(2'') & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & d(3'') & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & d(4'') \end{pmatrix}.$$

There exist at least one $i, j \in \{1, 2, 3, 4\}$ such that $d(i'), d(j'') \geq 3$, otherwise we get a contradiction to the Remark 4 with $\lambda_{\min}(Q_p(\Gamma_0)) \leq 0.9317 < 1$. Without any loss of generality, let $d(1'), d(1'') \geq 3$ and $S_0 = \{5', 6', 5'', 6''\} \subseteq V(G) \setminus \Gamma_0$ be a set of distinct vertices such that $5', 6'$ (resp. $5'', 6''$) are adjacent to $1''$ (resp. $1'$) in G. Thus the principal submatrix $Q_p(\Gamma_1)$ where $\Gamma_1 = \Gamma_0 \cup S_0$, is given by

$$Q_p(\Gamma_1) = \begin{pmatrix} Q_p(\Gamma_0) & A_{\Gamma_0, S_0} \\ A_{S_0, \Gamma_0} & Q_p(S_0) \end{pmatrix},$$

$$\text{where} \qquad A_{S_0,\Gamma_0} = \left(\begin{array}{c|cccc|cccc} \mathbf{0} & a_{1'5'} & a_{2'5'} & a_{3'5'} & a_{4'5'} & 1 & a_{2''5'} & a_{3''5'} & a_{4''5'} \\ & a_{1'6'} & a_{2'6'} & a_{3'6'} & a_{4'6'} & 1 & a_{2''6'} & a_{3''6'} & a_{4''6'} \\ \hline \mathbf{0} & 1 & a_{2'5''} & a_{3'5''} & a_{4'5''} & a_{1''5''} & a_{2''5''} & a_{3''5''} & a_{4''5''} \\ & 1 & a_{2'6''} & a_{3'6''} & a_{4'6''} & a_{1''6''} & a_{2''6''} & a_{3''6''} & a_{4''6''} \end{array}\right).$$

We have the following claims due to the fact that $\lambda_{\min}(Q_p(\Gamma_1)) \geq 1$ and $\rho(Q_p(\Gamma_1)) \leq 7$:

- $a_{5'i} = a_{6'i} = a_{5''j} = a_{6''j} = 0$, for $i \in \Gamma_0 \setminus \{1''\}, j \in \Gamma_0 \setminus \{1'\}$;
- $d(1') = d(1'') = 3, a_{5'i''} = a_{6'i''} = 0, i = 5, 6$;
- $d(i') \geq 3$, for some $i \in \{2', 3', 4'\}$; say $d(2') \geq 3$.

Thus the matrix $A_{S_0,\Gamma_0}, Q_p(S_0)$ now becomes

$$A_{S_0,\Gamma_0} = \left(\begin{array}{c|c|c|c|c} \mathbf{0} & \begin{smallmatrix} 0 \\ 0 \end{smallmatrix} & \mathbf{0} & \begin{smallmatrix} 1 \\ 1 \end{smallmatrix} & \mathbf{0} \\ \hline \mathbf{0} & \begin{smallmatrix} 1 \\ 1 \end{smallmatrix} & \mathbf{0} & \begin{smallmatrix} 0 \\ 0 \end{smallmatrix} & \mathbf{0} \end{array}\right), \; Q_p(S_0) = \left(\begin{array}{cc|cc} d(5') & a_{5'6'} & \multicolumn{2}{c}{\mathbf{0}} \\ a_{5'6'} & d(6') & & \\ \hline \multicolumn{2}{c|}{\mathbf{0}} & d(5'') & a_{5''6''} \\ & & a_{5''6''} & d(6'') \end{array}\right).$$

Let $7''$ be a neighbor of $2'$ in G, where $S_1 = \{7''\} \subset V(G) \setminus \Gamma_1$, and thus $\Gamma_2 = \Gamma_1 \cup S_1$. Now for all the choices of $1 \leq d(\cdot) \leq 5$ and $a_{\cdot\cdot} \in \{0, 1\}$, we get either $\lambda_{\min}(Q_p(\Gamma_2)) < 1$ or $\rho(Q_p(\Gamma_2)) > 7$, which contradicts to the fact that $\lambda_{s2}(Q(G)) \geq 1$ and $q(G) = 7$. Therefore, G must be a bipartite graph if it contains γ_0 as an induced subgraph.

4. Conclusion

We have improved one of our earlier results from [9] on the structural characterization of Q-integral connected graph G having $q(G) = 7$ and maximum edge-degree 8. We have shown that G must be a bipartite graph. If $G \neq K_{1,4} \square K_2$, then G contains one of the three graphs, namely $\mathcal{S}_4^2(m)(m = 0, 1, 2)$ as an induced subgraph. Further, $0, 1, 6$ and 7 are Q-eigenvalues of G.
Thus, we conclude that whenever $G \notin \{C_3, C_6, \overline{K_{3,3} - e}\}$ is a connected Q-integral graph with maximum edge-degree $2q(G) - 6$, then G is bipartite and $\mathcal{S}^2_{q(G)-3}(q(G) - 4)$-free for $q(G) \leq 7$.

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