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Research Article

On the strength and independence number of powers of paths and cycles

Rikio Ichishima[1,*], Francesc Antoni Muntaner-Batle[2] and Yukio Takahashi[3]

[1]Department of Sport and Physical Education, Faculty of Physical Education, Kokushikan University, 7-3-1 Nagayama, Tama-shi, Tokyo 206-8515, Japan
*ichishim@kokushikan.ac.jp

[2]Graph Theory and Applications Research Group, School of Electrical Engineering and Computer Science, Faculty of Engineering and Built Environment, The University of Newcastle, NSW 2308 Australia
famb1es@yahoo.es

[3]Department of Science and Engineering, Faculty of Electronics and Informatics, Kokushikan University, 4-28-1 Setagaya, Setagaya-ku, Tokyo 154-8515, Japan
takayu@kokushikan.ac.jp

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Abstract: A numbering f of a graph G of order n is a labeling that assigns distinct elements of the set $\{1, 2, \ldots, n\}$ to the vertices of G. The strength $\mathrm{str}(G)$ of G is defined by $\mathrm{str}(G) = \min\left\{\mathrm{str}_f(G) \,|\, f \text{ is a numbering of } G\right\}$, where $\mathrm{str}_f(G) = \max\{f(u) + f(v) \,|\, uv \in E(G)\}$. Using the concept of independence number of a graph, we determine formulas for the strength of powers of paths and cycles. To achieve the latter result, we establish a sharp upper bound for the strength of a graph in terms of its order and independence number and a formula for the independence number of powers of cycles.

Keywords: strength, independence number, kth power of a graph, graph labeling, combinatorial optimization.

AMS Subject classification: 05C12, 05C38, 05C69, 05C78, 90C27.

1. Introduction

Only graphs without loops or multiple edges will be considered in this paper. Undefined graph theoretical notation and terminology can be found in [2] or [22]. The *vertex set* of a graph G is denoted by $V(G)$, while the *edge set* of G is denoted by $E(G)$. The *path*, *cycle* and *complete graph* of order n are denoted by P_n, C_n and K_n, respectively.

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* *Corresponding Author*

For a connected graph G, the *distance* $d_G(u,v)$ between two vertices u and v is defined to be the minimum length among all $u-v$ paths in G. The greatest distance between any two vertices of a connected graph G is called the *diameter* of G and is denoted by $\operatorname{diam}(G)$. For example, $\operatorname{diam}(P_n) = n-1$ $(n \geq 2)$ and $\operatorname{diam}(C_n) = \lfloor n/2 \rfloor$ $(n \geq 3)$. The kth *power* G^k of a graph G of order n, where $k \geq 1$, is the graph with $V(G^k) = V(G)$ for which $uv \in E(G^k)$ if and only if $1 \leq d_G(u,v) \leq k$. Thus, $G^1 = G$ and $G^k = K_n$ if $k \geq \operatorname{diam}(G)$.

An extensive survey on graph labeling problems as well as their applications has been given by Gallian [8]. Among all graph labeling problems, bandwidth numbering of graphs has perhaps attracted the most attention in the literature. The bandwidth numbering was proposed by Harper [12] and later rediscovered by Harary [11]. Since then, a considerable amount of papers have been published on this subject. Survey articles on bandwidth numberings and related topics can be found in Chinn et al. [3] and Chung [4]. Readers interested in more recent information on bandwidth numberings may consult the survey by Lai and Williams [21], which also includes information on other kinds of graph labeling problems and their applications.

For the sake of notational convenience, we will denote the interval of integers k such that $i \leq k \leq j$ by simply writing $[i,j]$. A *numbering* f of a graph G of order n is a labeling that assigns distinct elements of the set $[1,n]$ to the vertices of G. The *bandwidth* $\operatorname{band}_f(G)$ *of a numbering* $f : V(G) \to [1,n]$ of G is defined by

$$\operatorname{band}_f(G) = \max\left\{|f(u) - f(v)| \;\middle|\; uv \in E(G)\right\},$$

and the *bandwidth* $\operatorname{band}(G)$ of a graph G is

$$\operatorname{band}(G) = \min\{\operatorname{band}_f(G) \,|\, f \text{ is a numbering of } G\}.$$

Therefore, it follows that $1 \leq \operatorname{band}(G) \leq n-1$ for every nonempty graph G of order n. It is also known from [5] that a graph G of order n has bandwidth k $(k \in [1,n-1])$ if and only if k is the smallest positive integer for which G is a subgraph of P_n^k. It follows that $\operatorname{band}(P_n^k) = k$ $(k \in [1,n-1])$.

An additive analogous to bandwidth numberings of graphs has been introduced and studied in [13] as a generalization of the problem of determining whether a graph is super edge-magic or not (see [6] for the definition of a super edge-magic graph and its super edge-magic labeling, and also consult either [1] or [7] for alternative and often more useful definitions of the same concept). A necessary and sufficient condition for a graph to be super edge-magic established by Figueroa-Centeno et al. [7] gives rise to the concept of the consecutive strength labeling of a graph (see [13] for the definition of a consecutive strength labeling of a graph), which is equivalent to super edge-magic labeling. The *strength* $\operatorname{str}_f(G)$ *of a numbering* $f : V(G) \to [1,n]$ of a graph G is defined by

$$\operatorname{str}_f(G) = \max\{f(u) + f(v) \,|\, uv \in E(G)\},$$

and the *strength* str(G) of G is

$$\text{str}\,(G) = \min\left\{\text{str}_f\,(G)\,|\, f \text{ is a numbering of } G\right\}.$$

Therefore, it follows that $3 \leq \text{str}\,(G) \leq 2n-1$ for every nonempty graph G of order n.

The independence number is an essential parameter to assess the resilience of the interconnection networks of multiprocessor systems modeled by a graph. A set S of vertices in a graph G is *independent* if no two vertices in S are adjacent. The maximum number of vertices in an independent set of vertices of G is called the *independence number* of G and is denoted by $\beta\,(G)$.

The decision problem associated with determining the independence number of an arbitrary graph is known to be NP-complete (see [10]). Hence, the direction of research is either to find sharp bounds for the independence number of a graph or to find exact values for special classes of graphs. In this paper, we focus on the powers of paths and cycles and determine formulas for the strength of these classes of graphs.

Before concluding this introduction, it is worth mentioning that Ichishima et al. [15] investigated certain minimum degree conditions concerning the strength of graphs. In the same paper, they also determined certain degree sequences of graphs that naturally arise when studying the strength and proved that these degree sequences determine unique graph realizations. Furthermore, in [15], using a similar reasoning to the one on strength result, the authors were able to establish a parallel result relying on degree sequences for the bandwidth of graphs.

2. Results on powers of paths and cycles

To present some results involving powers of paths and cycles in this section, we introduce a few technical lemmas that will prove to be useful.

Several bounds for the strength of a graph have been found in terms of other parameters defined on graphs. Among others, the following result established in [13] that provides a lower bound for the strength of a graph G in terms of its order and minimum degree $\delta\,(G)$ has proven to be particularly useful.

Lemma 1. *For every graph G of order n with $\delta\,(G) \geq 1$,*

$$\text{str}\,(G) \geq n + \delta\,(G)\,.$$

It is worth to mention that the bound given in Lemma 1 is sharp in the sense that there are infinitely many graphs G for which $\text{str}\,(G) = |V\,(G)| + \delta\,(G)$ (see [9, 13, 14, 17–20] for a detailed list of such graphs and other sharp bounds). In fact, it was shown in [16] that for every $k \in [1, n-1]$, there exists a graph G of order n satisfying $\delta\,(G) = k$ and $\text{str}\,(G) = n + k$.

Answering an open question posed in [13], Gao et al. [9] found the next result, which establishes another lower bound for the strength of a graph in terms of its order and independence number.

Lemma 2. *For every graph G of order n,*

$$\operatorname{str}(G) \geq 2n - 2\beta(G) + 1.$$

Before presenting our first result, we introduce one more technical lemma discovered by Chvátal [5].

Lemma 3. *For every integer n with $n \geq k+1$,*

$$\beta\left(P_n^k\right) = \left\lceil \frac{n}{k+1} \right\rceil,$$

where $k \in [2, n-1]$.

With the aid of Lemmas 2 and 3, it is now possible to present a formula for $\operatorname{str}\left(P_n^k\right)$, which is the analogous result for the bandwidth of the same graph obtained by Chvátal [5]. This extends the result found in [13] that $\operatorname{str}(P_n) = n+1$ for integers $n \geq 2$.

Theorem 1. *For every two integers n and k with $n \geq k+1$,*

$$\operatorname{str}\left(P_n^k\right) = 2n - 2\left\lceil \frac{n}{k+1} \right\rceil + 1,$$

where $k \in [2, n-1]$.

Proof. For integers n and k with $n \geq k+1$, where $k \in [2, n-1]$, let $G = P_n^k$ and define the graph G with $V(G) = \{x_i \mid i \in [1, n]\}$ and

$$\begin{aligned} E(G) = {} & \{x_i x_{i+1} \mid i \in [1, n-1]\} \cup \{x_i x_{i+2} \mid i \in [1, n-2]\} \\ & \cup \{x_i x_{i+3} \mid i \in [1, n-3]\} \cup \cdots \cup \{x_i x_{i+k} \mid i \in [1, n-k]\}. \end{aligned}$$

Moreover, if we let $\beta = \lceil n/(k+1) \rceil$, then $V(G)$ can be partitioned into three sets

$$\begin{aligned} S_1 &= \left\{x_{(k+1)i-k} \mid i \in [1, \beta]\right\}, \\ S_2 &= \left\{x_{(k+1)i-k+1} \mid i \in [1, \beta]\right\}, \\ S_3 &= V(G) - (S_1 \cup S_2). \end{aligned}$$

Consequently, each S_i $(i = 1, 2)$ is an independent set of cardinality β (that is, every two vertices u and v of S_i are not adjacent) and $|S_3| = n - 2\beta$.

Suppose that $S_3 = \{x_{s_i} \mid i \in [1, n-2\beta]\}$, where the subindices $s_1, s_2, \ldots, s_{n-2\beta}$ are ordered in such a way that

$$s_1 < s_2 < \cdots < s_{n-2\beta}.$$

Note that if we rename the vertices of S_3 using the labels of elements in the set $\{y_i \mid i \in [1, n-2\beta]\}$ and define the bijective function $\pi : S_3 \to \{y_i \mid i \in [1, n-2\beta]\}$ by

$$\pi(x_{s_i}) = y_i \ (i \in [1, n-2\beta]),$$

then we have

$$S_3 = \{y_i \mid i \in [1, n-2\beta]\}.$$

Now, consider the numbering $f : V(G) \to [1, n]$ such that

$$f(v) = \begin{cases} n+1-i & \text{if } v = x_{(k+1)i-k} \text{ and } i \in [1, \beta] \\ n-2\beta+i & \text{if } v = x_{(k+1)i-k+1} \text{ and } i \in [1, \beta] \\ i & \text{if } v = y_i \text{ and } i \in [1, n-2\beta]. \end{cases}$$

Then we have

$$\begin{aligned} \{f(v) \mid v \in S_3\} &= [1, n-2\beta], \\ \{f(v) \mid v \in S_2\} &= [n-2\beta+1, n-\beta], \\ \{f(v) \mid v \in S_1\} &= [n-\beta+1, n]. \end{aligned}$$

With the preceding knowledge in hand, we consider four cases.

Case 1. Suppose that $uv \in E(G)$, where $u \in S_1$ and $v \in S_2$. Then, without loss of generality, assume that $u = x_{(k+1)i-k}$ and $v = x_{(k+1)i-k+1}$ for all $i \in [1, \beta]$; otherwise, $d_{P_n}(u, v) > k$, that is, $uv \notin E(G)$. Thus,

$$\begin{aligned} f(u) + f(v) &= f\left(x_{(k+1)i-k}\right) + f\left(x_{(k+1)i-k+1}\right) \\ &= (n+1-i) + (n-2\beta+i) = 2n-2\beta+1. \end{aligned}$$

Case 2. Suppose that $uv \in E(G)$, where $u \in S_1$ and $v \in S_3$. Then

$$f(u) + f(v) \le (n-\beta) + (n-2\beta+1) = 2n-3\beta+1 < 2n-2\beta+1.$$

Case 3. Suppose that $uv \in E(G)$, where $u \in S_2$ and $v \in S_3$. Then

$$f(u) + f(v) \le (n-\beta) + (n-2\beta) = 2n-3\beta < 2n-2\beta+1.$$

Case 4. Suppose that $uv \in E(G)$, where $u, v \in S_3$. Then

$$f(u) + f(v) \leq (n - 2\beta) + (n - 2\beta - 1) = 2n - 4\beta - 1 < 2n - 2\beta + 1.$$

Therefore, f has the property that for all $i \in [1, \beta]$,

$$\begin{aligned} \mathrm{str}_f(G) &= \max\{f(u) + f(v) \mid uv \in E(G)\} = f\left(x_{(k+1)i-k}\right) + f\left(x_{(k+1)i-k+1}\right) \\ &= (n + 1 - i) + (n - 2\beta + i) = 2n - 2\beta + 1 = 2n - 2\left\lceil \frac{n}{k+1} \right\rceil + 1, \end{aligned}$$

proving that $\mathrm{str}(G) \leq 2n - 2\lceil n/(k+1) \rceil + 1$. It remains only to observe that the reverse inequality is obtained from Lemmas 2 and 3. $\square$

Recall from Lemma 3 that $\beta = \beta\left(P_n^k\right) = \lceil n/(k+1) \rceil$. It is important to observe then that the labeling f of S_3 presented in the proof of Theorem 1 can be replaced by any labeling that assigns distinct elements of the set $[1, n - 2\beta]$ to the vertices of S_3. This means that the numbering f of P_n^k with the property that

$$\begin{aligned} \{f(v) \mid v \in S_3\} &= [1, n - 2\beta], \\ \{f(v) \mid v \in S_2\} &= [n - 2\beta + 1, n - \beta], \\ \{f(v) \mid v \in S_1\} &= [n - \beta + 1, n] \end{aligned}$$

is not unique. In fact, there are at least $(n - 2\beta)!$ such labelings. Observe also that the same labeling f has the additional property that there are exactly β edges for which

$$\mathrm{str}_f\left(P_n^k\right) = 2n - 2\beta + 1,$$

namely, $x_{(k+1)i-k}x_{(k+1)i-k+1}$ $(i \in [1, \beta])$. These observations together with the following two lemmas lead to an analogous proof to determine a formula for $\mathrm{str}\left(C_n^k\right)$. This extends the result found in [13] that $\mathrm{str}(C_n) = n + 2$ for integers $n \geq 3$.
The next lemma provides an upper bound for the strength of a graph in terms of its order and independence number.

Lemma 4. *For every graph G of order n,*

$$\mathrm{str}(G) \leq 2n - \beta(G).$$

Proof. Let S be a maximum independent set of vertices of G. Then $|S| = \beta(G)$. Let $S = \{v_i \mid i \in [1, \beta(G)]\}$, and consider a numbering $f : V(G) \to [1, n]$ such that $f(S) = [n - \beta(G) + 1, n]$. Since no two vertices in S are adjacent, it follows that the maximum edge label induced by $f(u) + f(v)$, where $uv \in E(G)$, is at most $n + (n - \beta) = 2n - \beta(G)$. $\square$

For two graphs G_1 and G_2 with disjoint vertex sets, the *join* $G = G_1 + G_2$ has $V(G) = V(G_1) \cup V(G_2)$ and

$$E(G) = E(G_1) \cup E(G_2) \cup \{uv \mid u \in V(G_1) \text{ and } v \in V(G_2)\}.$$

Now, consider the graph $G = K_n + nK_1$ $(n \geq 1)$. Then we have $|V(G)| = 2n$ and $\delta(G) = \beta(G) = n$. Thus, we obtain

$$\operatorname{str}(G) \geq 2n + n = 3n$$

by Lemma 1, whereas we obtain

$$\operatorname{str}(G) \leq 4n - n = 3n$$

by Lemma 2. Consequently, we have the following result, which shows the sharpness of the bound given in Lemma 4.

Corollary 1. *For every positive integer n,*

$$\operatorname{str}(K_n + nK_1) = 3n.$$

A simple but interesting consequence of Lemmas 1 and 4 concerns an efficiently computable upper bound for the independence number of a graph in terms of its order and minimum degree.

Corollary 2. *For every graph G of order n,*

$$\beta(G) \leq n - \delta(G).$$

The bound given in the preceding result is sharp since it is attained, for instance, by the families of complete graphs K_n $(n \geq 1)$ and the graph $K_n + nK_1$ $(n \geq 1)$.

Let G be a graph with clique number $\omega(G)$ (the order of the largest complete subgraph of G) and let f be any numbering of G. Let $v_1, v_2, \ldots, v_{\omega(G)}$ be the vertices of a clique of G. Then the minimum labels that the vertices that form the clique may have are $1, 2, \ldots, \omega(G)$. This implies that there exists an edge $v_i v_j$ $(1 \leq i < j \leq \omega(G))$ in the clique such that $f(v_i) + f(v_j) = 2\omega(G) - 1$. Thus, the strength and clique numbers are related as follows: $\operatorname{str}(G) \geq 2\omega(G) - 1$.

The next lemma provides a formula for the independence number of powers of cycles.

Lemma 5. *For every two integers n and k with $n \geq k + 2$,*

$$\beta\left(C_n^k\right) = \left\lfloor \frac{n}{k+1} \right\rfloor,$$

where $k \in [2, \lfloor n/2 \rfloor]$.

Proof. We begin by showing that $\beta\left(C_n^k\right) \leq \lfloor n/(k+1) \rfloor$. Assume, to the contrary, that $\beta\left(C_n^k\right) > \lfloor n/(k+1) \rfloor$, and define the cycle C_n with

$$V(C_n) = \{x_i \mid i \in [1,n]\} \text{ and } E(C_n) = \{x_1 x_n\} \cup \{x_i x_{i+1} \mid i \in [1, n-1]\}.$$

We consider two cases, depending on whether n is a multiple of $k+1$.

Case 1. Suppose that $n = (k+1)l$ for some positive integer l. Then $\lfloor n/(k+1) \rfloor = l$. Let $S = \{x_{s_i} \mid i \in [1, l+1]\}$ be an independent set of cardinality $l+1$ in C_n^k, where the elements are listed in clockwise order as $x_{s_1}, x_{s_2}, \ldots, x_{s_{l+1}}$. Since S is an independent set of vertices in C_n^k, it follows that $d_{C_n}\left(x_{s_i}, x_{s_{i+1}}\right) \geq k+1$ for each $i \in [1, l]$; otherwise, $x_{s_i} x_{s_{i+1}} \in E\left(C_n^k\right)$ and hence S is not an independent set of vertices in C_n^k. Also, $d_{C_n}\left(x_{s_1}, x_{s_{l+1}}\right) \geq k+1$ for the same reason. This implies that

$$|E(C_n)| \geq (k+1)(l+1) > (k+1)l = n = |E(C_n)|,$$

which is impossible.

Case 2. Suppose that $n = (k+1)l + r$ for some positive integers l and $r \in [1, k]$. Then $\lfloor n/(k+1) \rfloor = l$. Let $S = \{x_{s_i} \mid i \in [1, l+1]\}$ be an independent set of cardinality $l+1$ in C_n^k, where again, the elements are listed in clockwise order as $x_{s_1}, x_{s_2}, \ldots, x_{s_{l+1}}$. Since S is an independent set of vertices in C_n^k, it follows that $d_{C_n}\left(x_{s_i}, x_{s_{i+1}}\right) \geq k+1$ for each $i \in [1, l]$; otherwise, $x_{s_i} x_{s_{i+1}} \in E\left(C_n^k\right)$ and hence S is not an independent set of vertices in C_n^k. Also, $d_{C_n}\left(x_{s_1}, x_{s_{l+1}}\right) \geq k+1$ for the same reason. This implies that

$$\begin{aligned} |E(C_n)| &\geq (k+1)(l+1) = (k+1)l + (k+1) \\ &> (k+1)l + r = n = |E(C_n)|, \end{aligned}$$

which is impossible.

Therefore, we obtain $\beta\left(C_n^k\right) \leq \lfloor n/(k+1) \rfloor$.

Next, we establish the reverse inequality. Let $S = \left\{x_{(k+1)i-k} \mid i \in [1, \lfloor n/(k+1) \rfloor]\right\}$. Then

$$d_{C_n}\left(x_{(k+1)i-k}, x_{(k+1)(i+1)-k}\right) = k+1$$

for each $i \in [1, \lfloor n/(k+1) \rfloor - 1]$. Therefore, S is an independent set of cardinality $\lfloor n/(k+1) \rfloor$, implying that $\beta\left(C_n^k\right) \geq \lfloor n/(k+1) \rfloor$. □

We are now ready to prove the following theorem.

Theorem 2. *For every two integers n and k with $n \geq k+2 \geq 4$,*

$$\operatorname{str}\left(C_n^k\right) = \begin{cases} n+2k & \text{if } n = 2k+2 \\ 2n - 2\lfloor n/(k+1) \rfloor + 1 & \text{if } n \neq 2k+2 \text{ and } k \in [2, \lfloor n/2 \rfloor] \end{cases}.$$

Proof. First, we define the cycle C_n as in the proof of Lemma 5. We then let $G = C_n^k$ and define the graph G with $V(G) = V(C_n)$ and

$$E(G) = \{uv \mid 1 \le d_{C_n}(u,v) \le k \text{ and } k \in [2, \lfloor n/2 \rfloor]\},$$

where $n \ge k+2 \ge 4$. Furthermore, let $\beta = \lfloor n/(k+1) \rfloor$.
We now consider two cases, depending on the relation among the integers n and k.

Case 1. Suppose that $n = 2k+2$. Then Lemmas 4 and 5 yield that

$$\begin{aligned} \operatorname{str}(G) &\le 2n - \beta = 2n - \left\lfloor \frac{n}{k+1} \right\rfloor = 2n - \left\lfloor \frac{2k+2}{k+1} \right\rfloor \\ &= 2n - 2 = n + (n-2) = n + 2k. \end{aligned}$$

On the other hand, G is a $2k$-regular graph when $n = 2k+2$. To see this, consider the following. The cycle C_n is a 2-regular connected graph so that a 2-regular spanning subgraph is obtained by joining vertices of C_n at distance k ($k \in [1, n/2 - 1]$) by skipping $k-1$ vertices. Since this increments the degree of each vertex by 2 for each k, it follows that G is a $2k$-regular graph. It is now immediate from Lemma 1 that $\operatorname{str}(G) \ge n + 2k$. Consequently, $\operatorname{str}(G) = n + 2k$.

Case 2. Suppose that $n \ne 2k+2$ and $k \in [2, \lfloor n/2 \rfloor]$, and consider a partition $\{S_1, S_2, S_3\}$ of $V(G)$ such that

$$\begin{aligned} S_1 &= \left\{ x_{(k+1)i-k} \mid i \in [1, \beta] \right\}, \\ S_2 &= \left\{ x_{(k+1)i-k+1} \mid i \in [1, \beta] \right\}, \\ S_3 &= V(G) - (S_1 \cup S_2). \end{aligned}$$

Then $|S_1| = |S_2| = \beta$ and $|S_3| = n - 2\beta$.
Suppose that $S_3 = \{x_{s_i} \mid i \in [1, n-2\beta]\}$, where the subindices $s_1, s_2, \ldots, s_{n-2\beta}$ are ordered in such a way that

$$s_1 < s_2 < \cdots < s_{n-2\beta}.$$

Note that if we define the bijective function $\pi : S_3 \to \{y_i \mid i \in [1, n-2\beta]\}$ by

$$\pi(x_{s_i}) = y_i \ (i \in [1, n-2\beta]),$$

then we have

$$S_3 = \{y_i \mid i \in [1, n-2\beta]\}.$$

Thus, the numbering $f : V(G) \to [1, n]$ such that

$$f(v) = \begin{cases} n+1-i & \text{if } v = x_{(k+1)i-k} \text{ and } i \in [1, \beta] \\ n-2\beta+i & \text{if } v = x_{(k+1)i-k+1} \text{ and } i \in [1, \beta] \\ i & \text{if } v = y_i \text{ and } i \in [1, n-2\beta] \end{cases}$$

has the properties that

$$\begin{aligned}
\{f(v) \mid v \in S_3\} &= [1, n-2\beta], \\
\{f(v) \mid v \in S_2\} &= [n-2\beta+1, n-\beta], \\
\{f(v) \mid v \in S_1\} &= [n-\beta+1, n],
\end{aligned}$$

and

$$\begin{aligned}
\operatorname{str}_f(G) &= \max\{f(u)+f(v) \mid uv \in E(G)\} \\
&= f\left(x_{(k+1)i-k}\right) + f\left(x_{(k+1)i-k+1}\right) \\
&= (n+1-i) + (n-2\beta+i) \\
&= 2n-2\beta+1 = 2n-2\lfloor n/(k+1) \rfloor + 1
\end{aligned}$$

for all $i \in [1, \beta]$. This implies that $\operatorname{str}(G) \leq 2n-2\lfloor n/(k+1)\rfloor+1$. On the other hand, the reverse inequality is obtained readily by applying Lemmas 2 and 5. Consequently, $\operatorname{str}(G) = 2n-2\lfloor n/(k+1)\rfloor+1$. □

The proof of the preceding theorem shows that the graph C_n^k $(n = 2k+2)$ constitutes an example of an infinite family of graphs for which the bound given in Lemma 4 is sharp.

3. Conclusions

We have used the bounds given in Lemmas 1 and 2 to determine the strength of P_n^k and C_n^k. It is interesting to notice that we find subfamilies of these graphs for which the bound given in Lemma 2 is sharp, and observe that the value for the strength of P_n^{n-1} coincides with the strength of K_n already established in [13] as expected. Notice also that $\left|E\left(P_n^k\right)\right| = k(2n-k-1)/2$ for integers n and k with $n \geq k+1$, where $k \in [2, n-1]$. This suggests that Lemma 2 may be potentially helpful in computing the strength of dense graphs. By using the bounds for the strength given in Lemmas 1, 2 and 4, we have obtained Corollaries 1 and 2.
We conclude this paper by stating the next problem for future research.

Problem 1. What is the probability that the strength of a given graph G of order n coincides with $n + \delta(G)$?

Conflict of interest. The authors declare that they have no conflict of interest.

Data Availability. Data sharing is not applicable to this article as no datasets were generated or analyzed during the current study.

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