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Research Article

Set colorings of the Cartesian product of some graph families

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Abstract: Neighbor-distinguishing colorings, which are colorings that induce a proper vertex coloring of a graph, have been the focus of different studies in graph theory. One such coloring is the set coloring. For a nontrivial graph G, let $c : V(G) \to \mathbb{N}$ and define the neighborhood color set $NC(v)$ of each vertex v as the set containing the colors of all neighbors of v. The coloring c is called a set coloring if $NC(u) \neq NC(v)$ for every pair of adjacent vertices u and v of G. The minimum number of colors required in a set coloring is called the set chromatic number of G and is denoted by $\chi_s(G)$. In recent years, set colorings have been studied with respect to different graph operations such as join, comb product, middle graph, and total graph. Continuing the theme of these previous works, we aim to investigate set colorings of the Cartesian product of graphs. In this work, we investigate the gap given by $\max\{\chi_s(G), \chi_s(H)\} - \chi_s(G \,\square\, H)$ for graphs G and H. In relation to this objective, we determine the set chromatic numbers of the Cartesian product of some graph families.

Keywords: set coloring, Cartesian product, neighbor-distinguishing coloring

AMS Subject classification: 05C15, 05C76, 05C38.

1. Introduction

One of the most well-studied topics in graph theory is the notion of proper vertex colorings, whose definition we present below. For this paper, we restrict our attention to simple, undirected, connected graphs.

Definition 1. Let G be a graph. A map $c : V(G) \to \mathbb{N}$ is called a *proper vertex coloring* if and only if $c(u) \neq c(v)$ whenever u and v are adjacent. If, in addition, $|c(V(G))| = k$, then c is called a *proper vertex k-coloring*. The smallest k for which G has a proper vertex k-coloring is called the *chromatic number* of G and is denoted by $\chi(G)$.

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In [3, 4], Chartrand et al. defined the notion of a neighbor-distinguishing coloring, which can be seen as a generalization of the notion of proper vertex colorings: A *neighbor-distinguishing coloring* of a graph is a coloring (of its vertices and/or edges) that induces a vertex labelling for which any two adjacent vertices are assigned distinct labels.

An example of a neighbor-distinguishing vertex coloring is multiset coloring, introduced by Chartrand et al. in [2]. Given a vertex coloring c of a graph G that is not necessarily proper and for each vertex v of G, the multiset $M(v)$ is defined to be the multiset of colors of the neighbors of v. The coloring c is said to be a *multiset coloring* if $M(v) \neq M(w)$ for any two distinct vertices v, w. It is worth noting that in [3], it has been established that multiset coloring and sigma coloring [4] are, in fact, equivalent notions.

In this paper, we focus on another neighbor-distinguishing coloring called *set coloring*. As indicated in its definition below, for set colorings, we consider the set (i.e., instead of multiset) of colors of the neighbors of each vertex. This implies, particularly, that repetition of colors is not taken into account in such sets.

Definition 2 (Chartrand et al., [1]). For a graph G, let $c : V(G) \to \mathbb{N}$ be a vertex coloring that is not necessarily proper. For any subset S of $V(G)$, note that $c(S) = \{c(v) : v \in S\}$.

1. The *neighborhood color set* (or NC) of v, denoted by $NC(v)$, is the set $c(N(v))$; that is, $NC(v)$ is the set of colors of the neighbors of v.

2. The coloring c is called *set neighbor-distinguishing*, or simply a *set coloring*, if $NC(u) \neq NC(v)$ for every pair u, v of adjacent vertices of G. If, in addition, we have $|c(V(G))| = k$, then c is called a *set k-coloring* of G.

3. The minimum number of colors required in a set coloring of G is called the *set chromatic number* of G and is denoted by $\chi_s(G)$.

The following results from [1] will prove to be useful in this work.

Proposition 1 (Chartrand et al., [1]). *Let G be a graph.*

1. *Any proper vertex coloring of G is also a set coloring of G; and $\chi_s(G) \leq \chi(G)$.*

2. *If G is nonempty, then $\chi_s(G) = 2$ if and only if G is bipartite.*

3. *If G is connected and $\chi(G) \geq 3$, then $\chi_s(G) \geq 3$.*

Set colorings have been studied with respect to different topics. Beginning with the fundamental work in [1], set colorings have been studied in relation to perfect graphs [8], random graphs [5], and a bound on the set chromatic number [12]. A number of research works have focused on set colorings and different graph operations. With respect to unary graph operations, for example, set colorings have been studied in relation to middle graph [6, 13] and total graph [14].

As of this writing, only two studies [7, 10] have dealt with set colorings in relation to binary graph operations (i.e., join and comb product). Thus, continuing the theme of previous studies, we aim to investigate set colorings of the Cartesian product of graphs. We will use the following definition.

Definition 3 ([9]). The *Cartesian product* $G = G_1 \square G_2$ of two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, where $V_1 \cap V_2 = \emptyset$ and $E_1 \cap E_2 = \emptyset$, is the graph whose vertex set is $V_1 \times V_2$ and two vertices $u = (u_1, u_2)$ and $v = (v_1, v_2)$ are adjacent whenever $[u_1 = v_1$ and $u_2v_2 \in E_2]$ or $[u_2 = v_2$ and $u_1v_1 \in E_1]$.

In the case of proper vertex colorings, the relationship between $\chi(G), \chi(H)$, and $\chi(G \square H)$ for arbitrary graphs G and H has been clearly established as follows.

Theorem 1 ([11]). *For arbitrary graphs G and H, we have*

$$\chi(G \square H) = \max\{\chi(G), \chi(H)\}.$$

The following result is then an immediate consequence of the preceding theorem and Proposition 1(1).

Corollary 1. *For nontrivial connected graphs G and H,*

$$\chi_s(G \square H) \leq \max\{\chi(G), \chi(H)\}.$$

Theorem 1 leads us to a natural question about the set chromatic number of the Cartesian product of graphs. For arbitrary graphs G and H, are $\chi_s(G), \chi_s(H)$, and $\chi_s(G \square H)$ also related in a relatively straightforward way?

To help answer this question, we investigate the set chromatic number of the Cartesian product of some families of graphs. We then observe the difference or *gap* given by $\max\{\chi_s(G), \chi_s(H)\} - \chi_s(G \square H)$ for these different graph families. In the succeeding sections, we will show that this gap can become negative, zero, or positive, and that this gap can become arbitrarily large.

Throughout this paper, we will use $\mathbb{N}$ to denote the set of positive integers and $\mathbb{N}_k$ to denote the set $\{1, 2, \ldots, k\}$. Moreover, the vertex and edge sets of the path graph P_m are to be given by $V(P_m) = \{1, 2, \ldots, m\}$ and $E(P_m) = \{\{i, i+1\} : i = 1, 2, \ldots, m-1\}$, respectively.

2. Set Chromatic Numbers of $P_m \square P_n$, $P_m \square C_n$, and $P_m \square K_n$

It is quite straightforward to identify graphs G and H for which the gap $\max\{\chi_s(G), \chi_s(H)\} - \chi_s(G \square H)$ is equal to 0. We begin with the following result on grids (i.e., $P_m \square P_n$) and cylindrical graphs (i.e., $P_m \square C_n$).

Proposition 2. *Let m, n be positive integers.*

1. *If $m + n \geq 3$, then $\chi_s(P_m \,\square\, P_n) = 2$.*

2. *If $n \geq 3$, then $\chi_s(P_m \,\square\, C_n) = \begin{cases} 2 & \text{if } n \text{ is even,} \\ 3 & \text{if } n \text{ is odd.} \end{cases}$*

Proof. Clearly, if $m + n \geq 3$, then the grid $P_m \,\square\, P_n$ is bipartite. Thus, (1) follows immediately from Proposition 1(2).

Now, let $n \geq 3$. If n is even, then the cylindrical graph $P_m \,\square\, C_n$ is bipartite and Proposition 1(2) implies $\chi_s(P_m \,\square\, C_n) = 2$ as well. On the other hand, if n is odd, then $\chi(P_m \,\square\, C_n) = 3$; thus, it follows that $\chi_s(P_m \,\square\, C_n) = 3$, by Proposition 1(1 & 3). □

From [1], it is known that $\chi_s(P_m) = 2$ for any $m \geq 2$ and that $\chi_s(C_n) = 2$ if $n \geq 4$ is even while $\chi_s(C_n) = 3$ if $n \geq 3$ is odd. Thus, given Proposition 2, we see that if G is a path and H is a path or a cycle, we have $\max\{\chi_s(G), \chi_s(H)\} - \chi_s(G \,\square\, H) = 0$.

Notice that in the preceding discussion, the set chromatic numbers involved are only at most 3. We now turn to the following result.

Theorem 2. *Let m, n be positive integers. If $n \geq 4$, then $\chi_s(P_m \,\square\, K_n) = n$.*

Proof. Let $G = P_m \,\square\, K_n$. By Corollary 1, we have $\chi_s(G) \leq \max\{\chi(P_m), \chi(K_n)\} = n$. Thus, we are left to show that $\chi_s(G) \geq n$. Suppose, on the contrary, that there is a set coloring c of G that uses $r \leq n - 1$ colors.

Let $V_1 = \{(1, v) : v \in V(K_n)\} \subseteq V(G)$ and set $k = |c(V_1)|$; that is, k is the number of colors used by c to color the vertices in V_1. We may assume that $c(V_1) = \mathbb{N}_k$. Let

$$X = \big\{(1, x) \in V_1 : c(1, x) = c(1, y) \ \exists (1, y) \in V_1 \setminus \{(1, x)\}\big\}.$$

Then $n - |X|$ is the number of vertices in V_1 whose colors do not repeat in V_1; it follows that $k = (n - |X|) + |c(X)| \geq n - |X| + 1$ and $|X| - 1 \geq n - k$.

Now, observe that for each $(1, v) \in X$, we have $NC(1, v) = \mathbb{N}_k \cup \{c(2, v)\}$. But the vertices in X are adjacent to each other; thus, they must have distinct NCs. For this to hold, only at most one vertex $(1, v')$ in X may have NC equal to $\mathbb{N}_k$ (i.e., $c(2, v')$ is also in $\mathbb{N}_k$). This means that the other $|X| - 1$ vertices in X must have $c(2, v) \notin \mathbb{N}_k$. Hence, there must be at least $|X| - 1$ colors not in $\mathbb{N}_k$; that is, $r - k \geq |X| - 1$, which implies that $n - 1 - k \geq |X| - 1$. But this contradicts the previously established inequality $|X| - 1 \geq n - k$.

Therefore, a set coloring of G that uses $r \leq n - 1$ colors cannot exist. □

Combining Proposition 2, Theorem 2, and the fact, from [1], that $\chi_s(K_n) = n$ for all n, we have the following remark.

Remark 1. For every positive integer $n \geq 2$, there are graphs G and H such that

$$\max\{\chi_s(G), \chi_s(H)\} = \chi_s(G \,\square\, H) = n.$$

3. Set chromatic number of $P_m \,\square\, W_n$

In this section, we will find graphs G and H such that $\max\{\chi_s(G), \chi_s(H)\} < \chi_s(G \,\square\, H)$. For this purpose, we turn to wheel graphs, which are defined as follows: the *wheel graph* W_n, where $n \geq 4$, is the graph of order n obtained by taking the join of K_1 and C_{n-1}. Alternatively, W_n is the graph obtained by adding a new vertex x to the cycle graph C_{n-1} and connecting x to all the vertices of C_{n-1}.

Note that $\chi(W_n) = 3$ if n is odd and $\chi(W_n) = 4$ if n is even. Moreover, the set chromatic number of wheel graphs has been completely determined in [7].

Proposition 3 ([7]). *Let W_n be the wheel graph of order n. Then $\chi_s(W_4) = 4$ and $\chi_s(W_n) = 3$ for $n \geq 5$.*

We will completely determine the set chromatic number of the Cartesian product $P_m \,\square\, W_n$, where $m \geq 2$ and $n \geq 4$. First, the result below immediately follows.

Corollary 2. *Let m, n be positive integers. If $n \geq 5$ and n is odd, then $\chi_s(P_m \,\square\, W_n) = 3$.*

Proof. First, note that $\chi(P_m \,\square\, W_n) = \max\{\chi(P_m), \chi(W_n)\} = 3$, by Theorem 1. Then by Proposition 1(3), we have $\chi_s(P_m \,\square\, W_n) \geq 3$. At the same time, Proposition 1(1) implies $\chi_s(P_m \,\square\, W_n) \leq 3$. $\square$

We are left to consider Cartesian products $P_m \,\square\, W_n$, where $m \geq 2$, $n \geq 4$, and n is even. In this case, note that $\chi(P_m \,\square\, W_n) = \max\{\chi(P_m), \chi(W_n)\} = 4$, by Theorem 1. Thus, by Proposition 1(1 & 3), we have $\chi(P_m \,\square\, W_n) = 3$ or 4 if n is even. We first consider the possibility that $\chi(P_m \,\square\, W_n) = 3$.

Consider the partial 3-coloring of $P_m \,\square\, W_{10}$, where $m \geq 4$, shown in Fig. 1 below. It is evident that, so far, the NCs of adjacent vertices in $P_m \,\square\, W_{10}$ are distinct. Thus, if 3 colors would not be enough to construct a set coloring of $P_m \,\square\, W_{10}$ (or $P_m \,\square\, W_n$, where n is even), conflicting NCs may not arise until we consider the graph's fourth "layer" or beyond.
In the succeeding lemmas, we establish different properties of any set 3-coloring of $P_m \,\square\, W_n$, assuming that such a coloring exists.

First, we introduce several notations. We will denote the vertex and edge sets of W_n as follows:

$$V(W_n) = \{v_0, v_1, ..., v_{n-1}\},$$
$$E(W_n) = \{v_0v_k : k = 1, 2, ..., n-1\} \cup \{v_1v_2, v_2v_3, \ldots, v_{n-2}v_{n-1}, v_{n-1}v_1\}.$$

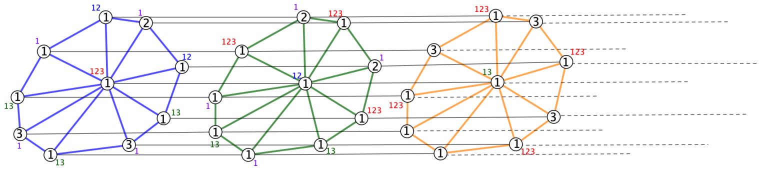

Figure 1. **A partial 3-coloring for $P_m \,\square\, W_{10}$, where $m \geq 4$**

Moreover, for each $i \in \{1, 2, ..., m\}$, we introduce the following notations, as illustrated in Fig. 2:

- The ith *layer* of $P_m \,\square\, W_n$ is the set $L_i = \{(i, v) : v \in V(W_n)\}$.
- The vertex $m_i = (i, v_0)$ is referred to as the *middle vertex* of the ith layer of $P_m \,\square\, W_n$.
- The vertices in $\mathcal{C}_i = \{(i, v_k) : k = 1, 2, ..., n-1\}$ are referred to as the *cycle vertices* of the ith layer of $P_m \,\square\, W_n$.

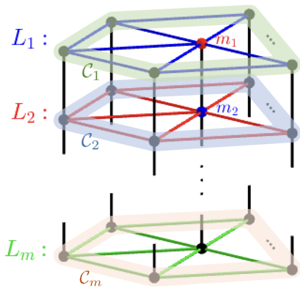

Figure 2. **Notations for the Cartesian product $P_m \,\square\, W_n$.**

Now, suppose we have a vertex coloring c of $P_m \,\square\, W_n$. We also introduce the notion of blocks (of each layer of $P_m \,\square\, W_n$) with respect to this coloring c. For each $i \in \{1, 2, ..., m\}$, we define the cyclic sequence s_i as follows:

$$s_i := (c(i, v_1), c(i, v_2), ..., c(i, v_{n-1}), c(i, v_1)).$$

By a *block* of s_i we mean a maximal subsequence of s_i of the same color. Further if the vertices in a block are colored α, we refer to the block as an α-block. The *length* of a block is the number of vertices in the block.

We are now ready to prove our first lemma.

Lemma 1. *Let m, n be positive integers, where $m \geq 2$, $n \geq 4$ and n is even. Suppose $P_m \,\square\, W_n$ has a set 3-coloring c for which $c(m_1) = 1$. Then the following statements hold:*

1. *$c(m_i) \in c(\mathcal{C}_i)$ for each $i \in \{1, 2, \ldots, m\}$;*
2. *$c(m_i) = 1$ for each $i \in \{1, 2, \ldots, m\}$;*
3. *in any s_i, where $i \in \{1, 2, \ldots, m\}$, any 2-block or 3-block must be of length 1;*
4. *$c(\mathcal{C}_1) = c(\mathcal{C}_m) = \{1, 2, 3\}$.*

Proof. Let $G = P_m \,\square\, W_n$.

1. Suppose, on the contrary, that $\alpha := c(m_i) \notin c(\mathcal{C}_i)$ for some $i \in \{1, 2, ..., m\}$. Then $|c(\mathcal{C}_i)| \leq 2$ so we consider two cases. First, suppose $|c(\mathcal{C}_i)| = 1$ and that $c(\mathcal{C}_i) = \{\beta\}$, where $\beta \neq \alpha$. Then for any $(i, v_k) \in \mathcal{C}_i$, there are only two possibilities for $NC(i, v_k)$: $\{\alpha, \beta\}$ or $\{1, 2, 3\}$. However, $|\mathcal{C}_i| = n - 1$ is odd; thus, two adjacent vertices in $\mathcal{C}_i$ will have the same NC, a contradiction.

   Now, suppose $|c(\mathcal{C}_i)| = 2$. Let $\{\beta, \gamma\} = c(\mathcal{C}_i)$, where $\alpha \notin \{\beta, \gamma\}$. Let b be any block of s_i with length at least 2. Then the NCs in b will have to be alternating with the first and last vertices having NC equal to $\{1, 2, 3\}$. This implies that b must be of odd length. But since s_i must have an even number of blocks, s_i must have an even number of elements. However, the number of elements in s_i is $n - 1$, which is odd, a contradiction.

2. Suppose there is an $i \in \{1, 2, ..., m - 1\}$ such that $\alpha := c(m_i) \neq c(m_{i+1}) =: \beta$. By (A1), we must have $\alpha \in c(\mathcal{C}_i)$ and $\beta \in c(\mathcal{C}_{i+1})$. Then $\{\alpha, \beta\}$ must be a subset of both $NC(m_i)$ and $NC(m_{i+1})$. Since these NCs must be distinct, we may assume that $NC(m_i) = \{1, 2, 3\}$ and $NC(m_{i+1}) = \{\alpha, \beta\}$. It follows that $|c(\mathcal{C}_{i+1})| \leq 2$.

   Let γ be the third element of $\{1, 2, 3\}$ that is neither α nor β. Recall that $\beta \in c(\mathcal{C}_{i+1})$; hence, $\gamma \notin c(\mathcal{C}_{i+1})$ because if it were, we would have $NC(m_{i+1}) = \{1, 2, 3\} = NC(m_i)$, a contradiction. Then $c(\mathcal{C}_{i+1}) = \{\beta\}$ or $\{\alpha, \beta\}$.

   In the latter case, note that any block of s_{i+1} with color α must be of length 1; otherwise, two adjacent vertices in $\mathcal{C}_{i+1}$ would both have α and β in their NCs. Then one of them would have NC equal to $\{\alpha, \beta\}$, which would equal $NC(m_{i+1}) = \{\alpha, \beta\}$, a contradiction. Since $n - 1$ is odd, s_{i+1} must have a β-block of length at least 2.

   Thus, whether $c(\mathcal{C}_{i+1}) = \{\beta\}$ or $c(\mathcal{C}_{i+1}) = \{\alpha, \beta\}$, it must be the case that s_{i+1} has a β-block of length at least 2. Thus, we may assume that $c(i + 1, v_1) = c(i+1, v_2) = \beta$. Consider the vertices (i, v_1) and (i, v_2) and recall that $c(m_i) = \alpha$. Then α and β are elements of both $NC(i, v_1)$ and $NC(i, v_2)$. Since these two NCs must be distinct, one of them would have NC equal to $\{1, 2, 3\} = NC(m_i)$, a contradiction.

   Therefore, for all $i \in \{1, 2, \ldots, m - 1\}$, we must have $c(m_i) = c(m_{i+1})$. The result then follows from the assumption that $c(m_1) = 1$.

3. Let $i \in \{1, 2, \ldots, m\}$ and suppose s_i has a 2-block with length at least 2. Let (i, v_k) and (i, v_{k+1}) be 2 vertices in $\mathcal{C}_i$ with color 2. Then $\{1, 2\}$ is a subset of $NC(m_i), NC(i, v_k)$, and $NC(i, v_{k+1})$. So two of these NCs must be equal, which is a contradiction since all three of them must be distinct. Thus, s_i cannot have a 2-block (similarly, 3-block) with length at least 2.

4. It is sufficient to prove that $c(\mathcal{C}_1) = \{1, 2, 3\}$. Suppose not; then by (A1), $c(\mathcal{C}_1)$ will have to be $\{1\}$, $\{1, 2\}$, or $\{1, 3\}$.

   Case A4.1. Suppose $c(\mathcal{C}_1) = \{1\}$.

   By (A2), we must have $NC(m_1) = \{1\}$. Then for any $(1, v_k) \in \mathcal{C}_1$, we have $NC(1, v_k) = \{1, 2\}$ or $\{1, 3\}$; that is, only 2 NCs are available for vertices in $\mathcal{C}_1$. But since $|\mathcal{C}_1|$ is odd, these 2 NCs will not be sufficient, a contradiction.

   Case A4.2. Suppose $c(\mathcal{C}_1) = \{1, 2\}$ or $\{1, 3\}$. We may assume $c(\mathcal{C}_1) = \{1, 2\}$.

   By (A2), we must have $NC(m_1) = \{1, 2\}$. By (A3), 2-blocks in s_1 must all be of length 1. Now, we will prove that 1-blocks in s_1 must all be of odd length. Let b be such a block; we may assume that the length of b is at least 2. Suppose $b = (c(1, v_1), c(1, v_2), \ldots, c(1, v_k))$, where $k \geq 2$.

   Since $c(1, v_{n-1}) = c(1, v_{k+1}) = 2$ and $NC(m_1) = \{1, 2\}$, we have $NC(1, v_1) = NC(1, v_k) = \{1, 2, 3\}$ and $c(2, v_1) = c(2, v_k) = 3$. Consequently, we have $k \geq 3$, $NC(2, v_{n-1}) = \{1, 2, 3\}$, and $2 \notin c(\mathcal{C}_2)$. Consider the sequence

$$s = (c(2, v_2), c(2, v_3), \ldots, c(2, v_{k-1})),$$

   where, by (A3), we have $c(2, v_2) = c(2, v_{k-1}) = 1$. Clearly, s cannot have 2 consecutive elements that are both equal to 1. Thus, s must be the alternating sequence $(1, 3, 1, 3, \ldots, 3, 1)$ and $k - 2$ must be odd. Thus, the length k of b must also be odd.

   Therefore, all the blocks in s_1 are of odd length. Since s_1 must have an even number of blocks, it follows that s_1 must also have an even number of elements, a contradiction.

□

Our second lemma provides further properties of any set 3-coloring, if such exists, of $P_m \square W_n$ if n is even and with the added condition that $m \geq 3$.

Lemma 2. *Let m, n be positive integers, where $m \geq 3$, $n \geq 4$ and n is even. Suppose $P_m \square W_n$ has a set 3-coloring c for which $c(m_1) = 1$. Then the following statements hold:*

1. *For $i \in \{2, m-1\}$, we have $c(\mathcal{C}_i) = \{1, 2\}$ or $\{1, 3\}$.*
2. *If $i \in \{2, 3, \ldots, m-1\}$ and $c(\mathcal{C}_i) = \{1, 2\}$ or $\{1, 3\}$, then $c(\mathcal{C}_{i+1}) \neq \{1\}$.*

3. If $c(\mathcal{C}_2) = \{1,2\}$, then $c(\mathcal{C}_3) = \{1,3\}$.

Proof. Let $G = P_m \,\Box\, W_n$.

1. It is sufficient to prove the result only for $c(\mathcal{C}_2)$. By (A4), we have $c(\mathcal{C}_1) = \{1,2,3\}$ and so $c(\mathcal{C}_2) \neq \{1,2,3\}$. We are now left to show that $c(\mathcal{C}_2) \neq \{1\}$. Assume the contrary; then $NC(m_2) = \{1\}$.

   We will first consider the blocks in s_1. Recall that, by (A3), all 2- and 3-blocks in s_1 must be of length 1. Now, with the assumption that $c(\mathcal{C}_2) = \{1\}$, it is easy to establish further that 1-blocks can only be of length 1, 2, or 3. Furthermore, the 1-blocks in s_1 satisfy the following:

   (a) A 1-block of length 1 or 3 must be between identical colors.
   Proof. It is clear that a 1-block of length 1 cannot be between distinct colors. Now, by way of contradiction, suppose there is a 1-block of length 3 that is between distinct colors. In this case, we must have $n \geq 5$. Let us assume that $(c(1,v_1), c(1,v_2), c(1,v_3), c(1,v_4), c(1,v_5)) = (2,1,1,1,3)$. Then $NC(1,v_2) = \{1,2\}$, $NC(1,v_3) = \{1\}$, and $NC(1,v_3) = \{1,3\}$. Since $NC(m_2) = \{1\}$, we must also have $NC(2,v_2) = \{1,3\}$ and $NC(2,v_4) = \{1,2\}$. Then the vertex $(2,v_3)$ is adjacent to vertices with NCs $\{1\}, \{1,2\}$, and $\{1,3\}$. Moreover, only at most one neighbor of $(2,v_3)$ is not colored 1. This implies the contradiction that $(2,v_3)$ will have NC equal to one of its neighbors' NCs.

   (b) A 1-block of length 2 must be between distinct colors.
   Proof. Suppose there is a 1-block of length 2 that is between identical colors. Then clearly, the two vertices in the 1-block will have the same NC, a contradiction.

   We now introduce the following: For $\beta \in \{2,3\}$, we define a *β-sequence* to be a maximal subsequence of s_1 such that the subsequence starts with β and all of its elements are in $\{1,\beta\}$. Since s_1 has at least one of each of the colors 2 and 3, there is at least one 2-sequence and at least one 3-sequence. Moreover, it is easy to see that 2- and 3-sequences must alternate in s_1. Thus, the total number of 2- and 3-sequences is even.

   Now, (B1.1) and (B1.2) imply that in any β-sequence, any occurrence of the color β, except at the end of the sequence, must be followed by a $(1,\beta)$, or a $(1,1,1,\beta)$, or a $(1,1)$ that is not the start of $(1,1,1,\beta)$. (Note that, in the case of $(1,1)$, this may only occur at the end of the subsequence.) This implies that any β-sequence must be of odd length.

   Therefore, s_1 has an even number of β-sequences, all of which are of odd length. This implies that s_1 has an even number of elements, a contradiction.

2. Suppose there is an $i \in \{2, 3, \ldots, m-1\}$ such that $c(\mathcal{C}_i) = \{1, 2\}$ and $c(\mathcal{C}_{i+1}) = \{1\}$. Then $NC(m_i) = \{1, 2\}$ and $NC(m_{i+1}) = \{1\}$. Recall that, by (A3), any 2-block in s_i must be of length 1.

   Suppose $(c(i, v_1), c(i, v_2), \ldots, c(i, v_k))$ is a 1-block in s_i. Since $NC(m_i) = \{1, 2\}$, which is a subset of both $NC(i, v_1)$ and $NC(i, v_k)$, then $NC(i, v_1) = NC(i, v_k) = \{1, 2, 3\}$ and $c(i-1, v_1) = c(i-1, v_k) = 3$. Thus, $k \geq 3$. Now, if $k \geq 4$, then $c(\mathcal{C}_{i+1}) = \{1\}$ implies that $(NC(i, v_2), NC(i, v_3), \ldots, NC(i, v_{k-1})$ must be an alternating sequence of $\{1\}$ and $\{1, 3\}$. Moreover, $NC(i, v_2)$ and $NC(i, v_{k-1})$ cannot be $\{1, 3\}$ because this would produce a 3-block of length at least 2 in s_{i-1}. Therefore, $NC(i, v_2) = NC(i, v_{k-1}) = \{1\}$ and so k must be odd; that is, any 1-block in s_i must be of odd length.

   Consequently, s_i only has blocks of odd length. Since the number of blocks in s_i must be even and s_i has an odd number of elements, this leads to a contradiction.

3. Suppose $c(\mathcal{C}_2) = \{1, 2\}$. By (B2), we know that $c(\mathcal{C}_3) \neq \{1\}$. Thus, we are left to show that $c(\mathcal{C}_3) \neq \{1, 2, 3\}$. Assume the contrary; then $NC(m_3) = \{1, 2, 3\}$.

   By (A3), any 2-block in s_2 must be of length 1. Then we may assume that

   $$(c(2, v_1), c(2, v_2), c(2, v_3)) = (1, 2, 1).$$

   Since $NC(m_2) = \{1, 2\} \subseteq NC(2, v_1)$, we must have $NC(2, v_1) = \{1, 2, 3\}$. Then $c(1, v_1)$ or $c(3, v_1)$ must be equal to 3. Then at least one of $NC(1, v_2)$ or $(NC(3, v_2)$ will be equal to $NC(m_1) = NC(m_3) = \{1, 2, 3\}$, a contradiction.

□

Using Lemmas 1 and 2, we can now prove that a set 3-coloring of $P_m \,\square\, W_n$ cannot exist if $m \geq 2$ and n is even.

Lemma 3. *Let m, n be integers, where $m \geq 2$, $n \geq 4$, and n is even. Then $\chi_s(P_m \,\square\, W_n) \geq 4$.*

Proof. Let $G = P_m \,\square\, W_n$. Suppose, on the contrary, that G has a set 3-coloring c. We assume, without loss of generality, that $c(m_1) = 1$.

If $m = 2$, then (A4) implies that $NC(m_1) = NC(m_2) = \{1, 2, 3\}$, which cannot be true. If $m = 3$, then by (B1), we may assume that $c(\mathcal{C}_2) = \{1, 2\}$. But in this case, (A4) and (B3) will lead to a contradiction. Thus, we only consider the case where $m \geq 4$.

Again, we assume without losing generality that $c(\mathcal{C}_2) = \{1, 2\}$. So far, we have $c(\mathcal{C}_1) = \{1, 2, 3\}$, $c(\mathcal{C}_2) = \{1, 2\}$, and $c(\mathcal{C}_3) = \{1, 3\}$ (by (B3)). Now, (B2) implies that $c(\mathcal{C}_4) \neq \{1\}$. Hence, $c(\mathcal{C}_4) = \{1, 2\}$ or $\{1, 2, 3\}$.

Suppose $c(\mathcal{C}_4) = \{1, 2\}$. (Note that, by (A4), we must have $m \geq 5$ in this case.) We will consider the blocks in s_3. By (A3), 3-blocks in s_3 are all of length 1. We will show that any 1-block in s_3 must be of odd length. Suppose, on the contrary, that s_3 has a 1-block, say $(c(3, v_1), c(3, v_2), \ldots, c(3, v_k))$, of even length. Then $NC(3, v_1) = NC(3, v_k) = \{1, 2, 3\}$. Since $c(\mathcal{C}_2) = c(\mathcal{C}_4) = \{1, 2\}$, we must have $NC(3, v_j) = \{1\}$ or $\{1, 2\}$ for $j \in \{2, 3, \ldots, k-1\}$. Thus, the sequence $(c(3, v_2), c(3, v_3), \ldots, c(3, v_{k-1}))$ is an alternating sequence of $\{1\}$ and $\{1, 2\}$. Since the 1-block is of even length, we may assume that $NC(3, v_2) = \{1, 2\}$. Thus, $c(2, v_2)$ or $c(4, v_2)$ is 2. We may assume $c(2, v_2) = 2$; then $\{1, 2\} \subseteq NC(2, v_1)$. But this is not possible since $NC(m_2) = \{1, 2\}$ and $NC(3, v_1) = \{1, 2, 3\}$. Therefore, any block in s_3 must be of odd length. Since there must be an even number of blocks in s_3, we have a contradiction; that is, $c(\mathcal{C}_4) \neq \{1, 2\}$.

Now, suppose $c(\mathcal{C}_4) = \{1, 2, 3\}$. We will consider the blocks in s_2. By (A3), 2-blocks in s_2 are all of length 1. We will show that all 1-blocks are also of length 1. Suppose, on the contrary, that $(c(2, v_1), c(2, v_2), c(2, v_3)) = (2, 1, 1)$. Then $NC(2, v_2) = \{1, 2, 3\}$ and at least one $c(1, v_2)$ or $c(3, v_2)$ is 3. But $c(1, v_2) \neq 3$ since otherwise, we will have $NC(1, v_1) = \{1, 2, 3\} = NC(m_1)$. Thus, $c(3, v_2) = 3$. Since $NC(m_3) = \{1, 3\}$, we have $NC(3, v_3) = \{1, 2, 3\}$ and $c(4, v_3) = 2$. But this will imply that $NC(4, v_2) = \{1, 2, 3\} = NC(m_4)$, which is a contradiction. Therefore, $c(\mathcal{C}_4) \neq \{1, 2, 3\}$ as well.

As there is no more option for $c(\mathcal{C}_4)$, we can conclude that G has no set 3-coloring. The desired conclusion follows. □

Given Corollary 2 and Lemma 3, our main result on the set chromatic number of the Cartesian product of path graphs and wheel graphs is now complete.

Theorem 3. *Let P_m be the path graph of order m, where $m \geq 2$, and W_n be the wheel graph of order n, where $n \geq 4$. Then*

$$\chi_s(P_m \,\square\, W_n) = \begin{cases} 3, & \text{if } n \text{ is odd,} \\ 4, & \text{if } n \text{ is even.} \end{cases}$$

Remark 2. Let $G = P_m$, where $m \geq 2$, and $H = W_n$, where $n \geq 6$ and n is even. Then, by the preceding theorem, we have $\max\{\chi_s(G), \chi_s(H)\} = 3 < 4 = \chi_s(G \,\square\, H)$. That is, we have found graphs G, H for which the gap $\max\{\chi_s(G), \chi_s(H)\} - \chi_s(G \,\square\, H)$ is negative.

4. Set chromatic number of $P_m \,\square\, G_{n,t}$

In this section, we consider the following graphs that have been introduced in [1].

Definition 4 ([1]). For an integer $n \geq 2$ and an integer t $(0 \leq t \leq n)$, let $G_{n,t}$ denote the graph of order $n + t$ obtained from K_n with $V(K_n) = \{v_1, v_2, ..., v_n\}$ by adding t new vertices $u_1, u_2, ..., u_t$ (if $t \geq 1$) and joining each u_i to v_i for $1 \leq i \leq t$.

The set chromatic number of this graph family has also been determined in [1].

Proposition 4 ([1]). *For $n \geq 2$ and $0 \leq t \leq n$, we have $\chi_s(G_{n,t}) = n$.*

When $m \geq 1$, $n \geq 2$, and $0 \leq t \leq n$, the preceding proposition and Theorem 1 imply that

$$\chi(P_m \,\square\, G_{n,t}) = \max\{\chi(P_m), \chi(G_{n,t})\} = n.$$

Thus, by Proposition 1(1), we have $\chi_s(P_m \,\square\, G_{n,t}) \leq n$. Naturally, we ask whether there are values of n and t for which this inequality becomes strict.

First, we establish the following lower bound for $\chi_s(P_m \,\square\, G_{n,t})$.

Lemma 4. *Let m, n, t be positive integers, where $m \geq 2$, $n \geq 3$, and $1 \leq t \leq n$. Then $\chi_s(P_m \,\square\, G_{n,t}) \geq n - t$.*

Proof. Let $G = P_m \,\square\, G_{n,t}$. Suppose, on the contrary, that c is a set coloring of G that uses k colors, where $k < n - t$.

Let $p_1, p_2, ..., p_t$ be the pendant vertices of $G_{n,t}$ and let $w_1, w_2, ..., w_t$ be the vertices in $G_{n,t}$ that are adjacent to $p_1, p_2, ..., p_t$, respectively. Let $V_1 = \big\{(1, v) \in V(G) : v \in V(G_{n,t}) \smallsetminus \{p_1, p_2, ..., p_t\}\big\}$. Set $k_1 := |c(V_1)|$ and without losing generality, assume $c(V_1) = \mathbb{N}_{k_1} := \{1, 2, ..., k_1\}$. Let

$$X = \big\{(1, x) \in V_1 : c(1, x) = c(1, y) \ \exists (1, y) \in V_1 \smallsetminus \{(1, x)\}\big\}.$$

Since $k_1 \leq k < n - t < |V_1|$, we must have $|X| > 0$. Moreover, $n - |X| + 1 \leq k_1$ or $|X| \geq n + 1 - k_1$.

Then for all $(1, x) \in X$, we have

$$NC(1, x) = \begin{cases} \mathbb{N}_{k_1} \cup \{c(2, x)\}, & \text{if } x \notin \{w_1, w_2, \ldots, w_t\}, \\ \mathbb{N}_{k_1} \cup \{c(2, x), c(1, p_i)\}, & \text{if } x = w_i \ \exists i \in \{1, 2, \ldots, t\}. \end{cases}$$

Thus, there are three types of possible NCs for vertices in X.

Type 1: $\mathbb{N}_{k_1}$
Type 2: $\mathbb{N}_{k_1} \cup \{1 \text{ color not in } \mathbb{N}_{k_1}\}$
Type 3: $\mathbb{N}_{k_1} \cup \{2 \text{ colors not in } \mathbb{N}_{k_1}\}$

Note that vertices in X must have distinct NCs. Let q_i be the number of vertices in X that have Type i NC, where $i = 1, 2, 3$. Then

$$q_1 \in \{0, 1\}, \quad q_3 \in \{0, 1, \ldots, t\}, \quad q_2 = |X| - q_1 - q_3.$$

Given that there are q_2 vertices in X with Type 2 NC, the number of colors not in $\mathbb{N}_{k_1}$ must be at least q_2. Then

$$k - k_1 \geq q_2 = |X| - q_1 - q_3 \geq |X| - 1 - t \geq (n + 1 - k_1) - 1 - t = n - k_1 - t;$$

thus, $k \geq n - t$, which is a contradiction. Therefore, G has no set coloring that uses less than $n - t$ colors. $\square$

The succeeding lemma shows that the preceding lower bound turns out to be optimal for a family of values of n and t.

Lemma 5. *Let m, n, t be positive integers, where $m \geq 2, n \geq 7$, and $n \geq 3t + 3$. Then $\chi_s(P_m \,\square\, G_{n,t}) \leq n - t$.*

Proof. Let $G = P_m \,\square\, G_{n,t}$, where $m \geq 2, n \geq 7$, and $n \geq 3t + 3$. For each $i \in \{1, 2, \ldots, m\}$, let $G_{n,t}(i)$ be the subgraph of $P_m \,\square\, G_{n,t}$ induced by the vertices in $V(G_{n,t}(i)) := \{(i, v) : v \in V(G_{n,t})\} \subseteq V(P_m \,\square\, G_{n,t})$. Then each $G_{n,t}(i)$ is isomorphic to $G_{n,t}$.

We will construct a vertex coloring c of G that uses exactly $k = n - t$ colors. We first color the vertices in $G_{n,t}(1)$ and $G_{n,t}(2)$. For convenience, we introduce the following notations:

$$V(G_{n,t}(1)) = \{a_1, a_2, \ldots, a_n, p_1, p_2, \ldots, p_t\},$$

where $a_1, a_2, \ldots, a_n$ are the vertices that form a clique in $G_{n,t}(1)$ and p_i is the pendant connected to a_i, for each $i = 1, 2, \ldots, t$. Similarly, let

$$V(G_{n,t}(2)) = \{b_1, b_2, \ldots, b_n, q_1, q_2, \ldots, q_t\},$$

where $b_1, b_2, \ldots, b_n$ are the vertices that form a clique in $G_{n,t}(2)$ and q_i is the pendant connected to b_i, for each $i = 1, 2, \ldots, t$.

Let $r = \left\lceil \frac{k-2}{2} \right\rceil$. We color the vertices of $G_{n,t}(1)$ and $G_{n,t}(2)$ by following the procedure below:

1. For $i = 1, 2, \ldots, t, t + 1, \ldots, n - r$, assign $c(a_i) = 1$.
2. For $i = 1, 2, \ldots, r$, assign $c(a_{n-r+i}) = 1 + i$.
3. For $i = 1, 2, \ldots, t$, assign $c(p_i) = k - i$.
4. For $i = 1, 2, \ldots, k - r - 1$, assign $c(b_{t+i}) = r + i$.
5. For $i = 1, 2, \ldots, t, t + k - r, t + k - r + 1, \ldots, n$, assign $c(b_i) = k$.
6. For $i = 1, 2, \ldots, t$, assign $c(q_i) = 1 + i$.

We then complete the coloring c as follows:

- Color the vertices in $G_{n,t}(3)$, $G_{n,t}(5)$, $G_{n,t}(7)$, and so on by copying the coloring of $G_{n,t}(1)$; that is, for each $i \in \{3, 5, 7, \ldots\}$, set $c(i, v) = c(1, v)$.
- Color the vertices in $G_{n,t}(4)$, $G_{n,t}(6)$, $G_{n,t}(8)$, and so on by copying the coloring of $G_{n,t}(2)$; that is, for each $i \in \{4, 6, 8, \ldots\}$, set $c(i, v) = c(2, v)$.

The procedure above will yield the following NCs:

- $NC(a_i) = \{1, 2, \ldots, r+1\} \cup \{k\} \cup \{k-i\}$, for $i = 1, 2, \ldots, t$
- $NC(a_{t+i}) = \{1, 2, \ldots, r+1\} \cup \{r+i\}$, for $i = 1, 2, \ldots, k-r-1$
- $NC(a_{t+k-r}) = NC(a_{n-r}) = \{1, 2, \ldots, r+1\} \cup \{k\}$
- $NC(a_{n-r+i}) = [\{1, 2, \ldots, r+1\} \setminus \{i+1\}] \cup \{k\}$, for $i = 1, 2, ..., r$
- $NC(p_i) = \{1, 1+i\}$, for $i = 1, 2, \ldots, t$
- $NC(b_i) = \{r+1, r+2, \ldots, k\} \cup \{1\} \cup \{1+i\}$, for $i = 1, 2, \ldots, t$
- $NC(b_{t+i}) = [\{r+1, r+2, \ldots, k\} \setminus \{r+i\}] \cup \{1\}$, for $i = 1, 2, \ldots, k-r-1$
- $NC(b_{t+k-r}) = NC(b_{n-r}) = \{r+1, r+2, \ldots, k\} \cup \{1\}$
- $NC(b_{t+k-r+i}) = NC(b_{n-r+i}) = \{r+1, r+2, \ldots, k\} \cup \{i+1\}$, for $i = 1, 2, \ldots, r$
- $NC(q_i) = \{k, k-i\}$, for $i = 1, 2, \ldots, t$
- $NC(i, v) = NC(1, v)$ for each $i \in \{3, 5, 7, \ldots\}$
- $NC(i, v) = NC(2, v)$ for each $i \in \{4, 6, 8, \ldots\}$

Now, observe that, for $i = 1, 2, \ldots, t$, we have $NC(b_i) \neq NC(b_{n-r})$ if $1+i \notin \{r+1, r+2, \ldots, k\} \cup \{1\}$. This means that we need t colors not in $\{r+1, r+2, \ldots, k\} \cup \{1\}$; thus, it is necessary that $r-1 \geq t$. A similar argument can be made to ensure that $NC(a_1), NC(a_2), \ldots, NC(a_t)$ are all not equal to $NC(a_{n-m})$; that is, we need t colors not in $\{1, 2, \ldots, r+1\} \cup \{k\}$. Since there are only $k-(r+2)$ colors not in $\{1, 2, \ldots, r+1\} \cup \{k\}$, we must have $k-(r+2) \geq t$. Note that $\frac{k-2}{2} \leq \lceil \frac{k-2}{2} \rceil \leq \frac{k-1}{2}$. This gives us $k-(r+2) \geq k-2-\frac{k-1}{2} = \frac{k-3}{2} = \frac{n-t-3}{2}$. The condition $n \geq 3t+3$ ensures that $r-1 \geq t$ and $k-(r+2) \geq \frac{n-t-3}{2} \geq t$ both hold.

It is then easy to verify that c is indeed a set $(n-t)$-coloring of G. $\square$

In Figure 3, we present a set 7-coloring of $P_2 \,\square\, G_{9,2}$ that has been constructed using the procedure discussed in the proof of Lemma 5.

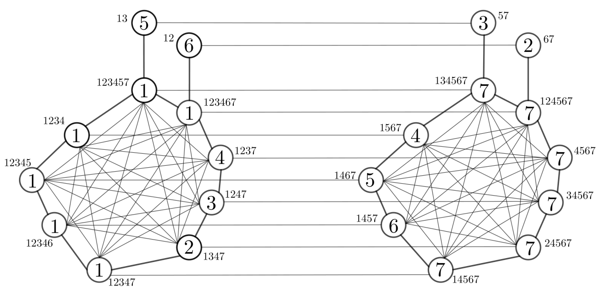

Figure 3. **A set 7-coloring of $P_2 \,\square\, G_{9,2}$**

Combining Lemmas 4 and 5, we have the following theorem.

Theorem 4. *Let m, n, t be positive integers, where $m \geq 2, n \geq 7$, and $n \geq 3t+3$. Then $\chi_s(P_m \square G_{n,t}) = n - t$.*

The preceding theorem allows us to make the following remark.

Remark 3. Let $G = P_m$ and $H = G_{n,t}$, where $m \geq 2, n \geq 7$, and $n \geq 3t+3$. We then have

$$\max\{\chi_s(G), \chi_s(H)\} - \chi_s(G \square H) = t.$$

Since t can be chosen arbitrarily (i.e, there exists an n corresponding to any t), this demonstrates that the gap $\max\{\chi_s(G), \chi_s(H)\} - \chi_s(G \square H)$ can become arbitrarily large.

5. Conclusion and Recommendation

In the literature, the notion of set colorings has been studied in relation to different graph operations such as middle graph, total graph, join, and comb product. In this work, we continued the theme of such previous papers by studying the set chromatic numbers of the Cartesian product of some graph families. Our general objective was to investigate the gap $\max\{\chi_s(G), \chi_s(H)\} - \chi_s(G \square H)$ for graphs G and H. We have proved that this gap can become negative, zero, or positive, and that it can become arbitrarily large. Moreover, in relation to this objective, we determined the exact set chromatic numbers of the Cartesian product of some families of graphs.

The following questions may prove to be interesting for future research: (1) Can the gap become arbitrarily large in the negative direction? More precisely, for any positive integer t, are there graphs G and H for which $\max\{\chi_s(G), \chi_s(H)\} - \chi_s(G \square H) \leq -t$? (2) Can we establish sharp lower and upper bounds for $\chi_s(G \square H)$ for arbitrary graphs G and H?

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