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Research Article

On k-(total) limited packing in graphs

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Abstract: A set $B \subseteq V(G)$ is called a k-total limited packing set in a graph G if $|B \cap N(v)| \leq k$ for any vertex $v \in V(G)$. The k-total limited packing number $L_{k,t}(G)$ is the maximum cardinality of a k-total limited packing set in G. This concept introduced by Hosseini Moghaddam et al. in 2016. Here, we give some results on the k-total limited packing number of graphs emphasizing trees, especially when $k = 2$. We also study the 2-(total) limited packing number of some product graphs. Ahmadi et al. introduced the concept of k-limited packing partition (kLPP) in 2024. A kLPP of graph G is a partition of $V(G)$ into k-limited packing sets. The minimum cardinality of a kLPP is called the kLPP number of G and is denoted by $\chi_{\times k}(G)$, and we obtain some results for this parameter.

Keywords: limited packing, k-limited packing partition number, graph products.

AMS Subject classification: 05C05, 05C69, 05C76

1. Introduction and preliminaries

In this work, we consider $G = \big(V(G), E(G)\big)$ as a finite simple graph. $N_G(v)$ and $N_G[v] = N_G(v) \cup \{v\}$ are used to refer to the *open neighborhood* and *closed neighborhood* of a vertex $v \in V(G)$, respectively. The *minimum* and *maximum degrees* of a graph G are denoted by $\delta(G)$ and $\Delta(G)$, respectively. We refer to [15] as a source for terminology and notation that is not explicitly defined here.
G^- indicates the graph obtained from G by removing its isolated vertices. By $G[S]$, we mean the subgraph induced by the subset S of vertices in G.
A set of vertices $S \subseteq V(G)$ is called a *dominating set* (DS) in G if every vertex not in S is adjacent to at least one vertex in S. The *domination number* of G, denoted

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$\gamma(G)$, is the smallest number of vertices in a dominating set of G. A set $S \subseteq V(G)$ is a *total dominating set* (TDS) in the graph G if every vertex in $V(G)$ is adjacent to a vertex of S. The *total domination number* of G, denoted $\gamma_t(G)$, is the smallest number of vertices in a total dominating set of G.

A set of vertices $S \subseteq V(G)$ with $\delta(G) \geq k-1$ is said to be a *k-tuple dominating set* (kTD set) in G provided that for every $v \in V(G)$, we have $|N[v] \cap S| \geq k$. The *k-tuple domination number* $\gamma_{\times k}(G)$ of graph G is the number of vertices in a smallest kTD set in G. A *k-tuple domatic partition* (kTD partition) of a graph G is a partition of the vertices of G into kTD sets. The largest number of sets that can be obtained from a vertex partition of G into kTD sets is called the *k-tuple domatic number* and is denoted by $d_{\times k}(G)$. Notice that when $k = 1$, S and $\gamma_{\times 1}(G)$ are the usual dominating set and domination number $\gamma(G)$, respectively. Additionally, $d_{\times 1}(G) = d(G)$ refers to the well-studied domatic number (see [4]).

A vertex subset B of a graph G is called a *packing* (resp. an *open packing*) provided that $|B \cap N[v]| \leq 1$ (resp. $|B \cap N(v)| \leq 1$) for each vertex $v \in V(G)$. The *packing number* $\rho(G)$ and *open packing number* $\rho_o(G)$ are defined as the maximum cardinality of a packing set and an open packing set, respectively. To obtain additional information on these concepts, the reader can refer to [9] and [8].

In 2010, the concept of limited packing (LP) in graphs was introduced by Gallant et al. [7]. A *k-limited packing* (kLP) in a graph G is a set $B \subseteq V(G)$ such that for each vertex v of $V(G)$, the cardinality of the intersection of B and $N[v]$ is at most k. The maximum cardinality of a k-limited packing set in G is called the *k-limited packing number* $L_k(G)$. They also presented some real-world applications of this concept in network security, market situation, NIMBY and codes. This topic was next investigated in numerous papers, such as references [2], [3], [5], [6], [11] and [14]. Similarly, a *k-total limited packing* (kTLP) in G is a set $B \subseteq V(G)$ such that for each vertex v of $V(G)$, the cardinality of the intersection of B and $N(v)$ is at most k. The maximum cardinality of a k-total limited packing set in G is called the *k-total limited packing number* $L_{k,t}(G)$. This topic was initially studied in [10], and some theoretical applications of it were given in [1, 12]. It is worth noting that the latter two concepts are identical to packing and open packing when k equals 1. Notice that a kLP set is a kTLP set, too.

A *k-limited packing partition* (kLPP) of a graph G is a partition of the vertices of G into kLP sets. This topic was first studied in [1]. The smallest number of sets that can be obtained from a vertex partition of G into kLP sets is called the *k-limited packing partition number* (kLPP number) and is denoted by $\chi_{\times k}(G)$. This concept can also be considered as the dual of kTD partition problem. Our main focus for kTLP sets is on $k = 2$. This is because for larger values of k, we lose some significant families of graphs (for instance, $\gamma_{\times k}$ and $d_{\times k}$ cannot be defined for trees when $k \geq 3$) or we encounter trivial problems (for instance, $L_{\times k}(G) = |V(G)|$ and $\chi_{\times k}(G) = 1$ if $k \geq \Delta(G) + 1$). On the other side, many results for $k \in \{1, 2\}$ may be generalized to the general case k. In addition, stronger results may be obtained for small values of k.

For the *cartesian product* of graphs G and H, denoted $G \Box H$, and the *direct product* of graphs G and H, denoted $G \times H$, the vertex set of the product is $V(G) \times V(H)$.

Their edge sets are defined as follows. In $G\Box H$, two vertices are adjacent if they are adjacent in one coordinate and equal in the other. In $G\times H$ two vertices are adjacent if they are adjacent in both coordinates.
Suppose that G is a labeled graph on n vertices, and $\mathcal{H}$ is a sequence of n rooted graphs $H_1, H_2, \ldots, H_n$. If we identify the i^{th} vertex of G with the root of H_i, we obtain a new graph called the *rooted product* graph. This graph is denoted by $G(\mathcal{H})$. We here focus on the special case of rooted product graphs for which $\mathcal{H}$ consists of n isomorphic rooted graph. Assume that v is the root vertex of H, we define the rooted product graph $G\circ_v H=(V,E)$, such that $V=V(G)\times V(H)$ and

$$E=\bigcup_{i=1}^{n}\big\{(g_i,h)(g_i,h'):hh'\in E(H)\big\}\cup\big\{(g_i,v)(g_j,v):g_ig_j\in E(G)\big\}.$$

For $g\in V(G)$, $h\in V(H)$ and $*\in\{\Box,\times,\circ_v\}$, we call $G^h=\{(g,h)\in V(G*H)|\ g\in V(G)\}$ a G-layer through h, and ${}^gH=\{(g,h)\in V(G*H)|h\in V(H)\}$ an H-layer through g in $G*H$.
Notice that the subgraphs induced by the H-layers (resp. the G-layers) of $G\circ_v H$ (or $G\Box H$) are isomorphic to H (resp. to G). However, there are no edges between the vertices of G^h and the vertices of ${}^gH$ in direct product $G\times H$.
The *corona product* of two graphs G with $V(G)=v_1,...,v_n$ and H is defined as the graph created by taking one copy of G, $|V(G)|$ copies of H and joining $v_i\in V(G)$ to every vertex in the i^{th} copy of H. The corona product of the graphs G and H is denoted by $G\odot H$.
Here, we first discuss kTLP, especially when $k=2$, and give several sharp bounds for it. Then, we improve some of these inequalities for trees. In Section 3, we bound L_2 and $L_{2,t}$ for the cartesian product, direct product and rooted product graphs . In Section 4, we give a lower bound for $\chi_{\times k}$, and determine the values of $\chi_{\times 2}$ for the corona product. For the sake of convenience, for any graph G by an $\eta(G)$-set with $\eta\in\{L_k,\gamma_t,\rho,\rho_o,L_{k,t}\}$ we mean a kLP set, TD set, packing set, open packing set and kTLP set in G of cardinality $\eta(G)$, respectively.

2. Results on k-total limited packing

If G is a graph of order n and $k\geq n-1$, then $L_{k,t}(G)=n$. Note that $k\geq\Delta(G)$ is a weaker condition than the previous one. Therefore, we only need to compute the kTLP number for those graphs G such that $k<\Delta(G)$.
It was proved in [1] that the problem of computing the 2TLP number is NP-complete, even for bipartite graphs and for chordal graphs. Consequently, it would be desirable to bound this parameter in terms of several invariants of graphs. Several bounds on the kTLP number (emphasizing trees, especially when $k=2$) were given in the following. If $B\subseteq V(G)$ and $|B|=k$, then $|B\cap N(v)|\leq k$ for each vertex v of $V(G)$. So, $k\leq L_{k,t}(G)\leq n$.

Theorem 1. *Let G be a graph of order $n \geq 2$ with degree sequence $d_1, d_2, \ldots, d_n$ such that $d_1 \leq d_2 \leq \cdots \leq d_n$. Then*

$$L_{k,t}(G) \leq \max \ \{x| \ d_1 + d_2 + \cdots + d_x \leq kn\},$$

and this bound is sharp.

Proof. Let $\mathcal{B} = \{v_1, v_2, \ldots, v_{|\mathcal{B}|}\}$ be an $L_{k,t}(G)$-set. Then

$$d_1 + d_2 + \cdots + d_{|\mathcal{B}|} \leq \deg(v_1) + \deg(v_2) + \cdots + \deg(v_{|\mathcal{B}|}) \leq k|\mathcal{B}| + k(n - |\mathcal{B}|).$$

So $d_1 + d_2 + \cdots + d_{|\mathcal{B}|} \leq kn$. Therefore, $L_{k,t}(G) \in \{x| \ d_1 + d_2 + \cdots + d_x \leq kn\}$. The sharpness of this bound can be seen as follows. Suppose that G is a complete graph of order at least $k+2$. Then, it is easy to see that $L_{k,t}(G) = k$. On the other hand, $k = L_{k,t}(G) \leq max \ \{x| \ x(n-1) \leq kn\} = k$. □

Lemma 1. *If G is a graph of order n, then $L_{k,t}(G) \leq n + k - \Delta(G)$.*

Proof. Assume that w is a vertex of maximum degree in G. As we mentioned at the beginning of this section, we assume that $k < \Delta(G)$. Otherwise, $L_{k,t}(G) = n \leq n + k - \Delta(G)$. Let S be an $L_{k,t}(G)$-set. Since $|N(w) \cap S| \leq k$, there is at least $\Delta(G) - k$ vertices in $N(w) \backslash S$. Hence, $|\overline{S}| \geq \Delta(G) - k$. Therefore, we have $L_{k,t}(G) = |S| = n - |\overline{S}| \leq n + k - \Delta(G)$. □

We now define the family Ω consisting of all graphs G constructed as follows, and in the next theorem we show that Ω is the set of all graphs G of order n satisfying $L_{2,t}(G) = n+2-\Delta(G)$. Suppose that G is a graph of order n such that $V(G) = A \cup B$ has the following conditions:

(i) $|A \cap B| = 3$,

(ii) $G[A]$ has a spanning star, and each component of $G[B]$ is a path or a cycle, and

(iii) for every vertex $v \in \overline{B}$, we have $|N(v) \cap B| \leq 2$.

Figure 1 depicts a representative member of Ω.

Theorem 2. *If G is a graph of order n, then $L_{2,t}(G) \leq n + 2 - \Delta(G)$. Furthermore, $L_{2,t}(G) = n + 2 - \Delta(G)$ if and only if $G \in \Omega$.*

Proof. Suppose that S is an $L_{2,t}(G)$-set, and w is a vertex of maximum degree in G. Notice that each component of $G[S]$ is a path or a cycle, and we have $L_{2,t}(G) = |S| = n - |\overline{S}| \leq n + 2 - \Delta(G)$ by Lemma 1.
If $L_{2,t}(G) = n+2-\Delta(G)$, then $|\overline{S}| = \Delta(G)-2$, $\big(V(G) \backslash N[w]\big) \subseteq S$ and $|N[w] \cap S| = 3$. Based on the above argument, we have $G \in \Omega$ with $N[w] = A$ and $S = B$.

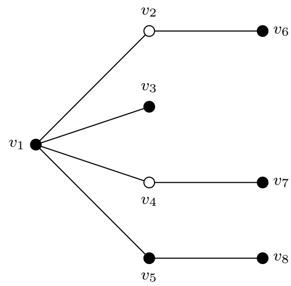

Figure 1. **A graph $H \in \Omega$ with $A = \{v_1, v_2, v_3, v_4, v_5\}$ and $B = \{v_1, v_3, v_5, v_6, v_7, v_8\}$.**

Let now $G \in \Omega$, then it suffices to prove that $L_{2,t}(G) \geq n + 2 - \Delta(G)$. Assume now that $A \cap B = \{u_1, u_2, u_3\}$ and $|A| = a+1$, where w is a vertex of degree a in $G[A]$. We claim that $\Delta(G) = a$. Each vertex $v \in B$ in $G[B]$ is at most of degree two. So, each of the vertices u_1, u_2 and u_3 is adjacent to at most two vertices in B. On the other hand, each of u_1, u_2, u_3 is adjacent to at most $a - 2$ vertices in $A \backslash \{u_1, u_2, u_3\}$. Thus, $\deg(u_1) \leq a$, $\deg(u_2) \leq a$ and $\deg(u_3) \leq a$. For each vertex $v \in A \backslash \{u_1, u_2, u_3\}$, v is adjacent to at most $a-3$ vertices in $A \backslash \{u_1, u_2, u_3, v\}$ and to at most two vertices in B. So, $\deg(v) \leq a - 1$ for every $v \in A \backslash \{u_1, u_2, u_3\}$. For each vertex $v' \in B \backslash \{u_1, u_2, u_3\}$, v' is adjacent to at most $a - 2$ vertices in $A \backslash \{u_1, u_2, u_3\}$ and to at most two vertices in B. Thus, $\deg(v') \leq a$ for every $v' \in B \backslash \{u_1, u_2, u_3\}$. Hence, $\Delta(G) \leq a$. But $\deg(w) \geq a$, which implies that $\Delta(G) = a$. Note that B is a 2TLP set of G with $|B| = n - |A| + 3 = n + 2 - \Delta(G)$. Therefore, we have $L_{2,t}(G) \geq n + 2 - \Delta(G)$. □

Theorem 3. *Let G be an r-regular graph of order n such that $L_{k,t}(G) = n + k - r$ for $k \leq r - 1$. Then, we have $r \geq \frac{n}{2}$.*

Proof. If $r = n - 1$, then G is a complete graph with $L_{k,t}(G) = k$ for $1 \leq k \leq n - 2$. So, let $r \leq n - 2$. Now assume that $w \in V(G)$ and that S is an $L_{k,t}(G)$-set with $|S| = n + k - r$. Since $|N(w) \cap S| \leq k$, it follows that $|N(w) \cap \overline{S}| \geq r - k$. Obviously, $|\overline{S}| = n - |S| = r - k$. Thus, there are exactly $r - k$ vertices, namely $v_1, v_2, \ldots, v_{r-k}$, in $N(w) \cap \overline{S}$. In particular, $\overline{S} = \{v_1, v_2, \ldots, v_{r-k}\}$ and $w \in S$. Let $U = V(G) \backslash N[w]$, clearly $U \subseteq S$, and $U \neq \emptyset$ since $r \leq n - 2$. If $u \in U$, then $|N(u) \cap S| \leq k$. So, any vertex $u \in U$ is adjacent to all vertices in $\overline{S}$, i.e., every vertex $v_i \in \overline{S}$ is adjacent to all $n - r - 1$ vertices in U. Note that v_i has at least one neighbor in $N[w]$, and $\deg(v_i) = r$. Therefore, $n - r - 1 + 1 \leq r$ and we get $r \geq \frac{n}{2}$. □

Remark 1. If G is an r-regular graph of order n with $r < \frac{n}{2}$, then $L_{k,t}(G) < n + k - r$ for $k \leq r - 1$. In fact, the bound of Lemma 1 is not sharp under these conditions.

Before we state the Theorem 4, we introduce a new concept needed in that theorem. For any tree T with order $n \geq 3$, $\delta'(T)$ denotes the minimum degree of T among all non-leaf vertices.

Theorem 4. *Let T be a tree of order $n \geq 3$ for which $\delta'(T) \geq c$ for some positive integer $c \geq 4$. Then, we have $L_{2,t}(T) \leq \frac{c-2}{c-1}n - c + 4$.*

Proof. We prove this theorem by induction on the order of tree T. Since $\delta'(T) \geq c$, we have $n \geq c+1$. If $n \in \{c+1, c+2, \ldots, 2c-1\}$, then $T \in \{K_{1,c}, K_{1,c+1}, \ldots, K_{1,2c-2}\}$, respectively. Hence, $L_{2,t}(T) = 3 \leq \frac{c-2}{c-1}n - c + 4$. Assume that for all tree T' of order $n' < n$ with $\delta'(T') \geq c$, we have $L_{2,t}(T') \leq \frac{c-2}{c-1}n' - c + 4$. Now let T be a tree of order $n \geq 2c$ such that $\delta'(T) \geq c$ and let S be an $L_{2,t}(T)$-set. We root T at r, and suppose v' is a leaf of T at the furthest distance from r, and v'' is the parent of v'. Assume that L is the set of all leaves in $N(v'')$. Since v'' is adjacent to at least $c-1$ leaves, it follows that $|L| \geq c-1$. Suppose that T'' be obtained from T by deleting all the vertices of L. By the induction hypothesis, we have $L_{2,t}(T'') \leq \frac{c-2}{c-1}|V(T'')| - c + 4 \leq \frac{c-2}{c-1}(n-(c-1)) - c + 4 = \frac{c-2}{c-1}n - 2c + 6$. On the other hand, $|L \cap S| \leq |N(v'') \cap S| \leq 2$. Therefore, we get $L_{2,t}(T) \leq L_{2,t}(T'') + 2 \leq \frac{c-2}{c-1}n - 2c + 8 \leq \frac{c-2}{c-1}n - c + 4$. $\square$

Proposition 1. *Let G be a graph without isolated vertices such that $\Delta(G) \geq 2$, then*

$$L_{2,t}(G) \leq \frac{\Delta(G)^2 + 1}{\delta(G)}\rho_o(G).$$

Proof. Let $v \in V(G)$ be an arbitrary vertex, then the set of all vertices at distance at most two from v has at most $\Delta(G)^2 + 1$ vertices. Thus, $\rho_o(G) \geq \frac{2n}{\Delta(G)^2+1}$, by the greedy algorithm. Moreover, $L_{k,t}(G) \leq \frac{kn}{\delta(G)}$ [10], and we get

$$\rho_o(G) \geq \frac{2n}{\Delta(G)^2 + 1} = \frac{2n\delta(G)}{(\Delta(G)^2 + 1)\delta(G)} \geq L_{2,t}(G)\frac{\delta(G)}{\Delta(G)^2 + 1}.$$

Therefore, we infer that $L_{2,t}(G) \leq \frac{\Delta(G)^2+1}{\delta(G)}\rho_o(G)$. $\square$

We can improve the above bounds for trees as follows.

Proposition 2. *If T is a given tree with $\Delta(T) \geq 2$, then*

$$L_{2,t}(T) \leq 2\rho_o(T).$$

Proof. We know that $L_{2,t}(T) \leq 2\gamma_t(T)$ ([10]). On the other hand, we have $\rho_o(T) = \gamma_t(T)$ for every tree T with at least two vertices ([13]). As a consequence, we have $L_{2,t}(T) \leq 2\gamma_t(T) = 2\rho_o(T)$. $\square$

We next state a relevant result on 2- total limited packings of graphs, which will be needed later.

Lemma 2. *(Hosseini Moghaddam et al. [10]) Let G be a graph of order n such that $\Delta(G) \geq 2$, then $\rho_o(G) + 1 \leq L_{2,t}(G)$.*

According to the previous lemma and the Proposition 2, we can say $\rho_o(T) + 1 \leq L_{2,t}(T) \leq 2\rho_o(T)$ for every tree T with $\Delta(T) \geq 2$. The next theorem shows how equality occurs in these bounds.

Theorem 5. *Let T be a tree such that $\Delta(T) \geq 2$. Then,*

(i) $\rho_o(T) + 1 = L_{2,t}(T)$ if and only if T is a star with at least three vertices, and

(ii) $L_{2,t}(T) = 2\rho_o(T)$ if and only if for every $L_{2,t}(T)$-set S and every $\gamma_t(T)$-set D, we have $|N(s) \cap D| = 1$ and $|N(d) \cap S| = 2$ for any $s \in S$ and any $d \in D$.

Proof. Let T be the star $K_{1,x}$ with $x \geq 2$. Then, $\rho_o(T) = 2$ and $L_{2,t}(T) = 3$. Therefore, $\rho_o(T) + 1 = L_{2,t}(T)$.
It remains for us to prove the converse. Assume now that T is a tree with $\rho_o(T)+1 = L_{2,t}(T)$. We claim that $diam(T) \leq 2$. Suppose to the contrary that there exist two vertices $v_1, v_4 \in V(T)$ such that $d(v_1, v_4) = 3$ and let $P = v_1v_2v_3v_4$ be the path between them. Assume that S_1 is a $\rho_o(T)$-set, then $|V(P) \cap S_1| \leq 2$. We now consider three cases as follows.

Case 1. Let $V(P) \cap S_1 = \emptyset$. Set $S_2 = S_1 \cup \{v_1, v_2\}$, we show that S_2 is a 2TLP set of T. Since $|N(v_i) \cap S_1| \leq 1$ and $|N(v_i) \cap \{v_1, v_2\}| \leq 1$ for $1 \leq i \leq 4$, it follows that $|N(v_i) \cap S_2| \leq 2$ for every $v_i \in V(P)$. Let now w be a vertex outside of P, so $|N(w) \cap \{v_1, v_2\}| \leq 1$ because T has no cycle. Thus, $|N(w) \cap S_2| \leq 2$ for every vertex w outside P. Therefore, we conclude that S_2 is a 2TLP set of T.

Case 2. Assume $V(P) \cap S_1 = \{v_i\}$ for $1 \leq i \leq 4$. First, let $i = 1$ or 4, by using similar techniques as in the previous case, $S_1 \cup \{v_2, v_3\}$ is a 2TLP set of T. If $i = 2$ or 3, then $S_1 \cup \{v_1, v_4\}$ is a 2TLP set of T.

Case 3. Suppose $V(P) \cap S_1 = \{v_i, v_j\}$ for some $1 \leq i \neq j \leq 4$. If $(i, j) \in \{(1, 2), (1, 4), (2, 3), (3, 4)\}$, then $S_1 \cup \{v_3, v_4\}$, $S_1 \cup \{v_2, v_3\}$, $S_1 \cup \{v_1, v_4\}$ and $S_1 \cup \{v_1, v_2\}$ are 2TLP sets of T, respectively.
In each case, we observe that $L_{2,t}(T) \geq \rho_o(T) + 2$, which contradicts the assumption $\rho_o(T) + 1 = L_{2,t}(T)$. Therefore, we deduce that $diam(T) \leq 2$, and T is a star with at least three vertices.
Let now T be a tree with $\Delta(T) \geq 2$. As mentioned earlier, we know that $L_{2,t}(T) = 2\rho_o(T)$ if and only if $L_{2,t}(T) = 2\gamma_t(T)$. Let S be an $L_{2,t}(T)$-set, and D be a $\gamma_t(T)$-set. We now restate the proof of Theorem 7 in [10]. We set $U = \{(s, d) \in V(G) \times V(G) | s \in S, d \in D \text{ and } s \in N(d)\}$, and count the members of U in two ways. Since $|N(s) \cap D| \geq 1$ for any $s \in S$, it follows that there is at least one vertex $d \in D$ such that $s \in N(d)$. Thus, $|S| \leq |U|$. On the other hand, for any $d \in D$ we have $|N(d) \cap S| \leq 2$. Hence, there exists at most two vertices $s_1, s_2 \in S$ such that $s_1, s_2 \in N(d)$. So, we get $|U| \leq 2|D|$, and $|S| \leq 2|D|$. Therefore, $L_{2,t}(T) = 2\gamma_t(T)$ if and only if the following statements hold:

(1) for any $s \in S$, we have $|N(s) \cap D| = 1$,

(ii) for any $d \in D$, we have $|N(d) \cap S| = 2$.

□

Theorem 6. *If G is a graph, then for any edge $e \in E(G)$,*

$$L_{k,t}(G) \leq L_{k,t}(G-e) \leq L_{k,t}(G) + 2.$$

Furthermore, these bounds are sharp.

Proof. Any kTLP set of G is also a kTLP set of $G-e$, so $L_{k,t}(G) \leq L_{k,t}(G-e)$. Moreover, if C is a cycle on n vertices, then $L_{2,t}(C) = L_{2,t}(C-e) = n$ for every edge $e \in E(C)$.
Suppose now that B is an $L_{k,t}(G-e)$-set and $e = uv$. If $u, v \in B$, then $B - \{u, v\}$ is a kTLP set of G, and hence $L_{k,t}(G) \geq |B| - 2$. If $u \in B$ and $v \notin B$, then $B - \{u\}$ is a kTLP set of G, and $L_{k,t}(G) \geq |B| - 1$. If $u, v \notin B$, then B is a kTLP set of G, and we have $L_{k,t}(G) \geq |B|$. Therefore, $L_{k,t}(G-e) \leq L_{k,t}(G) + 2$.
Let G be a double star $ST(x, y)$, which is the graph obtained by joining the centers of two stars $K_{1,x}$ and $K_{1,y}$ with an edge, such that $x, y \geq k+1$. Assume that the center of stars are u and v, respectively. Then, $L_{k,t}(G-e) = L_{k,t}(G) + 2$ for $e = uv$. □

Theorem 7. *Let G have a unique $L_{2,t}(G)$-set B. Then every leaf of G belongs to B.*

Proof. Let B be a unique $L_{2,t}(G)$-set, and let there exist a leaf $l \notin B$ with the support vertex v. If $v \in B$ and $|N(v) \cap B| \leq 1$, then $B' = B \cup \{l\}$ is a 2TLP set which is greater than B, a contradiction. So if $v \in B$, then $|N(v) \cap B| = 2$. Let $u \in N(v) \cap B$. We can easily see that $B'' = (B \backslash \{u\}) \cup \{l\}$ is an $L_{2,t}(G)$-set, which is impossible because B is unique. Hence $v \notin B$.
If some neighbor of v, say u', belongs to B, then $B'' = (B \backslash \{u'\}) \cup \{l\}$ is an $L_{2,t}(G)$-set. This contradicts the assumption. Therefore, we deduce that $N[v] \cap B = \emptyset$. So $B \cup \{l\}$ is a 2TLP set, which is a contradiction with the maximality of B. Hence $l \in B$. □

If $diam(G) = 1$, then G is a complete graph, and we know that $L_{2,t}(K_n) = 2$. What can be said about the 2TLP number of graphs with diameter 2? The following theorem is an answer to this question.

Theorem 8. *If $c \geq 3$ is a positive integer, then there exists a graph G with $diam(G) = 2$ such that $L_{2,t}(G) = c$.*

Proof. In what follows, we construct a graph G with diameter 2 for which $L_{2,t}(G) = c$. Suppose that $V_1 = \{v_1, v_2, \ldots, v_c\}$ and $V_2 = \{u_1, u_2, \ldots, u_{\frac{c(c-1)}{2}}\}$ with $V_1 \cap V_2 = \emptyset$. Let G be a graph with vertex set $V(G) = V_1 \cup V_2$ such that $G[V_1] = cK_1$, $G[V_2] =$

$K_{\frac{c(c-1)}{2}}$ and each pair of distinct vertices in V_1 has a unique common neighbor in V_2. Obviously, $diam(G) = 2$. It remains to show that $L_{2,t}(G) = c$. We know $|V(G)| = c + \frac{c(c-1)}{2}$ and $\Delta(G) = \frac{c(c-1)}{2} + 1$. Hence, by Theorem 2, $L_{2,t}(G) \leq |V(G)| + 2 - \Delta(G) = c + 1$. But $G \notin \Omega$, so $L_{2,t}(G) \leq c$. On the other hand, V_1 is a 2TLP set of G. Thus, $L_{2,t}(G) = c$. □

Theorem 9. *Assume that $a \geq 3$ and b are two integers with $a+1 \leq b \leq 2a$. Then, there exists a tree T for which $\rho_o(T) = a$ and $L_{2,t}(T) = b$.*

Proof. Let $a \geq 3$ and b be two integers such that $a+1 \leq b \leq 2a$, and $b = a+x$ with $1 \leq x \leq a$. In what follows, we construct a tree T with $\rho_o(T) = a$ and $L_{2,t}(T) = a+x$ for $a \geq 3$ and $1 \leq x \leq a$. We distinguish two cases based on the value of x.

Case 1. First, let $x = a$. Assume $P = v_1v_2 \dots v_a$ is a path. We add two leaves u_{i_1} and u_{i_2} to each v_i, and obtain tree T. Let S_1 be a $\rho_o(T)$-set. If $|S_1| \geq a+1$, by the Pigeonhole principle, there is at least one vertex v_i such that $|N(v_i) \cap S_1| \geq 2$, which is impossible. Hence, $\rho_o(T) \leq a$. On the other hand, $\{u_{1_1}, u_{2_1}, \dots, u_{a_1}\}$ is a 1TLP set of T, so $\rho_o(T) = a$.
Let S_2 be an $L_{2,t}(T)$-set. Similarly, if $L_{2,t}(T) \geq 2a+1$, there exists at least one vertex v_i such that $|N(v_i) \cap S_2| \geq 3$, a contradiction. Thus, $L_{2,t}(T) \leq 2a$. Moreover, $\{u_{1_1}, u_{1_2}, u_{2_1}, u_{2_2} \dots, u_{a_1}, u_{a_2}\}$ is a 2TLP set of T, hence $L_{2,t}(T) = 2a = b$.

Case 2. Suppose now that $1 \leq x \leq a-1$. Consider the star $T' = K_{1,a}$ with $V(T') = \{r, v_1, v_2, \dots, v_a\}$ and $\deg(r) = a$. Let T be the tree obtained from T' by adding two leaves u_i and u'_i to each v_i for $1 \leq i \leq x-1$ and one leaf u_i to each v_i for $x \leq i \leq a-1$. We show that $\rho_o(T) = a$ and $L_{2,t}(T) = b$. Since $T \notin \Omega$, it follows that $L_{2,t}(T) < |V(T)| + 2 - \Delta(T)$ by Theorem 2. Notice that $|V(T)| = 2a + x - 1$ and $\Delta(T) = a$, thus $L_{2,t}(T) \leq a + x$. On the other hand, $\{u_1, u_2, \dots, u_{a-1}, u'_1, u'_2, \dots, u'_{x-1}, v_1, v_a\}$ is a 2TLP set of T, so $L_{2,t}(T) = a + x = b$. Since T is a tree with at least two vertices, $\rho_o(T) = \gamma_t(T)$ [10]. Moreover, $\{r, v_1, v_2, \dots, v_{a-1}\}$ is a TD set of T, and hence $\gamma_t(T) \leq a$. Thus, $\rho_o(T) \leq a$. It is readily verified that $\{u_1, u_2, \dots, u_{a-1}, v_a\}$ is a 1TLP set of T. Therefore, $\rho_o(T) = a$. □

3. On 2-(total) limited packing number of some graph products

Theorem 10. *For any graphs G and H, $L_{2,t}(G\square H) \geq \max\{L_{2,t}(G)\rho(H), \rho(G)L_{2,t}(H)\}$. Moreover, this bound is sharp.*

Proof. Let P_G and P_H be an $L_{2,t}(G)$-set and a $\rho(H)$-set, respectively. Set $P = P_G \times P_H$, and suppose to the contrary that P is not a 2TLP set of $G\square H$. Therefore, there exists a vertex $(x, y) \in V(G) \times V(H)$ adjacent to three distinct vertices $(g_1, h_1), (g_2, h_2), (g_3, h_3) \in P$. We distinguish the following cases.

Case 1. $h_1 = h_2 = h_3$. If y is adjacent to h_1, then $x = g_1 = g_2 = g_3$, which is impossible. So, $y = h_1 = h_2 = h_3$. In such a situation, x is adjacent to $g_1, g_2, g_3 \in P_G$, which contradicts the fact that P_G is a 2TLP set in G.

Case 2. At least two vertices from $\{h_1, h_2, h_3\}$, say h_1 and h_2, are distinct. By the adjacency rule of the Cartesian product graphs, we deduce that $\{h_1, h_2\} \subseteq N_H[y] \cap P_H$. This contradicts the fact that P_H is a packing in H.

Therefore, P is a 2TLP set in $G\Box H$. Hence, $L_{2,t}(G\Box H) \geq |P| = L_{2,t}(G)\rho(H)$. Similarly, we have $L_{2,t}(G\Box H) \geq \rho(G)L_{2,t}(H)$.

We can show that this bound is sharp in the following way. Let G' be any connected graph on the set of vertices $\{v'_1, \dots, v'_n\}$. Let $G = G' \odot K_1$, in which $N_G(v'_i) \backslash N_{G'}(v'_i) = \{v_i\}$ for each $1 \leq i \leq n$. We now consider the graph $G\Box K_r$ for $r \geq 3$, and let Q be an $L_{2,t}(G\Box K_r)$-set. Clearly, $\rho(G) = n$ and $L_{2,t}(K_r) = 2$. It is not difficult to see that $|Q \cap (\{v_i, v'_i\} \times V(K_r))| \leq 2$ for each $1 \leq i \leq n$. This implies that

$$\begin{aligned} L_{2,t}(G) &= |Q| = |Q \cap V(G\Box K_r)| = |Q \cap \left(\cup_{i=1}^{n} (\{v_i, v'_i\} \times V(K_r)) \right)| \\ &= \textstyle\sum_{i=1}^{n} |Q \cap (\{v_i, v'_i\} \times V(K_r))| \leq 2n = L_{2,t}(K_r)\rho(G). \end{aligned} \tag{3.1}$$

On the other hand, $L_{2,t}(G) \leq 2n$ since G has $2n$ vertices. We also know $\rho(K_r) = 1$, so $\max\{L_{2,t}(G)\rho(K_r), \rho(G)L_{2,t}(K_r)\} = L_{2,t}(K_r)\rho(G)$.

According to this theorem, we have $L_{2,t}(G\Box K_r) \geq \rho(G)L_{2,t}(K_r)$. Therefore, this bound is sharp.

To show the sharpness of this bound, we present a simpler example by considering $G = K_{m,n}$ and $H = K_2$ for $m, n \geq 2$. We know that $L_{2,t}(K_{m,n}) = 4$, $\rho(K_2) = 1$ and $L_{2,t}(K_2) = \rho(K_{m,n}) = 2$. It is easy to see that $L_{2,t}(K_{m,n}\Box K_2) = 4$. □

Theorem 11. *Let G and H be graphs with i_G and i_H isolated vertices, respectively. Then,*

$$L_{2,t}(G \times H) \geq \max\ \{\rho_o(G^-)L_{2,t}(H^-), L_{2,t}(G^-)\rho_o(H^-)\} + i_G|V(H)| + i_H|V(G)| - i_G i_H$$

and this bound is sharp.

Proof. Suppose first that G and H are graphs without isolated vertices. Let P_G and P_H be a $\rho_o(G)$-set and an $L_{2,t}(H)$-set, respectively. Set $P = P_G \times P_H$, and assume for the sake of contradiction that P is not a 2TLP set of $G \times H$. Hence, there exists a vertex $(x, y) \in V(G \times H)$ adjacent to three distinct vertices $(g, h), (g', h'), (g'', h'') \in P$. Then $g = g' = g''$ because P_G is a $\rho_o(G)$-set. So $h \neq h' \neq h''$ and $|N(y) \cap P_H| \geq 3$, a contradiction. Therefore, P is a 2TLP set in $G \times H$, and $L_{2,t}(G \times H) \geq |P| = \rho_o(G)L_{2,t}(H)$. We have $L_{2,t}(G \times H) \geq L_{2,t}(G)\rho_o(H)$ by a similar fashion.

We have

$$L_{2,t}(G \times H) = L_{2,t}(G^- \times H^-) + i_G|V(H)| + i_H|V(G)| - i_G i_H.$$

Therefore,

$$L_{2,t}(G\times H) \geq \max\ \{\rho_o(G^-)L_{2,t}(H^-), L_{2,t}(G^-)\rho_o(H^-)\} + i_G|V(H)| + i_H|V(G)| - i_G i_H.$$

In what follows, we show that this bound is sharp. Let G be a bipartite graph without isolated vertices. Then, $L_{2,t}(G \times K_2) = L_{2,t}(2G) = 2L_{2,t}(G)$. On the other hand, $L_{2,t}(G \times K_2) \geq max\ \{\rho_o(G)L_{2,t}(K_2), L_{2,t}(G)\rho_o(K_2)\} = max\ \{2\rho_o(G), 2L_{2,t}(G)\} = 2L_{2,t}(G)$. □

Theorem 12. *Let G be a graph of order n with i_G isolated vertices. If H is a rooted graph at v, then*

$$n(L_{2,t}(H) - 1) + i_G \leq L_{2,t}(G \circ_v H) \leq nL_{2,t}(H).$$

Furthermore, these bounds are sharp.

Proof. Note that any H-layer in $G \circ_v H$ is isomorphic to H. So, each $L_{2,t}(G \circ_v H)$-set intersects every H-layer is at most $L_{2,t}(H)$ vertices. Hence, $L_{2,t}(G \circ_v H) \leq nL_{2,t}(H)$. On the other side, let P_H be an $L_{2,t}(H)$-set. We can readily observe that $P = \bigcup_{g\in V(G)} \big(\{g\} \times (P_H \backslash \{v\})\big)$ is a 2TLP set in $G \circ_v H$, so we have $n(L_{2,t}(H) - 1) \leq |P| \leq L_{2,t}(G \circ_v H)$.
Suppose that there exists an $L_{2,t}(H)$-set P_H not consisting of v. Notice that $P = \bigcup_{g\in V(G)} \big(\{g\} \times P_H\big)$ is a 2TLP set in $G \circ_v H$, and we have $nL_{2,t}(H) = |P| \leq L_{2,t}(G \circ_v H)$. Thus, we conclude that $L_{2,t}(G \circ_v H) = nL_{2,t}(H)$ in this case.
Assume now that v has degree two in all subgraphs induced by every $L_{2,t}(H)$-set P_H, that is $\deg_{H[P_H]}(v) = 2$. Suppose that for every 2TLP set P'_H in H, $|P'_H \backslash N[v]| \leq L_{2,t}(H) - 3$. Assume that P is an $L_{2,t}(G \circ_v H)$-set. Note that exactly i_G components of $G \circ_v H$ are isomorphic to H, which implies that each of these components has exactly $L_{2,t}(H)$ vertices in P.Moreover, we have one component isomorphic to $G^- \circ_v H$. Let $P^- = P \cap (G^- \circ_v H)$ and $P^-_g = P^- \cap {}^gH$ for every $g \in V(G^-)$. We now show that $L_{2,t}(G^- \circ_v H) = (n - i_G)(L_{2,t}(H) - 1)$. Assume that $L_{2,t}(G^- \circ_v H) > (n - i_G)(L_{2,t}(H) - 1)$. Hence, there exists at least one vertex $g_1 \in V(G^-)$ for which $|P^-_{g_1}| = L_{2,t}(H)$. Otherwise, $|P^-| = \sum_{g_i\in V(G^-)} |P^-_{g_i}| \leq \sum_{g_i\in V(G^-)} L(_{2,t}(H) - 1) = (n - i_G)(L_{2,t}(H) - 1)$, which is a contradiction.
Since G^- has no isolated vertex, there exists a vertex $g_2 \in V(G^-)$ for which $g_1g_2 \in E(G)$. If $|P^-_{g_2}| = L_{2,t}(H)$, then both (g_1, v) and (g_2, v) have three neighbors in P^-, which is impossible. Therefore $|P^-_{g_2}| \leq L_{2,t}(H) - 1$ for all $g_2 \in N_G(g_1)$. Now let g_2 be an arbitrary neighbor of g_1 in G^-. If $|P^-_{g_2}| = L_{2,t}(H) - 1$, then $|N_{(G\circ_v H)[^{g_2}H]}[(g_2, v)] \cap P^-_{g_2}| = 2$. This implies that $|N_{(G^-\circ_v H)}(g_1, v) \cap P^-| \geq 3$ or $|N_{(G^-\circ_v H)}(g_2, v) \cap P^-| \geq 3$, a contradiction. Thus, $|P^-_{g_2}| \leq L_{2,t}(H) - 2$ for all $g_2 \in N_G(g_1)$.
The above argument provides a guarantee that for every vertex $g_1 \in V(G)$ such that $|P^-_{g_1}| = L_{2,t}(H)$, we have $|P^-_{g_2}| \leq L_{2,t}(H) - 2$ for all $g_2 \in N_G(g_1)$. This implies that

$$L_{2,t}(G^- \circ_v H) \leq \frac{n - i_G}{2} L_{2,t}(H) + \frac{n - i_G}{2}(L_{2,t}(H) - 2) = (n - i_G)(L_{2,t}(H) - 1),$$

which contradicts the assumption $L_{2,t}(G^- \circ_v H) > (n-i_G)(L_{2,t}(H)-1)$. So $L_{2,t}(G^- \circ_v H) \leq (n-i_G)(L_{2,t}(H)-1)$. It follows that $L_{2,t}(G^- \circ_v H) = (n-i_G)(L_{2,t}(H)-1)$ by using the corresponding inequality obtained from the first steps of the proof. Therefore, $L_{2,t}(G \circ_v H) = L_{2,t}(G^- \circ_v H) + i_G L_{2,t}(H) = (n-i_G)(L_{2,t}(H)-1) + i_G L_{2,t}(H) = n(L_{2,t}(H)-1) + i_G$. □

Theorem 13. *For any graphs G and H,*

$$L_2(G\Box H) \leq \min\{L_2(G)|V(H)|, L_2(H)|V(G)|\},$$

and this bound is sharp.

Proof. Let $V(H) = \{v_1, v_2, \ldots, v_{|V(H)|}\}$. It is obvious that $G\Box H$ contains $|V(H)|$ disjoint G-layers. Suppose now that P is an $L_2(G\Box H)$-set, thus $P_i = P \cap G^{v_i}$ is a 2LP set in $(G\Box H)[G^{v_i}]$ for each $1 \leq i \leq |V(H)|$. Therefore, $|P_i| \leq L_2(G)$, which leads to

$$L_2(G\Box H) = |P| = \sum_{i=1}^{|V(H)|} |P_i| \leq L_2(G)|V(H)|.$$

Similarly, we have $L_2(G\Box H) \leq L_2(H)|V(G)|$.
For sharpness consider $G = P_2$ and $H = K_{m,n}$ for $m, n \geq 2$. We observe that $L_2(K_{m,n}) = 2$ [7], and $L_2(P_2) = 2$. It is easy to see that $L_2(G\Box H) = 4$. □

Theorem 14. *Let G and H be graphs with i_G and i_H isolated vertices, respectively. Then,*

$$\begin{aligned} L_2(G \times H) \geq \max\ &\{\rho_o(G^-)L_2(H^-), L_2(G^-)\rho_o(H^-), \rho(G^-)L_{2,t}(H^-), L_{2,t}(G^-)\rho(H^-)\} \\ &+ i_G|V(H)| + i_H|V(G)| - i_G i_H. \end{aligned}$$

Moreover, this bound is sharp.

Proof. Assume first that G and H are graphs without isolated vertices. Let P_G, P'_G, P_H and P'_H be an $L_{2,t}(G)$-set, a $\rho_o(G)$-set, a $\rho(H)$-set and an $L_2(H)$-set, respectively. Set $P = P_G \times P_H$ and $P' = P'_G \times P'_H$, and suppose to the contrary that P and P' are not 2LP sets of $G \times H$. So there exist vertices (x, y), $(x', y') \in V(G \times H)$ such that $|N[(x, y)] \cap P| \geq 3$ and $|N[(x', y')] \cap P'| \geq 3$, respectively.
If $(x, y) \in P$, then (x, y) is adjacent to two distinct vertices $(g, h), (g', h') \in P$. So $|N[y] \cap P_H| \geq 2$, which is impossible. If $(x, y) \in V(G \times H)\backslash P$, then (x, y) is adjacent to three distinct vertices $(g, h), (g', h'), (g'', h'') \in P$. We observe that $h = h' = h''$ because P_H is a $\rho(H)$-set. Hence $g \neq g' \neq g''$ and $|N(x) \cap P_G| \geq 3$, a contradiction. Therefore, P is a 2LP set in $G \times H$ and $L_2(G \times H) \geq |P| \geq L_{2,t}(G)\rho(H)$. We have $L_2(G \times H) \geq \rho(G)L_{2,t}(H)$ by a similar method.
If $(x', y') \in P'$, then (x', y') is adjacent to two distinct vertices $(g_1, h_1), (g_2, h_2) \in P'$. We have $g_1 = g_2$ because P'_G is a $\rho_o(G)$-set. Thus, $h_1 \neq h_2$ and $|N[y'] \cap P'_H| \geq 3$, which is impossible. If $(x', y') \in V(G \times H)\backslash P'$, then (x', y') is adjacent to three

distinct vertices $(g_1,h_1),(g_2,h_2),(g_3,h_3)\in P'$. $g_1=g_2=g_3$ since P'_G is a $\rho_o(G)$-set, so $h_1\neq h_2\neq h_3$ and $|N[y']\cap P'_H|\geq 3$, a contradiction. Therefore P' is a 2LP set in $G\times H$, and $L_2(G\times H)\geq \rho_o(G)L_2(H)$. We get $L_2(G\times H)\geq L_2(G)\rho_o(H)$ by a similar fashion. Therefore,

$$L_2(G\times H)\geq \max\ \{\rho_o(G)L_2(H), L_2(G)\rho_o(H), \rho(G)L_{2,t}(H), L_{2,t}(G)\rho(H)\}.$$

We now suppose that G and H are arbitrary graphs. Then,

$$L_2(G\times H)=L_2(G^-\times H^-)+i_G|V(H)|+i_H|V(G)|-i_Gi_H\geq$$

$$\max\ \{\rho_o(G^-)L_2(H^-), L_2(G^-)\rho_o(H^-), \rho(G^-)L_{2,t}(H^-), L_{2,t}(G^-)\rho(H^-)\}+$$

$$i_G|V(H)|+i_H|V(G)|-i_Gi_H.$$

In what follows, we show the sharpness of this bound. Let G be a bipartite graph without isolated vertices. Then, $L_2(G\times K_2)=L_2(2G)=2L_2(G)$. On the other hand,

$$L_2(G\times K_2)\geq \max\ \{\rho_o(G)L_2(K_2), L_2(G)\rho_o(K_2), \rho(G)L_{2,t}(K_2), L_{2,t}(G)\rho(K_2)\}=$$

$$\max\ \{2\rho_o(G), 2L_2(G), 2\rho(G), L_{2,t}(G)\}=2L_2(G).$$

□

We end this section by studying the 2LP number of rooted product graphs.

Theorem 15. *Let G be a graph of order n. If H is a graph with root v, then*

$$L_2(G\circ_v H)=\begin{cases} L_2(G)+n(L_2(H)-1) & \text{if } v\in P_H \text{ for every } L_2(H)\text{-set } P_H,\\ nL_2(H) & \text{if } v\notin P_H \text{ for some } L_2(H)\text{-set } P_H.\end{cases}$$

Proof. We consider two cases based on the membership of v to $L_2(H)$-sets.
Case 1. Assume that v belongs to any $L_2(H)$-set P_H, and P' be an $L_2(G)$-set. Set $P=(P'\times\{v\})\cup\big(V(G)\times(P_H\backslash\{v\})\big)$. It can be readily seen that P is a 2LP set in $G\circ_v H$. Therefore, $L_2(G\circ_v H)\geq|P'\times\{v\}|+|V(G)\times(P_H\backslash\{v\})|=L_2(G)+n(L_2(H)-1)$. On the other hand, let B be an $L_2(G\circ_v H)$-set. Then $B_g=B\cap{}^gH$ is a 2LP set in $(G\circ_v H)[{}^gH]$ for every $g\in V(G)$. Note that B_g is not an $L_2((G\circ_v H)[{}^gH])$-set for some $g\in V(G)$ since v belongs to every $L_2(H)$-set. Hence, $|B\cap{}^gH|=|B_g|\leq L_2(H)-1$ if $(g,v)\notin B$, which means $|B\cap({}^gH|\backslash\{(g,v)\})|\leq L_2(H)-1$. Also if $(g,v)\in B$, then $|B\cap({}^gH|\backslash\{(g,v)\})|\leq L_2(H)-1$ as well. In addition, $B\cap G^v$ is a 2LP set in $(G\circ_v H)[G^v]$. Thus, $|B\cap G^v|\leq L_2(G)$, and we have

$$L_2(G\circ_v H)=|B|=|B\cap G^v|+\sum_{g\in V(G)}|B\cap({}^gH|\backslash\{(g,v)\})|\leq L_2(G)+n(L_2(H)-1).$$

Therefore, $L_2(G \circ_v H) = L_2(G) + n(L_2(H) - 1)$.
Case 2. Assume that there exists an $L_2(H)$-set P_H for which $v \notin P_H$. Let ${}^gP_H = \{g\} \times P_H$ for every $g \in V(G)$, and let $P'' = \cup_{g \in V(G)} {}^gP_H$. We can easily see that P'' is a 2LP set in $G \circ_v H$, so $L_2(G \circ_v H) \geq |P''| = nL_2(H)$. On the other hand, let P be an $L_2(G \circ_v H)$-set. We can easily observe that the set $P_g = P \cap {}^gH$ is a 2LP set in $(G \circ_v H)[{}^gH]$ for every $g \in V(G)$. So $L_2(H) = L_2(G \circ_v H)[{}^gH] \geq |P_g|$. Therefore, $L_2(G \circ_v H) = |P| = \sum_{g \in V(G)} |P_g| \leq \sum_{g \in V(G)} L_2(H) = nL_2(H)$. This leads to $L_2(G \circ_v H) = nL_2(H)$. □

4. Results on vertex partitioning into k-limited packings

In the previous sections, we studied about k-limited packings in graphs. In the following, we state two theorems for the vertex partitioning into k-limited packing sets. As we said before, $\chi_{\times k}(G)$ is the minimum cardinality of kLPP in G. In the next theorem, we discuss the relationship between $L_k(G)$ and $\chi_{\times k}(G)$.

Theorem 16. *If G is a graph of order $n \geq 2$, then $\chi_{\times k}(G) \geq 2\sqrt{n} - L_k(G)$.*

Proof. We first prove that $\chi_{\times k}(G) \times L_k(G) \geq n$. Let $\{B_1, B_2, \ldots, B_{\chi_{\times k}(G)}\}$ be a kLPP of G. Then,

$$\chi_{\times k}(G) \times L_k(G) = \sum_{i=1}^{\chi_{\times k}(G)} L_k(G) \geq \sum_{i=1}^{\chi_{\times k}(G)} |B_i| = n$$

and equality holds when each set B_i is an $L_k(G)$-set. So $\chi_{\times k}(G) + L_k(G) \geq \chi_{\times k}(G) + \frac{n}{\chi_{\times k}(G)}$.
On the other hand, $\chi_{\times k}(G) \leq \frac{n}{k}$ because every subset of $V(G)$ of cardinality at most k is a kLP set. We observe that the function $g(x) = x + \frac{n}{x}$ is decreasing for $1 \leq x \leq \sqrt{n}$, and it is increasing for $\sqrt{n} \leq x \leq \frac{n}{k}$. Therefore $\chi_{\times k}(G) + L_k(G) \geq 2\sqrt{n}$.
This bound is sharp for the complete graph K_4, the cycle C_4 and the star S_4. □

We end with a study of the 2LPP number of corona product graphs. If v is a vertex of maximum degree in G, then $|N_{G \odot H}[v]| = \Delta(G) + 1 + |V(H)|$. So we need at least $\lceil \frac{\Delta(G)+1+|V(H)|}{2} \rceil$ 2-limited packing sets in every 2LPP of $G \odot H$. Therefore $\chi_{\times 2}(G \odot H) \geq \lceil \frac{\Delta(G)+1+|V(H)|}{2} \rceil$.

Theorem 17. *If G and H are two graphs, then*

$$\chi_{\times 2}(G \odot H) \in \{\chi_{\times 2}(G), \chi_{\times 2}(G) + 1, \chi_{\times 2}(G) + 2, \ldots, \chi_{\times 2}(G) + \lceil \frac{|V(H)|}{2} \rceil\}.$$

Proof. Let $\mathbb{P} = \{P_1, P_2, \ldots, P_{\chi_{\times 2}(G)}\}$ be a 2LPP of G, and let $V(G) = \{v_1, v_2, \ldots, v_n\}$ and $V(H) = \{u_1, u_2, \ldots, u_{n'}\}$.

Since G is a subgraph of $G \odot H$, $\chi_{\times 2}(G) \leq \chi_{\times 2}(G \odot H)$. That $\chi_{\times 2}(G \odot H) = \chi_{\times 2}(G)$ can be seen as follows. If $|N_G[v_i] \cap (\cup_{j=1}^{\chi_{\times 2}(G)} P_j)| \leq 2\chi_{\times 2}(G) - n'$ for each $v_i \in V(G)$, then we place the vertices of each copy of H in the members of $\mathbb{P}$ such that $|N_{G \odot H}[v_i] \cap P_j)| \leq 2$ for each $1 \leq i \leq n$ and $1 \leq j \leq \chi_{\times 2}(G)$. So the equality holds.

We now have two cases based on the behavior of $|V(H)|$ and prove the upper bound.

• Let $|V(H)|$ be even. In the worst case, if there exists a vertex $v_i \in V(G)$ such that $2\chi_{\times 2}(G) - 1 \leq |N_G[v_i] \cap (\cup_{j=1}^{\chi_{\times 2}(G)} P_j)| \leq 2\chi_{\times 2}(G)$, then we add new sets $P_{\chi_{\times 2}(G)+1}, P_{\chi_{\times 2}(G)+2}, \ldots, P_{\chi_{\times 2}(G)+\lceil \frac{|V(H)|}{2} \rceil}$ to $\mathbb{P}$ and put the vertices of each copy of H in these sets two by two until there are no vertices left. Therefore $\chi_{\times 2}(G \odot H) = \chi_{\times 2}(G) + \lceil \frac{|V(H)|}{2} \rceil$.

• Let $|V(H)|$ be odd. If there exists a vertex $v_i \in V(G)$ such that $|N_G[v_i] \cap (\cup_{j=1}^{\chi_{\times 2}(G)} P_j)| = 2\chi_{\times 2}(G)$, then $\chi_{\times 2}(G \odot H) = \chi_{\times 2}(G) + \lceil \frac{|V(H)|}{2} \rceil$ as before. Hence $\chi_{\times 2}(G) \leq \chi_{\times 2}(G \odot H) \leq \chi_{\times 2}(G) + \lceil \frac{|V(H)|}{2} \rceil$.

In what follows, we show that $\chi_{\times 2}(G \odot H)$ can take all values between $\chi_{\times 2}(G)$ and $\chi_{\times 2}(G) + \lceil \frac{|V(H)|}{2} \rceil$. It is enough to consider the graph G as $K_{a,a}$ and graph H as $\overline{K_{a+b-1}}$ for $a \geq 1$ and $b \geq 0$. Let $\mathbb{P} = \{P_1, P_2, \ldots, P_{\chi_{\times 2}(G)}\}$ be a 2LPP of $G = K_{a,a}$. Now we define a labeling for the vertices in each 2-limited packing set P_j in such a way that if $v_i \in V(G)$ belongs to P_j, then v_i has the label j for every $1 \leq i \leq n$ and $1 \leq j \leq \chi_{\times 2}(G)$. For example, we assign the labels from $\{1, 2, \ldots, a\}$ to the vertices of G as shown in Figure 2, which is equivalent to a 2LPP for G with the smallest cardinality.

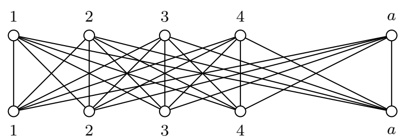

Figure 2. **An example for the labeling vertices of** $V(K_{a,a})$.

We first show that $\chi_{\times 2}(G \odot H) = \chi_{\times 2}(G) + \lceil \frac{b}{2} \rceil$. We have $|N_{G \odot H}[v_i]| = |N_G(v_i)| + 1 + |V(H)| = 2a + b$ for each $v_i \in V(G)$. $(a+1)$ closed neighbors of each $v_i \in V(G)$ are labeled as above. If $v_i \in P_k$, i.e., v_i has label k, then $|N_G[v_i] \cap P_k| = 2$ and $|N_G[v_i] \cap P_j| = 1$ for each $j \neq k$, as we see in figure 2. It is clear that j has $a-1$ values. We put $(a-1)$ unlabeled neighbors of v_i one by one in the sets P_j with the previous condition. Thus, b vertices are still unlabeled. We consider the new labels $a+1, a+2, \ldots, a+\lceil \frac{b}{2} \rceil$, and then label the remaining b vertices two by two with them. Therefore, $\chi_{\times 2}(G \odot H) = \chi_{\times 2}(G) + \lceil \frac{b}{2} \rceil$.

Note that if $a = 1$ and $b \geq 1$, then $\chi_{\times 2}(G \odot H) = \chi_{\times 2}(G) + \lceil \frac{b+1-1}{2} \rceil = \chi_{\times 2}(G) + \lceil \frac{|V(H)|}{2} \rceil$. If $b = 0$, then $\chi_{\times 2}(G \odot H) = \chi_{\times 2}(G)$. By putting the appropriate values of $a \neq 1$ and $b \neq 0$, the rest of the possible values for $\chi_{\times 2}(G \odot H)$ can be obtained. □

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References

[1] A.S. Ahmadi, N. Soltankhah, and B. Samadi, *Limited packings: Related vertex partitions and duality issues*, Appl. Math. Comput. **472** (2024), Article ID: 128613.
https://doi.org/10.1016/j.amc.2024.128613.

[2] X. Bai, H. Chang, and X. Li, *More on limited packings in graphs*, J. Comb. Optim. **40** (2020), no. 2, 412–430.
https://doi.org/10.1007/s10878-020-00606-z.

[3] P.N. Balister, B. Bollobás, and K. Gunderson, *Limited packings of closed neighbourhoods in graphs*, arXiv preprint arXiv:1501.01833 (2015).

[4] E.J. Cockayne and S.T. Hedetniemi, *Towards a theory of domination in graphs*, Networks **7** (1977), no. 3, 247–261.
https://doi.org/10.1002/net.3230070305.

[5] M.P. Dobson, V. Leoni, and G. Nasini, *The multiple domination and limited packing problems in graphs*, Inform. Process. Lett. **111** (2011), no. 23-24, 1108–1113.
https://doi.org/10.1016/j.ipl.2011.09.002.

[6] A. Gagarin and V. Zverovich, *The probabilistic approach to limited packings in graphs*, Discrete Appl. Math. **184** (2015), 146–153.
https://doi.org/10.1016/j.dam.2014.11.017.

[7] R. Gallant, G. Gunther, B.L. Hartnell, and D.F. Rall, *Limited packings in graphs*, Discrete Appl. Math. **158** (2010), no. 12, 1357–1364.
https://doi.org/10.1016/j.dam.2009.04.014.

[8] T.W. Haynes, S. Hedetniemi, and P. Slater, *Fundamentals of Domination in Graphs*, CRC press, 2013.

[9] T.W. Haynes, S.T. Hedetniemi, and M.A. Henning, *Domination in Graphs: Core Concepts*, Springer, Cham, 2023.

[10] S.M. Hosseini Moghaddam, D.A. Mojdeh, and B. Samadi, *Total limited packing in graphs*, Fasc. Math. **56** (2016), 121–127.
https://doi.org/10.1515/fascmath-2016-0008.

[11] V. Leoni and G. Nasini, *Limited packing and multiple domination problems: Poly-*

nomial time reductions, Discrete Appl. Math. **164** (2014), 547–553. https://doi.org/10.1016/j.dam.2013.10.023.

[12] D.A. Mojdeh, I. Peterin, B. Samadi, and I.G. Yero, *(Open) packing number of some graph products*, Discrete Math. Theor. Comput. Sci. **22** (2020), no. 4, Article ID: #1
https://doi.org/10.23638/DMTCS-22-4-1.

[13] D.F. Rall, *Total domination in categorical products of graphs*, Discuss. Math. Graph Theory **25** (2005), no. 1-2, 35–44.
http://doi.org/10.7151/dmgt.1257.

[14] B. Samadi, *On the k-limited packing numbers in graphs*, Discrete Optim. **22** (2016), no. Part B, 270–276.
https://doi.org/10.1016/j.disopt.2016.08.002.

[15] D.B. West, *Introduction to Graph Theory (second edition)*, Prentice hall Upper Saddle River, USA, 2001.