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Research Article

Determining the locating rainbow connection numbers of vertex-transitive graphs

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Abstract: The locating rainbow connection number of a graph is defined as the minimum number of colors required to color vertices such that for every two vertices there exists a rainbow vertex path and every vertex has a distinct rainbow code. This rainbow code signifies a distance between vertices within a given set of colors in a graph. This paper aims to determine the locating rainbow connection number for vertex-transitive graphs. Three main theorems are derived, focusing on the locating rainbow connection number for cycles, $(n-2)$-regular graphs, and complement of cycles $\overline{C_n}$.

Keywords: cycle, locating rainbow coloring, rainbow code, regular graph, vertex-transitive graph

AMS Subject classification: 05C78, 05C15

1. Introduction

In the theory of graph, there is a concept known as chromatic coloring. According to this concept, with $G = (V(G), E(G))$ being a finite, undirected, and connected graph, the chromatic coloring of G involves assigning colors to vertices in such a way that no two adjacent vertices share the same color. To describe the minimum number of

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colors required for such a chromatic coloring of G, we utilize the symbol $\chi(G)$, which represents the chromatic number of G.
In addition to the concept of locating chromatic number, Chartrand et al. [5] also introduced the concept of rainbow coloring in 2008. This concept was inspired by the undercover communication techniques employed by government agencies to guarantee the secure transfer of classified information, especially in the aftermath of the 9/11 attacks in 2001 [7]. Since then, this concept has been widely studied, involving a variety of graph operations and graph classes (e.g., [8], [10], [12], [14], and [15]). Motivated by the concept of rainbow coloring, in 2010, Krivelevich and Yuster introduced rainbow vertex coloring of a graph [9]. Following this, the rainbow vertex connection number of several classes of graphs has been a focus of several studies (e.g., [1] and [13].
Motivated by the concepts of rainbow vertex coloring and dimension partition, a concept that combines both, called locating rainbow coloring of a graph, was introduced in 2021 [2]. For a natural number k, a coloring of the vertex set of G is termed a *rainbow vertex k-coloring* if there exists a function $c : V(G) \longrightarrow \{1, 2, \ldots, k\}$ such that, for any distinct pair of vertices $x, y \in V(G)$, there is a *rainbow vertex $x - y$-path*, whose internal vertices are assigned a different color. The *rainbow vertex connection number* of G, denoted by $rvc(G)$, is the smallest positive integer k, so G has a rainbow vertex k-coloring. For $i \in \{1, 2, \ldots, k\}$, let R_i denote the set of vertices that have the color i and let $\Pi = \{R_1, R_2, \ldots, R_k\}$ be an ordered partition of $V(G)$. Thus, $rc_\Pi(v) = (d(v, R_1), d(v, R_2), \ldots, d(v, R_k))$, where $d(v, R_i) = \min\{d(v, y) : y \in R_i\}$ for every $i \in \{1, 2, \ldots, k\}$. Further, $rc_\Pi(v)$ is called the *rainbow code* of v of G with respect to Π. If $rc_\Pi(v_j) \neq rc_\Pi(v_l)$ for distinct $j, l \in \{1, 2, \ldots, n\}$, then the coloring c is known as a *locating rainbow k-coloring* of G. The smallest positive integer k for which a locating rainbow k-coloring exists in the graph G, denoted by $rvcl(G)$, is called the locating rainbow connection number of graph G [2]. It is important to note that every locating rainbow k-coloring of G also serves as a rainbow vertex coloring of G, implying that

$$rvc(G) \leq rvcl(G). \tag{1.1}$$

Several results regarding the $rvcl(G)$ can be found in [2], [4], and [3] with some required results in this paper as follows.

Lemma 1. *[2] Let c be a locating rainbow coloring of G. Let u and v be two distinct vertices of G. If $d(u, x) = d(v, x)$ for all $x \in V(G) - \{u, v\}$, then $c(u) \neq c(v)$.*

Lemma 2. *[4] Let n be an integer with $n \geq 3$ and G be a connected graph of order n containing a cycle. Then, $rvcl(G) \geq 3$.*

Theorem 1. *[4] Let n be an integer with $n \geq 3$ and G be a connected graph of order $n \geq 3$. Then, $rvcl(G) = n$ if and only if G is isomorphic to a complete graph of order n.*

Theorem 2. *[2] Let n be an integer with $n \geq 3$ and G be a connected graph of order $n \geq 3$ with $rvcl(G) = r$. If $diam(G)$ denotes the diameter of G, then $n \leq r \times diam(G)^{r-1}$.*

Lemma 1 concludes that if c is a locating rainbow coloring of G, then two distinct vertices which have the same distance to other vertices would not share the same color. Meanwhile, Theorem 2 demonstrates the assignment of a specific value for $rvcl(G)$ to determine the maximum number of vertices in a graph G such that every vertex in G has a distinct rainbow code.
We aim to explore the $rvcl(G)$, specifically focusing on regular graphs where all vertices have the same degree. In this research, we focus on classes of vertex-transitive graphs, also known as node symmetric graphs, where every pair of vertices is equivalent to some elements of its automorphism group [6]. Let n be the order of a graph G and $t \geq 3$. An $(n-t)$-regular graph is a graph in which all its vertices have a degree of $n-t$. All graphs belonging to the classes of 2-regular graphs or cycles, $(n-1)$-regular graphs, $(n-2)$-regular graphs, and one specific class of $(n-3)$-regular graphs, namely complement of cycles $\overline{C_n}$, are included in the vertex-transitive graphs. We have determined the locating rainbow connection number of $(n-1)$-regular graphs or complete graphs in Theorem 1 [2]. Therefore, in this paper, we determine the locating rainbow connection number of cycles, $(n-2)$-regular graphs, and complement of cycles $\overline{C_n}$.

2. Main Results

The main results are specifically focused on two subsections: the 2-regular graphs or cycles, which are extensively discussed in Subsection 2.1, $(n-2)$-regular graphs and complement of cycles $\overline{C_n}$ in Subsection 2.2. For simplicity, represent the set $\{n \in \mathbf{Z} \mid x \leq n \leq y\}$ as $[x, y]$.

2.1. The Locating Rainbow Connection Number of Cycles

The locating rainbow connection number of cycles are closely tied to the rainbow vertex connection number of cycles. However, not all rainbow vertex colorings imply a locating rainbow vertex coloring. Thus, we introduce a new coloring to fulfill the requirements of locating rainbow colorings on cycles. Some cases require the values of rainbow vertex connection number of cycles as presented in Theorem 3.

Theorem 3. *[11] Let n be an integer with $n \geq 3$ and C_n be a cycle of order n. Then, the rainbow vertex connection number of C_n is*

$$rvc(C_n) = \begin{cases} \lceil \frac{n}{2} \rceil - 2, & \text{for } n \in \{3,5,9\}; \\ \lceil \frac{n}{2} \rceil - 1, & \text{for } n \in \{4,6,7,8,10,11,12,13,15\}; \\ \lceil \frac{n}{2} \rceil, & \text{for } n = 14 \text{ or } n \geq 16. \end{cases}$$

In Theorem 4, we see that for relatively small orders, $rvcl(C_n)$ differs from $rvc(C_n)$. For larger orders, $rvcl(C_n) = rvc(C_n)$.

Theorem 4. *Let n be an integer with $n \geq 3$ and C_n be a cycle of order n. Then, the locating rainbow connection number of C_n is*

$$rvcl(C_n) = \begin{cases} 3, & for\ n \in [3,7]; \\ \lceil \frac{n}{2} \rceil - 1, & for\ n \in \{9, 15\}\ or\ n \in [11, 13]; \\ \lceil \frac{n}{2} \rceil, & for\ n \in \{8, 10, 14\}\ or\ n \geq 16. \end{cases}$$

Proof. Let $n \geq 3$ and $C_n = v_1, v_2, ..., v_n, v_{n+1}$ with $v_{n+1} = v_1$. We consider six cases.

Case 1. $n \in [3, 7]$

Based on Lemma 2 and Figure 1, we obtain $rvcl(C_n) = 3$ for $n \in [3, 7]$.

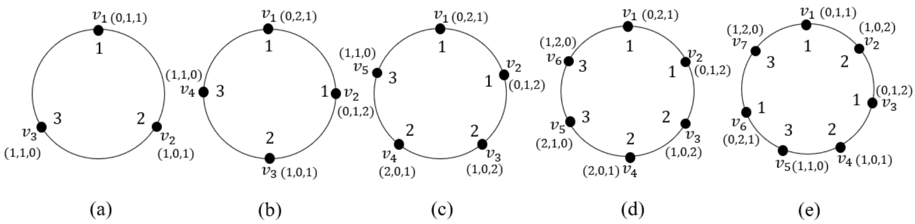

Figure 1. **Locating rainbow 3-colorings of (a) C_3, (b) C_4, (c) C_5, (d) C_6, and (e) C_7.**

Case 2. $n = 8$

Suppose that $rvcl(C_8) = 3$. Let c be a locating rainbow 3-coloring of C_8. We begin by considering vertices v_1 and v_5. Since the vertex coloring of C_8 represents the rainbow vertex coloring, there exist two possible rainbow vertex $v_1 - v_5$ paths: $P_1 = v_1, v_2, v_3, v_4, v_5$ and $P_2 = v_1, v_8, v_7, v_6, v_5$. Without loss of generality, let's assume the rainbow vertex path is P_1 and $c(v_i) = i - 1$ for $i \in [2, 4]$. Next, we focus on vertices v_3 and v_7. There are two possible $v_3 - v_7$ paths: $P_3 = v_3, v_2, v_1, v_8, v_7$, and $P_4 = v_3, v_4, v_5, v_6, v_7$. Let the rainbow vertex $v_3 - v_7$ path be P_3. Since $c(v_2) = 1$, we get $rc_\Pi(v_1) = rc_\Pi(v_3)$ if $c(v_1) = 2$ and $c(v_8) = 3$. Hence $c(v_1) = 3$ and $c(v_8) = 2$.
Next, we consider vertices v_2 and v_6. To ensure the existence of a rainbow vertex path between v_2 and v_6, we must have $c(v_5) = 1$ or $c(v_7) = 1$. If $c(v_5) = 1$, then $rc_\Pi(v_1) = rc_\Pi(v_4)$. If $c(v_7) = 1$, then $rc_\Pi(v_3) = rc_\Pi(v_8)$. Both cases lead to contradictions. In the case where the $v_3 - v_7$ path is P_4, we observe a similar contradiction. Therefore, we can conclude that $rvcl(C_8) \geq 4$. By defining a locating rainbow 4-coloring of C_8, as demonstrated in Figure 2(a), we establish that $rvcl(C_8) = 4$.

Case 3. $n = 9$

Suppose $rvcl(C_9) = 3$. For $i \in [1, 9]$, to ensure the existence of a rainbow vertex path between v_i and $v_{((i+4) \pmod 9))+1}$, it is necessary that every three consecutive vertices should be assigned distinct colors. However, this requirement leads to a contradiction, as there would inevitably be at least two vertices sharing both the same color and rainbow codes. Consequently, we deduce that $rvcl(C_9) \geq 4$.
To establish $rvcl(C_9) = 4$, we demonstrate a locating rainbow 4-coloring of C_9, as depicted in Figure 2(b).

Case 4. $n = 10$

Suppose that $rvcl(C_{10}) = 4$. Let c be a locating rainbow 4-coloring of C_{10}. We start by examining vertices v_1 and v_6. Since the vertex coloring of C_{10} is the rainbow vertex coloring, there exist two possible rainbow vertex v_1-v_6 paths: $P_1 = v_1, v_2, v_3, v_4, v_5, v_6$ and $P_2 = v_1, v_{10}, v_9, v_8, v_7, v_6$. Without loss of generality, assume that the rainbow vertex path is P_1 and $c(v_i) = i-1$ for $i \in [2, 5]$. Since we have four colors, there must be at least one color used by at least three distinct vertices and the distance between two vertices with the same color must be more than 2. Next, we categorize this case into four subcases based on the colors employed by at least three distinct vertices in C_{10}.

1. Color 1.
   Since $c(v_2) = 1$, it follows that $c(v_6) = c(v_9) = 1$. Consequently, $c(v_7) = 2$ and $c(v_1) = 4$. Therefore, $rc_\Pi(v_2) = rc_\Pi(v_6) = (0, 1, 2, 1)$, leading to a contradiction.

2. Color 2.

   Since $c(v_3) = 2$, there are three possible combinations of the remaining two vertices that can be colored with 2. First, if $c(v_7) = c(v_{10}) = 2$, then vertex v_6 and vertex v_8 must be colored with either 1 or 3. Consequently, $rc_\Pi(v_3) = rc_\Pi(v_7)$. Second, if $c(v_6) = c(v_{10}) = 2$, vertices v_7 and v_8 have to be colored with either 1 or 3. In this case, $c(v_9) = 4$ and $c(v_1) = 3$. Therefore, if $c(v_8) = 1$, then $rc_\Pi(v_6) = rc_\Pi(v_{10})$ and if $c(v_8) = 3$, then $rc_\Pi(v_5) = rc_\Pi(v_9)$. Third, $c(v_6) = c(v_9) = 2$, which implies $c(v_1) = 4$ and $rc_\Pi(v_{10}) = rc_\Pi(v_4)$. We get a contradiction.

3. Color 3.
   By employing a similar argument with color 2, we arrive at a contradiction.

4. Color 4.
   By employing a similar argument with color 1, we arrive at a contradiction.

Therefore, $rvcl(C_{10}) \geq 5$. To prove that $rvcl(C_{10}) \leq 5$, we define a locating rainbow 5-coloring of C_{10} as shown in Figure 2(c). Thus, $rvcl(C_{10}) = 5$.

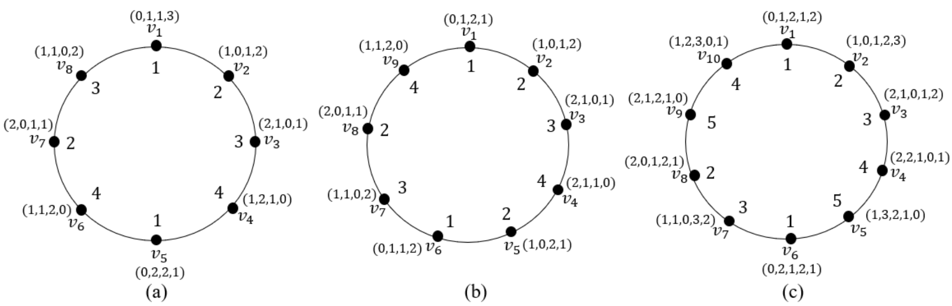

Figure 2. **A locating rainbow coloring (a) C_8, (b) C_9, and (c) C_{10}.**

Case 5. $n \in \{11, 12, 13, 15\}$.
Since $rvc(C_n) = \lceil \frac{n}{2} \rceil - 1$ by Theorem 3, it follows by Equation (1.1), we have $rvcl(C_n) \geq rvc(C_n) = \lceil \frac{n}{2} \rceil - 1$. Next, we show $rvcl(C_n) \leq \lceil \frac{n}{2} \rceil - 1$ by defining a locating rainbow $\lceil \frac{n}{2} \rceil - 1$-coloring of C_n as shown in Figures 3. Thus, we have $rvcl(C_n) = \lceil \frac{n}{2} \rceil - 1$ for $n \in \{11, 12, 13, 15\}$.

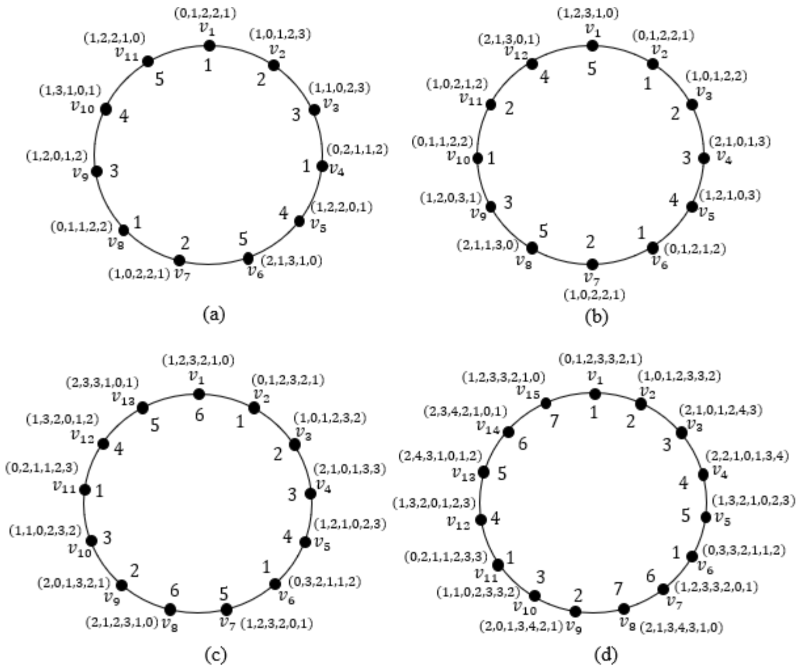

Figure 3. **A locating rainbow coloring of (a) C_{11}, (b) C_{12}, (c) C_{13}, and C_{15}.**

Case 6. $n = 14$ or $n \geq 16$
Since $rvc(C_n) = \lceil \frac{n}{2} \rceil$ by Theorem 3, it follows by Equation (1.1), we have $rvcl(C_n) \geq rvc(C_n) = \lceil \frac{n}{2} \rceil$ for $n = 14$ or $n \geq 16$. Next, we demonstrate the upper bound by defining a rainbow vertex coloring $c : V(C_n) \longrightarrow [1, n]$ as follows.

1. For odd n, we define

$$c(v_i) = \begin{cases} i \ mod \lceil \frac{n}{2} \rceil, & \text{for } i \in [1, n], i \neq \lceil \frac{n}{2} \rceil; \\ \lceil \frac{n}{2} \rceil, & \text{for others.} \end{cases}$$

   Utilizing the vertex coloring described above, we observe that color $\lceil \frac{n}{2} \rceil$ is exclusively assigned to $v_{\lceil \frac{n}{2} \rceil}$, and $c(v_i) = c(v_{i+\lceil \frac{n}{2} \rceil})$ for $i \in [1, \lceil \frac{n}{2} \rceil - 1]$. Therefore, for any two vertices of C_n, a rainbow vertex path connecting them exists and we have $d(v_i, v_{\lceil \frac{n}{2} \rceil}) \neq d(v_{i+\lceil \frac{n}{2} \rceil}, v_{\lceil \frac{n}{2} \rceil})$. Thus, $rc_\Pi(v_i) \neq rc_\Pi(v_j)$ for distinct $i, j \in [1, n]$.

2. For even n, we define

$$c(v_i) = \begin{cases} \frac{n}{2} - 1, & \text{for } i = n; \\ \frac{n}{2}, & \text{for } i = n-1; \\ i \ mod \ \frac{n}{2}, & \text{for others.} \end{cases}$$

Utilizing the vertex coloring described above, we have $c(v_i) = c(v_{i+\frac{n}{2}})$ for $n \in [1, \frac{n}{2}-2] \cup [\frac{n}{2}+1, n-2]$, $c(v_{\frac{n}{2}-1}) = c(v_n)$, and $c(v_{\frac{n}{2}}) = c(v_{n-1})$. Consequently, for any two vertices of C_n, there exists a rainbow vertex path. Moreover, we observe $d(v_i, R_{\frac{n}{2}}) \neq d(v_{i+\frac{n}{2}}, R_{\frac{n}{2}})$ for $n \in [1, \frac{n}{2}-2] \cup [\frac{n}{2}+1, n-2]$. Additionally, $d(v_{\frac{n}{2}-1}, R_1) \neq d(v_n, R_1)$, and $d(v_{\frac{n}{2}}) \neq d(v_{n-1})$. Hence, $rc_\Pi(v_i) \neq rc_\Pi(v_j)$ for distinct $i, j \in [1, n]$.

Since $rvcl(C_n) \geq \lceil \frac{n}{2} \rceil$ and $rvcl(C_n) \leq \lceil \frac{n}{2} \rceil$, we conclude that $rvcl(C_n) = \lceil \frac{n}{2} \rceil$. □

2.2. The Locating Rainbow Connection Numbers of $(n-t)$-Regular Graphs for $t = \{2, 3\}$

In this subsection, we present two main theorems concerning the locating rainbow connection number of $(n-2)$-regular graphs and complement of cycles $\overline{C_n}$. However, before delving into the theorems, we provide Lemma 3 to aid in the proof process of both theorems. For simplification, we use the term "entry" to refer to the distance from a vertex to a set of colors and the combination formula C_k^r is expressed as $\frac{r!}{k!(r-k)!}$.

Lemma 3. *Let $n, t,$ and r be three integers with $n \geq 5$, $t = \{2, 3\}$, and $r \geq 2$. Let $R_{(n,t)}$ be a regular graph of order n with all vertices have a degree of $n - t$, and let r be the locating rainbow connection number of $R_{(n,t)}$ with $r \geq 3$. Then:*

(1) *each rainbow code contains at most $t-1$ of entries 2;*

(2) *every color can be used for at most $1 + \sum_{k=1}^{t-1} C_{t-k}^{r-1}$ vertices;*

(3) *for every color w, the maximum number of vertices v, such that $d(v, R_w) = 2$ is $t-1$.*

Proof.

(1) Since $diam(R_{(n,t)}) = 2$, the graph only contains entries of $0, 1,$ and 2. Each vertex in $R_{(n,t)}$ is non-adjacent to exactly $t-1$ other vertices, resulting in the rainbow code possibly containing at most $t-1$ entries of 2.

(2) Based on the first point in this proof, there are three possible rainbow codes: those without entry 2 and those with at most $t-1$ entries of 2. For codes without entry 2, there is only one possible rainbow code, namely the one consisting of a single entry 0 and the remaining entries being 1. Next, consider codes that include entry 2. For $t = 2$, each vertex is adjacent to only one other vertex, meaning entry 2 can appear exactly once, resulting in C_1^{r-1} or C_{t-1}^{r-1} possible rainbow codes. For $t = 3$, each vertex is adjacent to two other vertices, allowing

two types of rainbow codes that include entry 2: those with exactly one entry 2, contributing C_2^{r-1} or C_{t-1}^{r-1}, and those with exactly two entries 2, contributing C_1^{r-1} or C_{t-2}^{r-1}. By generalizing this pattern, for any t, the total number of rainbow codes involving entry 2 is given by $\sum_{k=1}^{t-1} C_{t-k}^{r-1}$. Hence, every color can be used for at most $1 + \sum_{k=1}^{t-1} C_{t-k}^{r-1}$ times.

(3) Suppose there are t distinct vertices $v_1, v_2, ..., v_t$ such that $d(v_i, R_w) = 2$ for $i \in [1, t]$. Consequently, there exist $v_1', v_2', ..., v_t'$ such that $v_i v_i' \notin E(G)$ and $c(v_i') = w$ for $i \in [1, t]$. Since $v_i v_j' \in E(G)$ for $i \in [1, t]$ and $i \neq j$, it follows that $d(v_i, R_w) = 1$, which is a contradiction. Therefore, for every color w, the maximum number of vertices v such that $d(v, R_w) = 2$ is $t - 1$.

□

2.2.1. The Locating Rainbow Connection Number of $(n-2)$-Regular Graphs

Consider a graph G of order $n \geq 4$ and n is even. If all vertices in G are of degree $n-2$, it is termed an $(n-2)$-regular graph and denoted by $R_{(n,2)}$. Earlier, in Theorem 1, we derived the locating rainbow connection number for $(n-1)$-regular graph. In this section, we focus on determining the *rvcl* of $(n-2)$-regular graph.
In actuality, $(n-2)$-regular graph is obtained by removing $\frac{n}{2}$ disjoint edges from an $(n-1)$-regular graph. The resulting $(n-2)$-regular graph has the following vertices and edges: $V(R_{(n,2)}) = \{v_i | i \in [1, n]\}$ and $E(R_{(n,2)}) = \{v_i v_j | i, j \in [1, n], i \neq j, j \neq i + \frac{n}{2}\}$.

Theorem 5. *Let n be an even integer with $n \geq 4$. If $R_{(n,2)}$ is an $(n-2)$-regular graph of order n, then $rvcl(R_{(n,2)}) = \frac{n}{2} + 1$.*

Proof. Suppose that $rvcl(R_{(n,2)}) = \frac{n}{2}$. Based on Lemma 3, the number of rainbow codes that do not contain entry 2 is at most $\frac{n}{2}$ and the number of codes that contain entry 2 is at most $\frac{n}{2} - 1$. Thus, the maximum number of different rainbow codes is $n - 1$, leading to a contradiction. Therefore, we have $rvcl(R_{(n,2)}) \geq \frac{n}{2} + 1$.
Furthermore, we demonstrate that $rvcl(R_{(n,2)}) \leq \frac{n}{2} + 1$ by defining vertex coloring $c : V(R_{(n,2)}) \longrightarrow [1, \frac{n}{2} + 1]$ as follows.

$$c(v_i) = \begin{cases} i, & \text{for } i \in [1, \frac{n}{2} + 1]; \\ 1, & \text{otherwise.} \end{cases}$$

Since $diam(R_{(n,2)}) = 2$, using $\frac{n}{2} + 1$ colors will ensure that for any two vertices u and v, there exists a $u - v$ rainbow vertex path. Furthermore, we show that all vertices in $R_{(n,2)}$ have distinct rainbow codes by considering the following.

1. $c(v_i) \neq c(v_j)$ for distinct $i, j \in [1, \frac{n}{2} + 1]$.

2. $c(v_i) = c(v_j) = 1$ for distinct $i, j \in [\frac{n}{2} + 2, n] \cup \{v_1\}$, but $d(v_1, R_{\frac{n}{2}+1}) = 2$ and $d(v_i, R_{\frac{n}{2}+1}) = 1$ for $i \in [\frac{n}{2} + 2, n]$. Besides that, $d(v_i, R_{i-\frac{n}{2}}) = 2$ and $d(v_i, R_a) = 1$ for $a \in [1, \frac{n}{2}]$, $a \neq i - \frac{n}{2}$.

Thus, $rc_{\Pi}(v_i) \neq rc_{\Pi}(v_j)$ for distinct $i, j \in [1, n]$. Therefore, we have $rvcl(R_{(n,2)}) \leq \frac{n}{2} + 1$. Hence, the proof is complete. $\square$

For illustration, in Figure 4 we provide a locating rainbow coloring of the regular graph $R_{(16,2)}$.

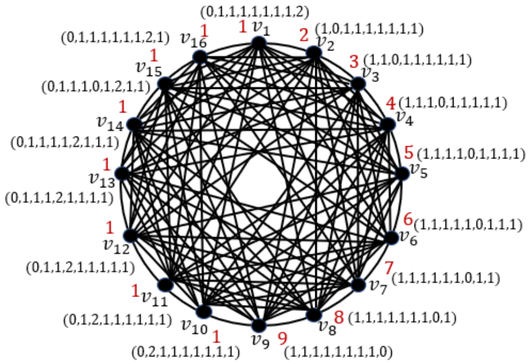

Figure 4. **A locating rainbow coloring of** $R_{(16,2)}$.

2.2.2. The Locating Rainbow Connection Number of the Complement of Cycle Graphs $\overline{C_n}$

The complement of cycle graph, denote by $\overline{C_n}$, is obtained by removing a cycle from a $(n-1)$-regular graph. It is easily observed that the complement of cycle $\overline{C_n}$ is connected if and only if $n \geq 5$ and it lies within the class $R_{(n,t)}$ with $t = 3$. The complement of cycle $\overline{C_n}$ has the following vertices and edges: $V(\overline{C_n}) = \{v_i | i \in [1, n]\}$ and $E(\overline{C_n}) = \{v_1 v_j | j \in [3, n-1]\} \cup \{v_i v_j | i \in [2, n], j \in [1, n], j \neq i-1, j \neq i, j \neq (i+1) \ mod \ n\}$. In Theorem 6, we determine $rvcl(\overline{C_n})$.

Theorem 6. *Let n be an integer with $n \geq 5$. If $\overline{C_n}$ is an complement of cycle graph of order n, then.*

$$rvcl(\overline{C_n}) = \begin{cases} \lceil \frac{n}{2} \rceil - \lceil \frac{n}{10} \rceil + 2, & \textit{for } n \equiv 2 \pmod{10} \textit{ or } n \equiv 4 \pmod{10}; \\ \lceil \frac{n}{2} \rceil - \lceil \frac{n}{10} \rceil + 1, & \textit{for others.} \end{cases}$$

Proof. The proof is partitioned into two cases as outlined below.

1. $n \in \{5, 6\}$
   Based on Lemma 2, Theorem 2, and Figure 5, we obtain $rvcl(\overline{C_n}) = 3$.

2. $n \geq 7$
   Suppose $rvcl(\overline{C_n}) = \lceil \frac{n}{2} \rceil - \lceil \frac{n}{10} \rceil + 1$ for $n \equiv 2 \pmod{10}$ or $n \equiv 4 \pmod{10}$. Based on Lemma 3(2), the number of rainbow codes that do not include entry 2 is at most $\lceil \frac{n}{2} \rceil - \lceil \frac{n}{10} \rceil + 1$. Since $rvcl(\overline{C_n}) < \lceil \frac{n}{2} \rceil$, there must be at least one

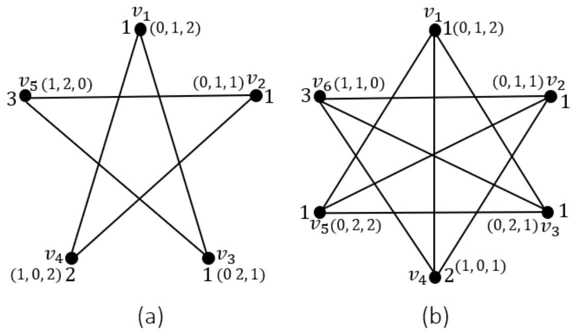

Figure 5. $\overline{C_5}$, $\overline{C_6}$.

color used at least three timesen. Hence there are only $\lceil\frac{n}{2}\rceil - \lceil\frac{n}{10}\rceil$ sets of colors that produce a rainbow code containing entries of 2.

Thus, according to Lemma 3(3), $2(\lceil\frac{n}{2}\rceil - \lceil\frac{n}{10}\rceil)$ is the maximum number of vertices with a rainbow code containing entries of 2. Meanwhile, for two pair sets of colors R_a and R_b that produce entry 2 in a rainbow code, there exist three distinct vertices u, v, w such that $d(u, R_a) = 2$ and $d(u, R_b) = 1$, $d(v, R_a) = 1$ and $d(v, R_b) = 2$, and $d(w, R_a) = d(w, R_b) = 2$. Consequently, any two sets of colors result in at most three distinct rainbow codes containing entries of 2.

Therefore, the maximum number of distinct rainbow codes is $(\lceil\frac{n}{2}\rceil - \lceil\frac{n}{10}\rceil + 1) + ((\lfloor\frac{(\lceil\frac{n}{2}\rceil - \lceil\frac{n}{10}\rceil)}{2}\rfloor) \times 3)$. Since $((\lceil\frac{n}{2}\rceil - \lceil\frac{n}{10}\rceil + 1) + ((\lfloor\frac{(\lceil\frac{n}{2}\rceil - \lceil\frac{n}{10}\rceil)}{2}\rfloor) \times 3)) < n$, which leads to a contradiction. Similarly, for other values of n, contradictions are obtained. Therefore, $rvcl(\overline{C_n}) \geq \lceil\frac{n}{2}\rceil - \lceil\frac{n}{10}\rceil + 2$ for $n \equiv 2 \pmod{10}$ or $n \equiv 4 \pmod{10}$, and $rvcl(\overline{C_n}) \geq \lceil\frac{n}{2}\rceil - \lceil\frac{n}{10}\rceil + 1$ for other values of n.

Furthermore, we will show that $rvcl(\overline{C_n}) \leq \lceil\frac{n}{2}\rceil - \lceil\frac{n}{10}\rceil + 2$ for $n \equiv 2 \pmod{10}$ or $n \equiv 4 \pmod{10}$, and $rvcl(\overline{C_n}) \leq \lceil\frac{n}{2}\rceil - \lceil\frac{n}{10}\rceil + 1$ for other values of n. using the following coloring steps. For simplification, r is assumed to represent the number of colors given to a graph $\overline{C_n}$.

(a) Assign colors $2, 3, ..., r-1$ to the vertices $v_{4+(i-1)5}, v_{6+(i-1)5}$ sequentially for $i \in [1, \frac{r-2}{2}]$ and even r.

(b) Assign colors $2, 3, ..., r-1$ to the vertices $v_{4+(i-1)5}, v_{6+(i-1)5}$ sequentially for $i \in [1, \frac{r-3}{2}]$ and odd r.

(c) In the graph colored with even r, set $c(v_n) = r$.

(d) In the graph colored with odd r, set $c(v_n) = r$. Furthermore, if $n - (6 + (\frac{r-3}{2} - 1)5) \in \{3, 4\}$, then $c(v_{n-1}) = r - 1$; and if $n - (6 + (\frac{r-3}{2} - 1)5) = 5$, then $c(v_{n-2}) = r - 1$.

(e) Finally, color all remaining uncolored vertices with color 1.

Since $diam(\overline{C_n}) = 2$, there will always be a rainbow path connecting any two vertices in the graph $\overline{C_n}$.

Furthermore, the colors $2, 3, ..., r$ that are assigned only once to the vertices in the graph $\overline{C_n}$ result in all vertices assigned with these colors having distinct rainbow codes. For color 1, $c(v_1) = 1$ and it is adjacent to the vertices colored with $2, 3, ..., r - 1$. Additionally, for any two distinct vertices v_i and v_j with $c(v_i) = c(v_j) = 1$ for $i, j \in [2, n]$, there is always at least one set of colors R_a for $a \neq 1$, such that $d(v_i, R_a) = 2$ and $d(v_j, R_a) = 1$. As a result, all vertices in the graph $\overline{C_n}$ have distinct rainbow codes. Therefore, we obtain $rvcl(\overline{C_n}) = \lceil \frac{n}{2} \rceil - \lceil \frac{n}{10} \rceil + 2$ for $n \equiv 2 \pmod{10}$ or $n \equiv 2 \pmod{10}$; and $rvcl(\overline{C_n}) = \lceil \frac{n}{2} \rceil - \lceil \frac{n}{10} \rceil + 1$ for other values of n.

$\square$

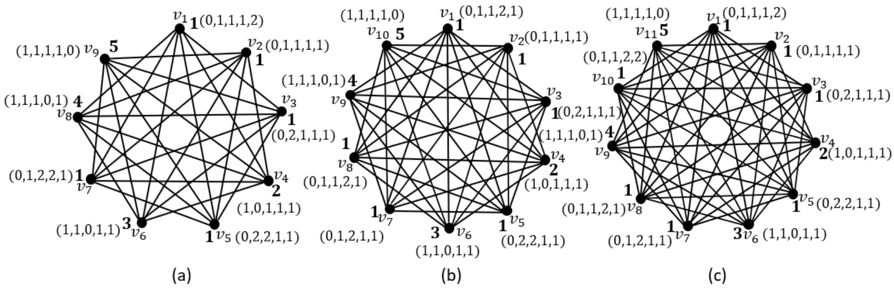

Figure 6. **A locating rainbow coloring of (a) $\overline{C_9}$, (b) $\overline{C_{10}}$, and (c) $\overline{C_{11}}$.**

All regular graphs discussed in this research are vertex-transitive graphs. Hence, we conclude this paper with an open problem: What is the locating rainbow connection number of any regular graph which is not vertex-transitive?

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