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Research Article

Characterizing arc-colored digraphs with an Eulerian trail with restrictions in the color transitions

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Abstract: Let H be a digraph possibly with loops, and D a multidigraph without loops. An H-coloring of D is a function $c : A(D) \to V(H)$. We say that D is an H-colored multidigraph whenever we are taking a fixed H-coloring of D. A trail $W = (v_0, e_0, v_1, e_1, v_2, \ldots, v_{n-1}, e_{n-1}, v_n)$ in D is an H-trail if and only if $(c(e_i), c(e_{i+1}))$ is an arc in H, for each $i \in \{0, \ldots, n-2\}$. We say that an H-colored multidigraph is H-trail-connected if and only if there is an H-trail starting with arc f_1 and ending with arc f_2, for any pair of arcs f_1 and f_2 in D. Let D be an H-colored multidigraph and u a vertex of D, the auxiliary digraph D_u is the digraph of allowed transition throughout u.

In this paper we give the following characterization: Let D be an H-colored multidigraph such that the underlying graph of D_u is a disjoint union of complete bipartite graphs, for every $u \in V(D)$. Then D has a Euler H-trail if and only if D is H-trail-connected and, for every $u \in V(D)$, the underlying graph of D_u has a perfect matching. As a consequence we obtain the well-known characterization of the 2-arc-colored multidigraphs containing properly colored Euler trail. Finally, we give an infinite family of digraphs H such that for every multidigraph D without isolated vertices, and every H-coloring of D, the underlying graph of D_u is a disjoint union of complete bipartite graphs and, possibly, isolated vertices, for every $u \in V(D)$.

Keywords: Euler trails, arc-colored digraphs, forbidden transitions

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1. Introduction

For basic concepts, terminology and notation not defined here, we refer the reader to [3] and [4]. Throughout this work, we will consider finite directed graphs, and

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directed multigraphs (directed graphs allowing parallel arcs). Let D be a directed graph, $V(D)$ and $A(D)$ will denote the sets of vertices and arcs of D, respectively.

A *(directed) spanning circuit* in a (directed) graph G is defined as a closed (directed) trail that contains each vertex of G. We will say that a (directed) graph is *supereulerian* if it contains a (directed) spanning circuit. Let T be a (directed) spanning circuit in G: if T visits each vertex of G exactly once, then T will be called *Hamilton cycle*; and if T visits each edge of G, then T will be called *Euler trail*. We will say that a (directed) graph is *hamiltonian* if it has a Hamilton cycle and *eulerian* if it contains a Euler trail.

The following graphs were introduced by Harary and Nash-Williams in [12]. Let G be a graph with p vertices and q edges, for $n \geq 2$, $L_n(G)$ will denote the graph with nq vertices that are obtained as follows: for each edge $e = uv$ of G, we take two vertices $f(e,u)$ and $f(e,v)$ in $L_n(G)$ and adding a path with $n-2$ new intermediate vertices connecting $f(e,u)$ and $f(e,v)$; finally, for each vertex u of G, we add an edge joining $f(e,u)$ and $f(g,u)$, whenever e and g are distinct edges with end point u. They also proved the following relationships between eulerian graphs and hamiltonian cycles of $L_n(G)$.

Theorem 1 ([12]). *Let G be a graph. The following assertions hold:*

1. *If G is eulerian, then $L_n(G)$ is hamiltonian, for every $n \geq 2$.*
2. *If $L_n(G)$ is hamiltonian, for some $n \geq 3$, then G is eulerian.*
3. *G is superulerian if and only if $L_2(G)$ is hamiltonian.*

For a digraph D, the *line digraph* of D, denoted by $L(D)$, is the digraph with vertex set $A(D)$ and $A(L(D)) = \{(a,b) \mid a,b \in V(L(D))$, the head of a coincides with the tail of $b\}$. Kasteleyn in [14] show that there is a one-to-one correspondence between the closed directed trails of D and the directed cycles of $L(D)$.

Theorem 2 ([14]). *There is a one-to-one correspondence between the set of closed directed trails in D and the set of directed cycles in $L(D)$.*

Imori et al. [13] introduced the notion of pancircularity as follows. A digraph D with q arcs is said to be *pancircular*, if D contains closed trails of every length L for all $2 \leq L \leq q$.

Corollary 1 ([13]). *Let D be a digraph. Then, D is pancircular if and only if $L(D)$ is pancyclic.*

A graph G is called *edge-colored* if each edge has an assigned color. A *properly colored (PC)* walk is a walk in which no two consecutive edges have the same color, including the last and first edges in a closed walk. Several authors have worked with this concept, for example, Bang-Jensen et al. [2]. Properly colored walks are of interest as a generalization of walk in undirected and directed graphs, see [3], as well as, in graph theory applications, for example, in genetic and molecular biology [7, 8, 17, 18], channel assignment in wireless networks [1, 19].

Kotzig [15] gave the following characterization of edge-colored multigraphs containing properly colored closed Euler trail.

Theorem 3 (Kotzig [15]). *Let G be an edge-colored eulerian multigraph. Then G has a properly colored closed Euler trail if and only if $\delta_i(x) \leq \sum_{j\neq i} \delta_j(x)$, where $\delta_i(x)$ is the number of edges with color i incident with x, for each vertex x of G.*

Properly colored directed walk in arc-colored directed graphs have also been studied by many authors, for example, Gutin et al. [11], Carraher and Hartke [5, 6] and Gourvès et al. [10]. In [21, 22], arc-colored directed graphs are used to model conflict resolution.

An arc-colored directed graphs D is PC *trail-connected*, if there is a PC trail starting with arc f_1 and ending with arc f_2, for any pair of arcs f_1, f_2 in D. Sheng et al. [20] characterized 2-arc-colored directed graphs containing PC Euler trails.

Theorem 4 ([20]). *Let D be a 2-arc-colored directed multigraph. Then D has PC Euler trail if and only if D is PC trail-connected and for every $v \in V(D)$, $d_i^+(v) = d_{3-i}^-(v)$, for $i \in \{1, 2\}$.*

Different kinds of edge-coloring in directed and undirected graphs have been studied, for example, in [16] the arcs of a tournament were colored with the vertices of a poset. We will consider the following edge-coloring. Let H be a directed graph possibly with loops and D a directed graph without loops. An *H-coloring* of D is a function $c : A(D) \to V(H)$. We will say that D is an *H-colored directed graph*, whenever we are taking a fixed H-coloring of D. A directed walk $W = (v_0, e_0, v_1, e_1, \ldots, e_{k-1}, v_k)$ in D, where $e_i = (v_i, v_{i+1})$ for every i in $\{0, \ldots, k-1\}$, is a *directed H-walk* if $(c(e_0), a_0, c(e_1), \ldots, c(e_{k-2}), a_{k-2}, c(e_{k-1}))$ is a directed walk in H, with $a_i = (c(e_i), c(e_{i+1}))$ for every $i \in \{0, \ldots, k-2\}$. Let W be a directed H-walk, if W is a directed trail then W will be called *H-trail*.

The concepts of H-coloring and H-walks were introduced, for the first time by Linek and Sands in [16], and have been worked mainly in the context of kernel theory and related topics.

A theoretical reason to study H-walks is that they generalize monochromatic walks and properly colored walks. To see that H-walks generalize properly colored walks, notice that if H is a complete graph without loops, then every H-walk is a properly colored walk. And, if H is a graph that only contains loops, then every H-walk is a monochromatic walk. Also, notice that if $W = (x_0, x_1, \ldots, x_n)$ is an H-walk such that $(c(x_0x_1), c(x_1x_2), \ldots, c(x_{n-1}x_n))$ is a path in H, then W is a rainbow walk.

A motivation for the study of H-walks, in H-colored digraphs, is their possible applications. For example, suppose that we are working with a communication network, represented by a digraph D, where each vertex represent a connection point, and an arc from one connection point A to another connection point B means a way to send information directly from A to B. Notice that more than one arc from A to B can exist since there could be different ways to send information. Moreover, due to different issues (namely risk; such as, damage, attack, virus, blockage, among many others) some arc transitions may be prohibited. In order to have a robust network against communications faults, it is desired to have communication routes with allowed arc transition. To represent this situation we can color the arcs of D, where $A(D) = \{f_1, \ldots, f_m\}$, with colors $\{c_1, \ldots, c_m\}$, such that $c(f_1) = c_1$; and we define the digraph H, that will determine what arc transitions are allowed, as follows: the set of vertices of H is the set of colors $\{c_1, \ldots, c_m\}$, and we add an arc in H from c_i to c_j whenever the transition from arc f_i to arc f_j is allowed. So, if it is required to send a message from point A to point B through the communication network in the most convenient way, we need to find an H-walk in D from A to B.

Another application can be found in roads, roads can be easily represented with digraphs placing a vertex at the intersection of two streets, and an arc from a vertex v to a vertex u if and only if there is a street from v to u without passing through any other corner. Due to traffic laws, it is not always possible to turn from one street to another. So, if we want to find a walk from one corner to another that respect the traffic law, it is not enough to find a walks in a non-arc-colored digraph. So, we can color the arcs of the digraph as stated above. In this case, the digraph H indicates which turns are allowed by traffic laws.

Galeana-Sánchez et al. [9] studied the problem of the existence of closed Euler H-trails in H-colored undirected graphs. They defined the graph G_u as follows: Let u be a vertex of G; G_u is the graph such that $V(G_u) = \{e \in E(G) \mid e \text{ is incident with } u\}$, and two different vertices a and b are joining by only one edge in G_u if and only if $c(a)$ and $c(b)$ are adjacent in H.
They also showed necessary and sufficient conditions for the existence of closed Euler H-trails, as follows.

Theorem 5 ([9]). *Let H be a graph possibly with loops and G be an H-colored multigraph without loops. Suppose that G is Eulerian and G_u is a complete k_u-partite graph, for every u in $V(G)$ and for some k_u in $\mathbb{N}$. Then G has a closed Euler H-trail if and only if $|C_i^u| \leq \sum_{j \neq i} |C_j^u|$ for every u in $V(G)$, where $\{C_1^u, ..., C_{k_u}^u\}$ is the partition of $V(G_u)$ into independent sets.*

The rest of the paper is organized as follows. Section 2 is devoted to give some notation and terminology, which will be used through the paper. In Section 3, we will study the problem of the existence of Euler H-trail. In order to obtain our results we define the auxiliary digraph $L_n^H(D)$. Then, we will prove that there exists a bijection between the set of closed Euler H-trails in D and the set of directed cycles in $L_2^H(D)$.

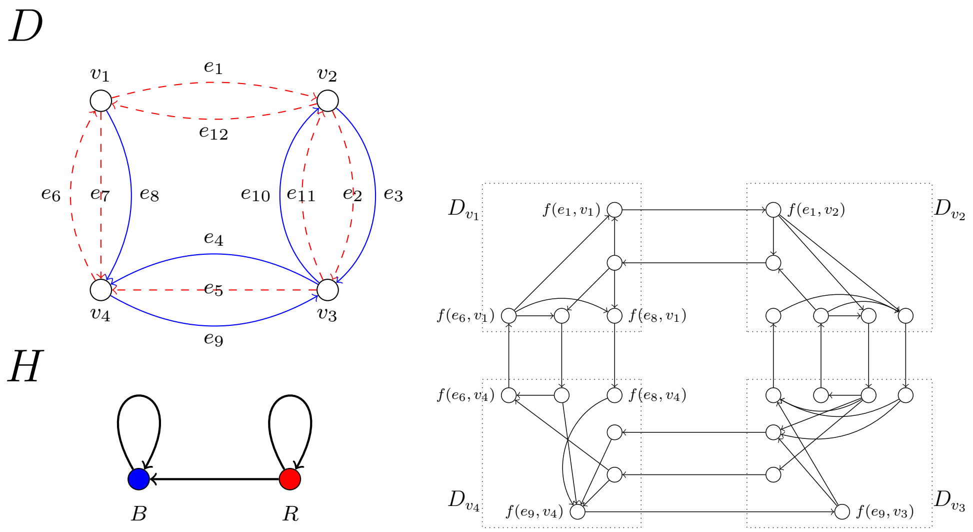
As a consequence, D has a closed Euler H-trail if and only if $L_n^H(D)$ is hamiltonian, for every $n \geq 2$. Finally, in Section 4 we give an infinite family of digraphs H such that for every multidigraph D without isolated vertices, and every H-coloring of D, the underlying graph of D_u is a disjoint union of complete bipartite graph and, possibly, isolated vertices, for every $u \in V(D)$.

2. Notation and Terminology

Let D be a multidigraph. If e is an arc and u and v are vertices such that $e = (u, v)$, then e is *incident* from u and to v, we will say that u and v are the tail and the head of e, respectively. If $u = v$, then the arc e is a *loop*. The *out-neighborhood* (*in-neighborhood*) of a vertex u, denoted by $N^+(u)$ ($N^-(u)$), is defined as the set of all arcs with tail (head) u. A digraph D' is a *subdigraph* of D if $V(D') \subseteq V(D)$ and $A(D') \subseteq A(D)$. Let S be a nonempty subset of $V(D)$; the subdigraph of D whose vertex set is S, and whose arc set is the set of those arcs of D that have both ends in S, is called the *subdigraph of* D *induced by* S, and is denoted by $D[S]$. The *underlying graph* of D is the graph G obtained from D by replacing every arc (u, v) with the edge uv.

A directed walk in a digraph D is a sequence $(v_0, e_0, v_1, e_1, \ldots, e_{k-1}, v_k)$, where $e_i = (v_i, v_{i+1})$ for every i in $\{0, \ldots, k-1\}$. We will say that the direct walk $(v_0, e_0, v_1, e_1, \ldots, e_{k-1}, v_k)$ is closed if $v_0 = v_k$. If $v_i \neq v_j$ for all i and j with $i \neq j$, it is called a directed path. A directed cycle is a closed directed walk $(v_0, e_0, v_1, e_1, \ldots, e_{k-1}, v_k, e_k, v_0)$, with $k \geq 2$, such that $v_i \neq v_j$ for all i and j with $i \neq j$. In a digraph D a directed walk in which no arc is repeated is a directed trail. A closed Euler directed trail in a digraph D, is a closed directed trail which traverses each arc of D exactly once. If $W = (v_0, e_0, v_1, e_1, \ldots, e_{k-1}, v_k)$ and $W' = (v_k, e_k, v_{k+1}, e_{k+1}, \ldots, e_{t-1}, v_t)$ are two directed walks, the directed walk $(v_k, e_{k-1}, \ldots, e_1, v_1, e_0, v_0)$, obtained by reversing W, is denoted by W^{-1} and the directed walk $(v_0, e_0, v_1, e_1, \ldots, e_{k-1}, v_k, e_k, v_{k+1}, e_{k+1}, \ldots, e_{t-1}, v_t)$, obtaining by concatenating W and W' at v_k, is denoted by $W \cup W'$. If there is no confusion we will omit adjective "directed".

A simple graph G is said to be multipartite, if for some positive integer k, there exists a partition $X_1, \ldots, X_k$ of $V(G)$, such that X_i is an independent set in G (that is no two vertices of X_i are adjacent) for every i in $\{1, \ldots, k\}$, in this case, also G is called k-partite. It said that G is a complete k-partite graph whenever G is k-partite and for every u in X_i and for every v in X_j , with $i \neq j$, we have that u and v are adjacent, denoted by $K_{n_1,\ldots,n_k}$ where $|X_i| = n_i$ for every i in $\{1, \ldots, k\}$. In the particular case when $k = 2$, the graph G is said to be bipartite graph. Let $G_1, G_2, \ldots, G_n$ be graphs, the *disjoint union*, denoted by $\bigcup_{i=1}^n G_i$, of $G_1, G_2, \ldots, G_n$ is the graph with vertex set, the disjoint union of $V(G_1), V(G_2), \ldots, V(G_n)$ and whose edge set, is the disjoint union of $E(G_1), E(G_2), \ldots, E(G_n)$. A matching in a graph G is a subset M of $E(G)$, such that no two elements of M are incident with the same vertex in G. A matching M saturates a vertex v if some edge of M is incident with v. If every vertex of G is



(a)D is an H-colored multidigraph (b)$L_2^H(D)$

Figure 1. **The auxiliary digraph** $L_2^H(D)$

saturated, the matching M is said to be perfect. We will need the following results.

Theorem 6 (Hall's Theorem). *A bipartite graph with bipartition (X, Y) has a matching that saturates every vertex in X if and only if $|N(S)| \geq |S|$, for every $S \subseteq X$.*

Lemma 1. *Let M and M' be two perfect matching of a graph G. If $M \cap M' = \emptyset$, then there exists a partition of the vertices of G into even cycles. Moreover, every cycle alternates edges between M and M'.*

3. H-trails

In what follows, H will be a digraph possibly with loops, and D will be a multidigraph without loops.
Throughout the paper we will use the auxiliary digraphs D_u and $L_n^H(D)$, which are defined below.

Definition 1. Let D be an H-colored multidigraph and u be a vertex of D. D_u is the digraph such that $V(D_u) = \{f(e, u) \mid e \in A(D)$ and e is incident with $u\}$, and two different vertices $f(e, u)$ and $f(g, u)$ are joining by only one arc from $f(e, u)$ to $f(g, u)$ in D_u if and only if $e = (x, u)$ and $g = (u, y)$, for some x and y in $V(D)$, and $(c(e), c(g)) \in A(H)$.

Observation 1. D_u is a bipartite digraph with bipartition (X, Y), where $X = \{f(e,u) \in V(D_u) \mid e = (x,u) \text{ for some } x \in V(D)\}$ and $Y = \{f(e,u) \in V(D_u) \mid e = (u,y) \text{ for some } y \in V(D)\}$. Moreover, if $(f(e,u), f(g,u)) \in A(D_u)$, then $f(e,u) \in X$ and $f(g,u) \in Y$.

Definition 2. Let D be an H-colored multidigraph with $|A(D)| = q$. For $n \geq 2$, $L_n^H(D)$ is the digraph with nq vertices, obtained as follows: for each arc $e = (u,v)$ of D, we take two vertices $f(e,u)$ and $f(e,v)$ in $L_n^H(D)$, and adding a directed path from $f(e,u)$ to $f(e,v)$ with $n-2$ new intermediate vertices. And the rest of the arcs of $L_n^H(D)$ are defined as follows: $(f(e,u), f(g,u)) \in A(L_n^H(D))$ if and only if $e = (x,u)$ and $g = (u,y)$, for some x and y in $V(D)$, and $(c(e), c(g)) \in A(H)$.

Notice that the digraph $L_n^H(D)$ can be constructed as follows: take the disjoint union of D_u, for every $u \in V(D)$, and for every $e = (u,v) \in A(D)$, we add a directed path from $f(e,u)$ to $f(e,v)$ with $n-2$ new intermediate vertices, see Figure 1.

Observation 7. For every $e = (u,v) \in A(D)$, we have that $d^+(f(e,u)) = d^-(f(e,v)) = 1$ in $L_n^H(D)$, for each $n \geq 2$. Moreover, when $n = 2$, we have that $N^+(f(e,u)) = \{f(e,v)\}$ and $N^-(f(e,v)) = \{f(e,u)\}$.

Theorem 8. *Let D be an H-colored multidigraph. Then there is a bijection between the set of closed H-trails in D and the set of directed cycles in $L_2^H(D)$.*

Proof. Let D be an H-colored multidigraph, $\mathcal{P}$ the set of closed H-trails in D and $\mathcal{C}$ the set of directed cycles in $L_2^H(D)$. Consider $T : \mathcal{P} \to \mathcal{C}$ defined by $T(x_0, e_0, x_1, e_1, x_2, \ldots, x_{n-1}, e_{n-1}, x_n) = (f(e_0,x_0), f(e_0,x_1), f(e_1,x_1), f(e_1,x_2), \ldots, f(e_{n-1},x_{n-1}), f(e_{n-1},x_n), f(e_0,x_0))$.

Claim 1. T is well-defined.
First we will prove that $T(P) = C \in \mathcal{C}$.
It follows from the definition of $L_2^H(D)$ that $(f(e_i,x_i), f(e_i,x_{i+1}))$ is an arc of $L_2^H(D)$, for every $i \in \{0, \ldots, n-1\}$.
Let e_i and e_{i+1} be consecutive arcs in P (if $i = n$, then $e_{i+1} = e_1$). Notice that e_i and e_{i+1} are incident with x_{i+1}. Since P is a closed H-trail, we have that $(c(e_i), c(e_{i+1}))$ is an arc in H and $(f(e_i,x_{i+1}), f(e_{i+1},x_{i+1})) \in A(L_2^H(D))$. Hence, C is a closed walk in $L_2^H(D)$. Since P is a closed H-trail, it follows that C does not repeat a vertex and C is a cycle, i.e., $T(P) = C \in \mathcal{C}$.
Next, we will prove that if $P_1 = P_2$, then $T(P_1) = T(P_2)$.
Let $P_1 = (x_0, e_0, x_1, e_1, x_2, \ldots, x_{n-1}, e_{n-1}, x_n)$ and $P_2 = (y_0, f_0, y_1, f_1, y_2, \ldots, y_{n-1}, f_{n-1}, y_n)$ in $\mathcal{P}$ such that $P_1 = P_2$.
Since $P_1 = P_2$, we have that $A(P_1) = A(P_2)$ and there exists $k \in \mathbb{N}$ such that $e_i = f_{i+k(mod n)}$ (since the arcs are traversed in the same order but the first arc is not necessarily the same). It follows from the definition of T that $V(T(P_1)) = V(T(P_2))$ and their are ordered the same. Then, we have that $T(P_1) = T(P_2)$. Therefore, T is well-defined.

Claim 2. T is injective.
Let P_1 and P_2 in $\mathcal{P}$ such that $P_1 \neq P_2$.
If $A(P_1) \neq A(P_2)$, then $V(T(P_1)) \neq V(T(P_2))$ and $T(P_1) \neq T(P_2)$. Otherwise, since $P_1 \neq P_2$, there exists $\{e_1, e_2\} \subseteq A(P_1) = A(P_2)$ such that $e_1 = (x_1, y_1)$ is the arc preceding $e_2 = (x_2, y_2)$ at P_1 and e_1 is not the arc preceding e_2 at P_2. Hence, $(f(e_1, y_1), f(e_2, x_2)) \in A(T(P_1))$ and $(f(e_1, y_1), f(e_2, x_2)) \notin A(T(P_2))$. Therefore, $T(P_1) \neq T(P_2)$ and T is injective.
Claim 3. T is surjective.
Let C be a cycle in $L_2^H(D)$. It follows from Observation 2 and the fact that $(f(e, x), f(e, y))$ is in $A(L_2^H(D))$, for every $e = (x, y) \in A(D)$, that C must be of the form $C = (f(e_0, x_0), f(e_0, y_0), f(e_1, x_1), f(e_1, y_1), \ldots, f(e_q, x_q), f(e_q, y_q), f(e_0, x_0))$, where $e_i = (x_i, y_i) \in A(D)$.
Consider the sequence $P = (x_0, e_0, x_1, e_1, x_2, \ldots, x_q, e_q, x_0)$.
Claim 3.1. P is a closed H-trail in D.
For each $k \in \{1, \ldots, q\}$, $(f(e_k, y_k), f(e_{k+1}, x_{k+1})) \in A(C) \subseteq A(L_2^H(D))$ (the subindices are taken modulo $q + 1$). So, it follows from the definition of $L_2^H(D)$ that $y_{k-1} = x_k$ and $(c(e_{k-1}), c(e_k)) \in A(H)$. Hence, P is a closed H-walk in D.
On the other hand, since C is a cycle, it follows that e_i appears just once in P, for every $i \in \{1, \ldots, q\}$, and P is a closed H-trail.
It follows from the definition of T that $T(P) = C$. Therefore, T is surjective.
By Claims 2 and 3, we have that T is a bijection. □

The underlying graph of $L_n^H(D)$ and D_x will be denoted by $UL_n^H(D)$ and UD_x, respectively. It follows from the definition of $UL_2^H(D)$ that $M_J = \{f(e, x)f(e, y) \in E(UL_2^H(D)) \mid e = (x, y) \in A(D)\}$ is a perfect matching, that we will called the *joint matching of* $UL_2^H(D)$.

Theorem 9. *Let D be an H-colored multidigraph. There exists a partition of the arcs of D into closed H-trails if and only if UD_x has a perfect matching, for every $x \in V(D)$.*

Proof. Let D be an H-colored multidigraph and M_J the joint matching of $UL_2^H(D)$. Suppose that there exists a partition of the arcs of D into closed H-trails, say $P = \{P_1, \ldots, P_k\}$.
It follows from Theorem 8 that $C = \{C_1, \ldots, C_k\}$ is a set of cycles in $L_2^H(D)$, where $T(P_i) = C_i$. Since $P = \{P_1, \ldots, P_k\}$ is a partition of the arcs of D, we have that $V(C_i) \cap V(C_j) = \emptyset$ (because $A(P_i) \cap A(P_j) = \emptyset$) and $V(L_2^H(D)) = \bigcup_{i=1}^k V(C_i)$ (because $A(D) = \bigcup_{i=1}^k A(P_i)$). Therefore, C is a partition of the vertices of $L_2^H(D)$ into cycles. Moreover, $C' = \{C'_1, \ldots, C'_k\}$ is a partition of the vertices of $UL_2^H(D)$ into cycles, where C'_i is the underlying cycle of C_i, for each $C_i \in C$.
By construction of C'_i, we have that C'_i alternate edges between M_J and $E(UL_2^H(D)) \setminus M_J$. Hence, $M = \bigcup_{i=1}^k E(C_i) \setminus M_J$ is a perfect matching of $UL_2^H(D) \setminus M_J$. Therefore, for every $x \in V(D)$, $M_x = M \cap E(UD_x)$ is a perfect matching of UD_x.

Conversely, suppose that M is a perfect matching of $UL_2^H(D) \setminus M_J$. Then, by Lemma 1, we have that there exists a partition of $UL_2^H(D)$ into even cycles, say $C' = \{C'_1, \ldots, C'_k\}$, such that every cycle alternate edges between M and M_J.
Since C'_i alternate edges between M_J and M, C'_i must be of the form $C'_i = (f(e_1,x_1), f(e_1,x_2), f(e_2,x_2), f(e_2,x_3), \ldots, f(e_n,x_n), f(e_n,x_1), f(e_1,x_1))$.
It follows from Observation 2 that if $(f(e_1,x_1), f(e_1,x_2)) \in A(L_2^H(D))$, then C'_i is a directed cycle in $L_2^H(D)$. Otherwise, $(f(e_1,x_2), f(e_1,x_1)) \in A(L_2^H(D))$ and $C_i^{'-1}$ is a directed cycle in $L_2^H(D)$. And then C_i will the directed cycle, i.e., $C_i = C'_i$ or $C_i = C_i^{'-1}$, for every $i \in \{1, \ldots, k\}$.
Let $C = \{C_1, \ldots, C_k\}$. Since C' is a partition of the vertices of $UL_2^H(D)$ into cycles, it follows that C is a partition of the vertices of $L_2^H(D)$ into directed cycles.
It follows from Theorem 8 that $P = \{P_1, \ldots, P_k\}$ is a set of closed H-trails, where $P_i = T^{-1}(C_i)$. Since C is a partition of the vertices of $L_2^H(D)$, it follows that P is a partition of the arcs of D into closed H-trails. □

Definition 3. Let D be an H-colored multidigraph. The H-line digraph, denoted by $L_1^H(D)$, is the digraph such that $V(L_1^H(D)) = A(D)$, and two different vertices $e = (x,y)$ and $g = (u,v)$ are joining by only one arc from e to g in $L_1^H(D)$ if and only if $y = u$ and $(c(e), c(g)) \in A(H)$.

Notice that $L_1^H(D)$ can be obtained from $L_2^H(D)$ by contracting the arc $(f(e,u), f(e,v))$ (and deleting the corresponding loop), for each $e = (u,v) \in A(D)$, in $L_2^H(D)$.
In view of the Theorem 9, we can conclude the following version of Theorem 1 for H-colored multidigraphs.

Theorem 10. *Let D be an H-colored multidigraph. Then,*

a. If $L_n^H(D)$ is hamiltonian, for some $n \geq 1$, then D has a closed Euler H-trail.

b. D has a closed Euler H-trail if and only if $L_n^H(D)$ is hamiltonian, for every $n \geq 1$.

Definition 4. Let D be an H-colored multidigraph with q arcs. We will say that D is an H-pancircular multidigraph if and only if it contains a closed H-trail of length L, for every $2 \leq L \leq q$.

Corollary 2. *Let D be an H-colored multidigraph with q arcs. Then, D is an H-pancircular multidigraph if and only if $L_1^H(D)$ is pancyclic.*

An H-colored multidigraph D is *H-trail-connected*, if and only if there is an H-trail starting with arc f_1 and ending with arc f_2, for any pair of arcs f_1 and f_2 in D.

Theorem 11. *Let D be an H-colored multidigraph such that $UD_u = \bigcup_{i=1}^{k_u} K_{n_i^u, m_i^u}$, for every u in $V(D)$ and some $k_u \geq 1$. Then D has a closed Euler H-trail if and only if D is H-trail-connected and, for every $u \in V(D)$, $n_i^u = m_i^u$ for each $i \in \{1, \ldots, k_u\}$.*

Proof. Let D be an H-colored multidigraph and $D_u = \bigcup_{i=1}^{k_u} K_{n_i^u, m_i^u}$, for every u in $V(D)$ and some $k_u \geq 1$.
Suppose that D has a closed Euler H-trail. Then, D is H-trail-connected and by Theorem 9, UD_u has a perfect matching, for every $u \in V(D)$, say M_u. Since $UD_u = \bigcup_{i=1}^{k_u} K_{n_i^u, m_i^u}$ and M_u is a perfect matching, we have that $n_i^u = m_i^u$, for every $i \in \{1, \ldots, k_u\}$.
Conversely, suppose that D is H-trail-connected and, for every $u \in V(D)$, $n_i^u = m_i^u$ for every $i \in \{1, \ldots, k_u\}$. Hence, $UD_u = \bigcup_{i=1}^{k_u} K_{n_i^u, n_i^u}$.
It follows from Theorem 6 that UD_u has a perfect matching, for every $u \in V(D)$. Then, by Theorem 9, D has a a partition of the arcs of D into closed H-trails, say $\mathcal{P} = \{P_1, \ldots, P_k\}$.
If $k = 1$, then D has a closed Euler H-trail. Otherwise, since D is H-trail-connected, there is an H-trail in D starting with arc $e_1 \in A(P_1)$ and ending with arc $e_2 \in A(D) \setminus A(P_1)$, say Q_1.
Let $g_1 = (x_1, x_2) \in A(Q_1)$ be the first arc in $A(Q_1) \setminus A(P_1)$ and $g_0 = (x_0, x_1)$ the arc prior to g_1 in Q_1. Hence, $g_0 \in A(P_1)$ and $(c(g_0), c(g_1)) \in A(H)$. Suppose, without loss of generality, that $g_1 \in A(P_2)$.
Let $P_1 = (x_1, f_1, \ldots, x_0, g_0, x_1)$ and $P_2 = (x_1, g_1, x_2, \ldots, f_0, x_1)$. By the definition of closed H-trail, we have that $\{(c(g_0), c(f_1)), (c(f_0), c(g_1))\} \subseteq A(H)$. By the definition of D_{x_1}, we have that the arcs $(f(g_0, x_1), f(f_1, x_1))$, $(f(f_0, x_1), f(g_1, x_1))$ and $(f(g_0, x_1), f(g_1, x_1))$ are in $A(D_{x_1})$. So, it follows from the Observation 1 and the fact that $UD_{x_1} = \bigcup_{i=1}^{k_{x_1}} K_{n_i^{x_1}, n_i^{x_1}}$ that $(f(f_0, x_1), f(f_1, x_1)) \in A(D_{x_1})$. Therefore, $(c(f_0), c(f_1)) \in A(H)$ and $T_1 = P_1 \cup P_2 = (x_1, f_1, \ldots, x_0, g_0, x_1, g_1, x_2, \ldots, f_0, x_1)$ is a closed H-trail.
If $A(T_1) = A(D)$, then T_1 is a closed Euler H-trail. Otherwise, there is an H-trail in D starting with arc $e_3 \in A(T_1)$ and ending with arc $e_4 \in A(D) \setminus A(T_1)$, say Q_2.
Let $g_3 = (v_1, v_2) \in A(Q_2)$ be the first arc in $A(Q_2) \setminus A(T_1)$ and $g_2 = (v_0, v_1)$ the arc prior to g_3 in Q_2. Hence, $g_2 \in A(T_1)$ and $(c(g_2), c(g_3)) \in A(H)$. Suppose, without loss of generality, that $g_3 \in A(P_3)$.
Let $T_1 = (v_1, f_3, \ldots, g_2, v_1)$ and $P_2 = (v_1, g_3, v_2, \ldots, f_2, v_1)$. By the definition of closed H-trail, it follows that $\{(c(g_2), c(f_3)), (c(f_2), c(g_3))\} \subseteq A(H)$. By the definition of D_{v_1}, we have that the arcs $(f(g_2, v_1), f(f_3, v_1))$, $(f(f_2, v_1), f(g_3, v_1))$ and $(f(g_2, v_1), f(g_3, v_1))$ are in $A(D_{v_1})$. It follows from the Observation 1 and the fact that $UD_{v_1} = \bigcup_{i=1}^{k_{v_1}} K_{n_i^{v_1}, n_i^{v_1}}$ that the arc $(f(f_2, v_1), f(f_3, v_1))$ is in $A(D_{v_1})$. Therefore, $(c(f_2), c(f_3)) \in A(H)$ and $T_2 = T_1 \cup P_3 = (v_1, f_3, \ldots, g_2, v_1, g_3, v_2, \ldots, f_2, v_1)$ is a closed H-trail.
If $A(T_2) = A(D)$, then T_2 is a closed Euler H-trail. Otherwise, we can repeat this procedure and after a finite number of steps we obtain that D has a closed Euler H-trail. □

Let D be an c-arc-colored multidigraph and $v \in V(D)$. We will say that $F_v = \{e \in A(D) \mid e \text{ is incident with } v\}$, and $c(F_v) = \{i \mid \text{there exists } e \in F_v \text{ such that } c(e) = i\}$.

Corollary 3. *Let D be a c-arc-colored multidigraph such that $c(F_v) = \{c_1^v, c_2^v\}$, for* *every $v \in V(D)$. Then, D is PC Euler if and only if D is PC trail-connected and for every $v \in V(D)$, $d^+_{c^v_i}(v) = d^-_{c^v_{3-i}}(v)$, for $i \in \{1,2\}$.*

Proof. Suppose that D is a c-arc-colored multidigraph. Then, D is an H-colored multidigraph, where H is the complete digraph without loops and $V(H) = \{1, 2, \ldots, c\}$. Notice that if P is a (closed) trail in D, then P is a PC (closed) trail if and only if P is a (closed) H-trail.
Suppose that $c(F_v) = \{c^v_1, c^v_2\}$, for every $v \in V(D)$. Let u be a vertex in $V(D)$.
Claim 1. $UD_u = K_{n_1,m_1} \cup K_{n_2,m_2}$, where $n_i = d^-_{c^u_i}(u)$ and $m_i = d^+_{c^u_{3-i}}(u)$.
Let $E^+_u = \{f = (u,x) \in A(D) \mid \text{for some } x \in V(D)\}$ and $E^-_u = \{f = (x,u) \in A(D) \mid \text{for some } x \in V(D)\}$. Hence, $F^+_{c^u_i} = \{f \in E^+_u \mid c(f) = c^u_i\}$ and $F^-_{c^u_i} = \{f \in E^-_u \mid c(f) = c^u_i\}$ are independent sets in UD_u, for every $i \in \{1,2\}$. Moreover, for every $i \in \{1,2\}$, $fg \in E(UD_u)$ if and only if $f \in F^-_{c^u_i}$ and $g \in F^+_{c^u_{3-i}}$.
Notice that $|F^+_{c^u_i}| = d^+_{c^u_i}(u)$ and $|F^-_{c^u_i}| = d^-_{c^u_i}(u)$. Therefore, $UD_u = K_{n_1,m_1} \cup K_{n_2,m_2}$, where $n_i = d^-_{c^u_i}(u)$ and $m_i = d^+_{c^u_{3-i}}(u)$.
Suppose that D is PC Euler. Then, D is PC trail-connected and D has a closed Euler H-trail. It follows from Theorem 11 and Claim 1 that $d^-_{c^u_i}(u) = d^+_{c^u_{3-i}}(u)$, for every $i \in \{1,2\}$.
Conversely, supposed that D is PC trail-connected and for every $v \in V(D)$, $d^+_{c^v_i}(v) = d^-_{c^v_{3-i}}(v)$, for $i \in \{1,2\}$. Since D is PC trail-connected, we have that D is H-trail-connected.
It follows from Claim 1 and $d^+_{c^u_i}(u) = d^-_{c^u_{3-i}}(u)$, for $i \in \{1,2\}$, that $UD_u = K_{n_1,n_1} \cup K_{n_2,n_2}$. Hence, by Theorem 11, D has a closed Euler H-trail, i.e., D is PC Euler. □

Corollary 4 ([20]). *Let D be a 2-arc-colored multidigraph. Then D is PC Euler if and only if D is PC trail-connected and for every $v \in V(D)$, $d^+_i(v) = d^-_{3-i}(v)$, for $i \in \{1,2\}$.*

Theorem 12. *Let D be an H-colored multidigraph. Then D is H-trail-connected if and only of $L^H_2(D)$ is strongly connected.*

Proof. Let D be an H-colored multidigraph.
Suppose that D is H-trail-connected. Let $f(e,u)$ and $f(g,x)$ in $V(L^H_2(D))$.
Since D is an H-trail-connected graph, there exists an H-trail such that $e = (u_1,u_2)$ is the first arc and $g = (x_1,x_2)$ is the last arc, say $P = (u_1, e, u_2, e_1, \ldots, x_1, g, x_2)$.
Notice that $T'(P) = (f(e,u_1), f(e,u_2), f(e_1,u_2), \ldots, f(g,x_1), f(g,x_2))$ is a path.
Since $f(e,u)$ and $f(g,x)$ are in $V(L^H_2(D))$, it follows that ($u = u_1$ or $u = u_2$) and ($x = x_1$ or $x = x_2$). Therefore, there exits a path from $f(e,u)$ to $f(g,x)$ and $L^H_2(D)$ is strongly connected.
Conversely, suppose that $L^H_2(D)$ is strongly connected. Let $e = (x,y)$ and $g = (u,v)$ in $A(D)$.
Since $L^H_2(D)$ is strongly connected, there exists a path from $f(e,x)$ to $f(g,v)$, say P. It follows from Observation 1 that P must be of the form $P = (f(e,x), f(e,y), f(e_1,y), \ldots, f(g,u), f(g,v))$. So, $T'^{-1}(P) = (x, e, y, e_1, \ldots, u, g, v)$

is an H-trail such that its first arc is e and its last arc is g. Therefore, D is H-trail-connected. □

The next assertion follows directly from the previous results.

Corollary 5. *Given a 2-arc-colored multidigraph, we can check in polynomial time whether D has a close PC Euler trail.*

4. H-coloring

After reading Theorem 11 arises naturally the following question: when does a multidigraph satisfy the hypothesis that $UD_u = \bigcup_{i=1}^{k_u} K_{n_i^u, m_i^u}$, for every u in $V(D)$? In this section we give an infinite family of digraphs H such that for every multidigraph D without isolated vertices, and every H-coloring of D, $UD_u = \bigcup_{i=1}^{k_u} K_{n_i^u, m_i^u}$, for every $u \in V(D)$.
Recall that $V(D_u)$ is the set of arcs in D incident with u, so in this section we will use a simplified notation for the vertices of D_u, the vertex $f(e,u)$ will denote by e.
In the remainder of this section we will denote the empty graph with n vertices by $K_{n,0}$.

Theorem 13. *Let H be a digraph, possibly with loops, such that for every pair of vertices u and v of H, $N_H^+(u) = N_H^+(v)$ or $N_H^+(u) \cap N_H^+(v) = \emptyset$. Then, for every multidigraph D without isolated vertices, and every H-coloring of D, $UD_u = \bigcup_{i=1}^{k_u} K_{n_i^u, m_i^u}$, for every $u \in V(D)$ and for some $k_u \geq 1$.*

Proof. Suppose that H is a digraph, possibly with loops, such that for every pair of vertices u and v of H, $N_H^+(u) = N_H^+(v)$ or $N_H^+(u) \cap N_H^+(v) = \emptyset$.
Let D be an H-colored multidigraph without isolated vertices and $u \in V(D)$.
We define $V_0 = \{e \in V(UD_u) : d_{UD_u}(e) = 0\}$. Notice that $UD_u[V_0] = K_{m,0}$, for a non-negative integer m. Thus, we only need to prove that each component in $UD_u - V_0$ is a complete bipartite graph.
Let B be a component of UD_u with at least two vertices. It follows from Observation 1 that B is a bipartite graph with partite sets $X_B = X \cap V(B)$ and $Y_B = Y \cap V(B)$, where X and Y are the partite sets of D_u defined in Observation 1.
Now we will prove that every vertex in X_B is adjacent to every vertex in Y_B. So, let $x \in X_B$ and $y \in Y_B$. Since B is a component of UD_u, there is an xy-path in B, namely $T = (x = e_1, e_2, \ldots, e_{2p} = y)$.
Notice that for every $i \in \{1, \ldots, p\}$, $e_{2i-1} \in X_B$ and $e_{2i} \in Y_B$. It follows from the definition of D_u that $(c(e_{2i-1}), c(e_{2i}))$ and $(c(e_{2(i+1)-1}), c(e_{2i}))$ are arcs of H, i.e., $N_H^+(c(e_{2i-1})) \cap N_H^+(c(e_{2(i+1)-1})) \neq \emptyset$. So, by hypothesis, $N_H^+(c(e_{2i-1})) = N_H^+(c(e_{2(i+1)-1}))$. Thus, $N_H^+(c(e_1)) = N_H^+(c(e_{2p-1}))$. Moreover, $c(e_{2p}) \in N_H^+(c(e_{2p-1})) = N_H^+(c(e_1))$. Hence, $x = e_1$ and $y = e_{2p}$ are adjacent in B and B is a complete bipartite graph.

Therefore, every component in $UD_u - V_0$ is a complete bipartite graph and $UD_u = \bigcup_{i=1}^{k_u} K_{n_i^u, m_i^u}$, for every $u \in V(D)$ and for some $k_u \geq 1$. □

Proposition 1. *Let H be a complete digraph with a loop in each vertex. Then, for every multidigraph D without isolated vertices, and every H-coloring of D, UD_u is a complete bipartite graph or an empty graph, for every $u \in V(D)$.*

Proof. Suppose that H is a complete digraph with a loop in each vertex. Let D be an H-colored multidigraph without isolated vertices, and u a vertex of D.
Case 1. $d^-(u) = 0$ or $d^+(u) = 0$.
It follows from Observation 1 that UD_u is an empty graph.
Case 2. $d^-(u) \neq 0$ and $d^+(u) \neq 0$.
Let $X = \{e \in A(D) : e = (x, u)$ for some $x \in V(D)\}$ and $Y = \{f \in A(D) : f = (u, x)$ for some $x \in V(D)\}$. Since $d^-(u) \neq 0$ and $d^+(u) \neq 0$, we have that $X \neq \emptyset$ and $Y \neq \emptyset$.
Consider $e \in X$ and $f \in Y$. Since H is a complete graph with a loop at each vertex, it follows that $(c(e), c(f))$ is an arc in H. Therefore, (e, f) is an arc in D_u.
It follows by Observation 1 that $V(UD_u) = X \cup Y$. Therefore, UD_u is a complete bipartite graph. □

Corollary 6. *Let H be a digraph such that $d^+(x) = 1$, for every vertex x in $V(H)$. Then, for every multidigraph D without isolated vertices, and every H-coloring of D, $UD_u = \bigcup_{i=1}^{k_u} K_{n_i^u, m_i^u}$, for every $u \in V(D)$ and for some $k_u \geq 1$.*

Proof. Let H be a digraph such that $d^+(x) = 1$, for every vertex x in $V(H)$. Hence, for every pair of vertices, u and v, in H, we have that $N_H^+(u) = N_H^+(v)$ or $N_H^+(u) \cap N_H^+(v) = \emptyset$.
Therefore, by Theorem 13, every multidigraph D without isolated vertices, and every H-coloring of D, $UD_u = \bigcup_{i=1}^{k_u} K_{n_i^u, m_i^u}$, for every $u \in V(D)$ and for some $k_u \geq 1$. □

Corollary 7. *Let H be a digraph such that $d^-(x) = 1$, for every vertex x in $V(H)$. Then, for every multidigraph D without isolated vertices, and every H-coloring of D, $UD_u = \bigcup_{i=1}^{k_u} K_{n_i^u, m_i^u}$, for every $u \in V(D)$ and for some $k_u \geq 1$.*

Corollary 8. *If H is a digraph with only loops, then for every multidigraph D without isolated vertices, and every H-coloring of D, $UD_u = \bigcup_{i=1}^{k_u} K_{n_i^u, m_i^u}$, for every $u \in V(D)$ and for some $k_u \geq 1$.*

Corollary 9. *If H is a cycle, then for every multidigraph D without isolated vertices, and every H-coloring of D, $UD_u = \bigcup_{i=1}^{k_u} K_{n_i^u, m_i^u}$, for every $u \in V(D)$ and for some $k_u \geq 1$.*

Corollary 10. *If H is a path with a loop only in the end vertices, then for every multidigraph D without isolated vertices, and every H-coloring of D, $UD_u = \bigcup_{i=1}^{k_u} K_{n_i^u, m_i^u}$, for every $u \in V(D)$ and for some $k_u \geq 1$.*

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