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Research Article

The r-dynamic chromatic number of the corona product of graphs

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Abstract: Let G be a graph. An *r-dynamic k-coloring* of G is a proper k-coloring of G such that every vertex v in $V(G)$ has neighbors in at least $\min\{r, d_G(v)\}$ different color classes. The *r-dynamic chromatic number* of G, denoted by $\chi_r(G)$, is the least k such that G has an r-dynamic k-coloring. We determine the r-dynamic chromatic number of the corona product $G \odot H$ of graphs G and H, in terms of the dynamic chromatic numbers of G and H.

Keywords: dynamic coloring, corona product.

AMS Subject classification: 05C15, 05C76

1. Introduction

We refer the book [7] for graph theoretical notation and terminology. Let G be a finite, simple, connected, undirected graph with vertex set $V(G)$ and edge set $E(G)$. For $v \in V(G)$, $d_G(v)$ and $N_G(v)$ denote, respectively, the degree of v, and the neighborhood of v. The minimum degree and the maximum degree of G are denoted by $\delta(G)$ and $\Delta(G)$, respectively.
A *k-coloring* of G is a map $c : V(G) \to S$, where $|S| = k$; it is *proper* if adjacent vertices receive different colors. An *r-dynamic k-coloring* is a proper k-coloring c of G such that on each vertex neighborhood $N_G(v)$ at least $\min\{r, d_G(v)\}$ colors are used, i.e., $|c(N_G(v))| \geq \min\{r, d_G(v)\}$. The *$r$-dynamic chromatic number*, introduced by Montgomery [23] and written as $\chi_r(G)$, is the minimum k such that G has an r-dynamic k-coloring.

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The 1-dynamic chromatic number of G is its chromatic number $\chi(G)$, i,e., $\chi_1(G) = \chi(G)$. The 2-dynamic chromatic number was introduced as *dynamic chromatic number* by Montgomery [23]. Given a graph G, form G^2 by adding edges joining nonadjacent vertices having a common neighbor in G. From the definition, we have the following (see [13]):
$\chi_r(G) = \chi_{\Delta(G)}(G)$, when $r \geq \Delta(G)$;
$\chi_{\Delta(G)}(G) = \chi(G^2)$;
$\chi_1(G) \leq \chi_2(G) \leq \chi_3(G) \leq \cdots \leq \chi_{\Delta(G)}(G)$; and
$\chi_r(G) \geq \min\{r, \Delta(G)\} + 1$.
Hence, $\chi_r(G) = \chi_{\Delta(G)}(G) \geq \Delta(G) + 1$, when $r \geq \Delta(G)$; and $\chi_r(G) \geq r + 1$, when $r \leq \Delta(G)$.
The r-dynamic chromatic number has been studied by several authors (see [2–6, 8–10, 12–15, 20–23, 25]). In particular, for the corona product, it has been studied in [1, 16–19, 24].
Given two simple graphs G and H with $V(G) = \{v_1, v_2, \ldots, v_n\}$ and n disjoint copies $H_1, H_2, \ldots, H_n$ of H, the *corona product of* G *and* H, denoted by $G \odot H$, is the simple graph obtained from the disjoint union $G \cup (H_1 \cup H_2 \cup \cdots \cup H_n)$ by making the vertex v_i of G adjacent to every vertex of H_i, $i \in \{1, 2, \ldots, n\}$ (defined by Frucht and Harary [11]). In considering the corona product $G \odot H$, assume throughout that G is a connected graph with at least two vertices.
Observe that, for $i \in \{1, 2, \ldots, n\}$, $d_{G \odot H}(v_i) = d_G(v_i) + |V(H)|$ and for $u \in V(H_i)$, $d_{G \odot H}(u) = d_H(u) + 1$. Hence, $\delta(G \odot H) = \delta(H) + 1$ and $\Delta(G \odot H) = \Delta(G) + |V(H)|$.
For disjoint graphs G_1 and G_2, the graph $G_1 \vee G_2$ is the simple graph obtained from the disjoint union $G_1 \cup G_2$ by joining each vertex of G_1 to each vertex of G_2.
In Section 2, for $1 \leq r \leq \Delta(G) + |V(H)|$, we compute the r-dynamic chromatic number of the corona product $G \odot H$.

2. Result

Theorem 1.

1. $\chi_1(G \odot H) = \chi(G \odot H) = \max\{\chi(G), 1 + \chi(H)\}$.

2. $\chi_2(G \odot H) = \begin{cases} 3 & \text{if } \chi(G) = 2 \text{ and } E(H) = \emptyset, \\ \chi_1(G \odot H) & \text{otherwise.} \end{cases}$

3. For $3 \leq r \leq \Delta(H) + 1$,
$$\chi_r(G \odot H) = \max\{\chi(G), 1 + \chi_{r-1}(H)\}.$$
4. For $\Delta(H) + 2 \leq r \leq \chi_{\Delta(H)}(H)$,
$$\chi_r(G \odot H) = \max\{\chi(G), 1 + \chi_{\Delta(H)}(H)\}.$$
5. For $\chi_{\Delta(H)}(H) + 1 \leq r \leq |V(H)|$,
$$\chi_r(G \odot H) = \max\{\chi(G), r + 1\}.$$
6. For $|V(H)| + 1 \leq r \leq |V(H)| + \Delta(G)$,
$$\chi_r(G \odot H) = \max\{\chi_{r-|V(H)|}(G), r + 1\}.$$

Proof. *Proof of 1.* Let $k = \max\{\chi(G), 1 + \chi(H)\}$.
As G and $K_1 \vee H$ are subgraphs of $G \odot H$, we have, $\chi(G \odot H) \geq \chi(G)$ and $\chi(G \odot H)$

$\geq \chi(K_1 \vee H) = 1 + \chi(H)$. Hence, $\chi(G \odot H) \geq k$.
Let c' be a proper $\chi(G)$-coloring of G with colors $\{1, 2, \ldots, \chi(G)\}$ and let c'' be a proper $\chi(H)$-coloring of H with colors $\{1, 2, \ldots, \chi(H)\}$. Define $c_1' : V(G \odot H) \to \{1, 2, \ldots, k\}$ as follows: for $i \in \{1, 2, \ldots, n\}$, $c_1'(v_i) = c'(v_i)$ and c_1' restricted to $V(H_i)$ is c''. We now recolor $G \odot H$ as follows. For $i \in \{1, 2, \ldots, n\}$, if $c'(v_i) \in \{1, 2, \ldots, \chi(H)\}$, recolor the vertices $u \in V(H_i)$ satisfying $c''(u) = c'(v_i)$ with the color $1+\chi(H)$. This yields a proper k-coloring $c_1 : V(G \odot H) \to \{1, 2, \ldots, k\}$ of $G \odot H$. Hence, $\chi(G \odot H) \leq k$.
Thus, $\chi_1(G \odot H) = \chi(G \odot H) = k$.

Proof of 2. Let $k = \max\{\chi(G), 1 + \chi(H)\}$. Since $\chi_1(G \odot H) \leq \chi_2(G \odot H)$, we have $k \leq \chi_2(G \odot H)$. We consider two cases.

Case 1. $E(H) \neq \emptyset$.
Then, consider the k-coloring c_1 defined in the proof of 1. Take $c_2 = c_1$. Let $i \in \{1, 2, \ldots, n\}$. Since $E(H) \neq \emptyset$, we have $|c_2(N_{G \odot H}(v_i))| \geq |c_2(V(H_i))| \geq 2$. As $\min\{2, d_{G \odot H}(v_i)\} = \min\{2, d_G(v_i) + |V(H)|\} = 2$, we have $|c_2(N_{G \odot H}(v_i))| \geq \min\{2, d_{G \odot H}(v_i)\}$. Also, let $u \in V(H_i)$. For $d_H(u) \geq 1$, as $c_2(v_i) \notin c_2(N_{H_i}(u))$, we have $|c_2(N_{G \odot H}(u))| \geq |\{c_2(v_i)\}| + |c_2(N_{H_i}(u))| \geq 1 + 1 = 2 = \min\{2, 1 + d_H(u)\} = \min\{2, d_{G \odot H}(u)\}$; and for $d_H(u) = 0$, we have $|c_2(N_{G \odot H}(u))| = 1 = \min\{2, 1\} = \min\{2, 1 + d_H(u)\} = \min\{2, d_{G \odot H}(u)\}$. Hence, c_2 is a 2-dynamic k-coloring of $G \odot H$, and so $\chi_2(G \odot H) \leq k$. Thus, $\chi_2(G \odot H) = k$.

Case 2. $E(H) = \emptyset$.
Then, $H = K_m^c$, the complement of K_m, and $k = \max\{\chi(G), 1 + \chi(H)\} = \chi(G)$. We consider two subcases.

Subcase 2.1. $\chi(G) \geq 3$.
Let c' be a proper $\chi(G)$-coloring of G with colors $\{1, 2, \ldots, \chi(G)\}$.
For $m \geq 2$, extend c' to obtain $c_2 : V(G \odot H) \to \{1, 2, \ldots, \chi(G)\}$ so that, for each $i \in \{1, 2, \ldots, n\}$, $c'(v_i) \notin c_2(V(H_i))$ and $|c_2(V(H_i))| \geq 2$. Such an extension is possible. Let $i \in \{1, 2, \ldots, n\}$. Then $|c_2(N_{G \odot H}(v_i))| \geq |c_2(V(H_i))| \geq 2$. Since $\min\{2, d_{G \odot H}(v_i)\} = \min\{2, d_G(v_i) + |V(H)|\} = 2$, we have $|c_2(N_{G \odot H}(v_i))| \geq \min\{2, d_{G \odot H}(v_i)\}$. Also, let $u \in V(H_i)$. Then $d_H(u) = 0$, and so $|c_2(N_{G \odot H}(u))| = 1 = \min\{2, 1\} = \min\{2, 1 + d_H(u)\} = \min\{2, d_{G \odot H}(u)\}$.
Now consider $m = 1$. Let $i \in \{1, 2, \ldots, n\}$, and let $v_j^{(i)}$ be an arbitrary vertex of G which is adjacent to v_i in G. Extend c' to obtain $c_2 : V(G \odot H) \to \{1, 2, \ldots, \chi(G)\}$ so that, for each $i \in \{1, 2, \ldots, n\}$, $c_2(V(H_i)) \in \{1, 2, \ldots, \chi(G)\} \setminus \{c'(v_i), c'(v_j^{(i)})\}$. Such an extension is possible. Then $|c_2(N_{G \odot H}(v_i))| \geq |\{c'(v_j^{(i)}), c_2(V(H_i))\}| \geq 2$. Since $\min\{2, d_{G \odot H}(v_i)\} = \min\{2, d_G(v_i) + |V(H)|\} = 2$, we have $|c_2(N_{G \odot H}(v_i))| \geq \min\{2, d_{G \odot H}(v_i)\}$. Also, let $V(H_i) = \{u\}$. Then $d_H(u) = 0$, and so $|c_2(N_{G \odot H}(u))| = 1 = \min\{2, 1\} = \min\{2, 1 + d_H(u)\} = \min\{2, d_{G \odot H}(u)\}$.
Hence, for $m \geq 1$, c_2 is a 2-dynamic $\chi(G)$-coloring of $G \odot H$, and so $\chi_2(G \odot H) \leq \chi(G)$. Thus $\chi_2(G \odot H) = \chi(G) = k$.

Subcase 2.2. $\chi(G) = 2$.
We have, $\chi_2(G \odot H) \geq \min\{2, \Delta(G \odot H)\} + 1 = \min\{2, \Delta(G) + |V(H)|\} + 1 = 3$.

Let c' be a proper 2-coloring of G with colors $\{1, 2\}$. Extend c' to obtain $c_2 : V(G \odot H) \to \{1, 2, 3\}$ so that, for each $i \in \{1, 2, \ldots, n\}$, $c_2(V(H_i)) = 3$. Let $i \in \{1, 2, \ldots, n\}$. Then $|c_2(N_{G \odot H}(v_i))| = 2$ and $\min\{2, d_{G \odot H}(v_i)\} = \min\{2, d_G(v_i) + |V(H)|\} = 2$. Also, let $V(H_i) = \{u\}$. Then $|c_2(N_{G \odot H}(u))| = 1$ and $\min\{2, d_{G \odot H}(u)\} = \min\{2, 1 + d_H(u)\} = \min\{2, 1\} = 1$. It follows that, c_2 is a 2-dynamic 3-coloring of $G \odot H$, and so $\chi_2(G \odot H) \leq 3$. Thus $\chi_2(G \odot H) = 3$. This completes the proof of 2.

Proof of 3. Any r-dynamic $\chi_r(G \odot H)$-coloring c of $G \odot H$ yields a proper $\chi_r(G \odot H)$-coloring for G and an $(r-1)$-dynamic $(\chi_r(G \odot H) - 1)$-coloring for H_i (since $c(v_i) \notin c(V(H_i)))$, where $i \in \{1, 2, \ldots, n\}$. Hence, $\chi(G) \leq \chi_r(G \odot H)$ and $\chi_{r-1}(H) \leq \chi_r(G \odot H) - 1$. Thus, $\chi_r(G \odot H) \geq \max\{\chi(G), 1 + \chi_{r-1}(H)\}$.
Let $k = \max\{\chi(G), 1 + \chi_{r-1}(H)\}$. Let c' be a proper $\chi(G)$-coloring of G with colors $\{1, 2, \ldots, \chi(G)\}$ and let c'' be an $(r-1)$-dynamic $\chi_{r-1}(H)$-coloring of H with colors $\{1, 2, \ldots, \chi_{r-1}(H).\}$ Define $c'_r : V(G \odot H) \to \{1, 2, \ldots, k\}$ as follows: for $i \in \{1, 2, \ldots, n\}$, $c'_r(v_i) = c'(v_i)$ and c'_r restricted to $V(H_i)$ is c''. We now recolor $G \odot H$ as follows. For $i \in \{1, 2, \ldots, n\}$, if $c'(v_i) \in \{1, 2, \ldots, \chi_{r-1}(H)\}$, recolor the vertices $u \in V(H_i)$ satisfying $c''(u) = c'(v_i)$ with the color $1 + \chi_{r-1}(H)$. This yields a proper k-coloring $c_r : V(G \odot H) \to \{1, 2, \ldots, k\}$ of $G \odot H$.
Let $i \in \{1, 2, \ldots, n\}$. Then $|c_r(N_{G \odot H}(v_i))| \geq |c_r(V(H_i))| = \chi_{r-1}(H) \geq r = \min\{r, d_G(v_i) + |V(H)|\} = \min\{r, d_{G \odot H}(v_i)\}$. Also, let $u \in V(H_i)$. Then $|c_r(N_{G \odot H}(u))| \geq |\{c_r(v_i)\}| + |c_r(N_{H_i}(u))| \geq 1 + \min\{r-1,\ d_H(u)\} = \min\{r, 1 + d_H(u)\} = \min\{r, d_{G \odot H}(u)\}$. Hence, c_r is an r-dynamic k-coloring of $G \odot H$, and so $\chi_r(G \odot H) \leq k$. Thus $\chi_r(G \odot H) = k$.
Proof of 4. Any r-dynamic $\chi_r(G \odot H)$-coloring c of $G \odot H$ yields a proper $\chi_r(G \odot H)$-coloring for G and a $\Delta(H)$-dynamic $(\chi_r(G \odot H) - 1)$-coloring for H_i (since $c(v_i) \notin c(V(H_i)))$, where $i \in \{1, 2, \ldots, n\}$. This implies that, $\chi(G) \leq \chi_r(G \odot H)$ and $\chi_{\Delta(H)}(H) \leq \chi_r(G \odot H) - 1$. Thus, $\chi_r(G \odot H) \geq \max\{\chi(G), 1 + \chi_{\Delta(H)}(H)\}$.
Consider the coloring $c_{\Delta(H)+1}$ obtained in the proof of 3. Take $c_r = c_{\Delta(H)+1}$. Let $i \in \{1, 2, \ldots, n\}$. Then, by hypothesis, we have $|c_r(N_{G \odot H}(v_i))| \geq |c_r(V(H_i))| \geq \chi_{\Delta(H)}(H) \geq r$. Since $r \leq \chi_{\Delta(H)}(H) \leq |V(H)|$, we have $\min\{r, d_{G \odot H}(v_i)\} = \min\{r, d_G(v_i) + |V(H)|\} = r$. Hence, $|c_r(N_{G \odot H}(v_i))| \geq \min\{r, d_{G \odot H}(v_i)\}$. Also, let $u \in V(H_i)$. Then $|c_r(N_{G \odot H}(u))| \geq |\{c_r(v_i)\}| + |c_r(N_{H_i}(u))| = 1 + d_H(u) = \min\{r, 1 + d_H(u)\} = \min\{r, d_{G \odot H}(u)\}$, since $1 + d_H(u) \leq 1 + \Delta(H) < \Delta(H) + 2 \leq r$. Hence, c_r is an r-dynamic k-coloring of $G \odot H$, where $k = \max\{\chi(G), 1 + \chi_{\Delta(H)}(H)\}$, and so $\chi_r(G \odot H) \leq k$. Thus $\chi_r(G \odot H) = k$.
Proof of 5. $\chi_r(G \odot H) \geq \chi(G)$ and $\chi_r(G \odot H) \geq \min\{r, \Delta(G \odot H)\} + 1 = \min\{r, \Delta(G) + |V(H)|\} + 1 = r + 1$ implies that $\chi_r(G \odot H) \geq \max\{\chi(G), r + 1\}$.
Consider the coloring $c_{\Delta(H)+1}$ obtained in the proof of 3. Note that, from the proof of 4, $c_{\chi_{\Delta(H)}(H)} = c_{\Delta(H)+1}$. For each r satisfying $\chi_{\Delta(H)}(H) + 1 \leq r \leq |V(H)|$, let $k = \max\{\chi(G), r + 1\}$. We define a k-coloring c_r from c_{r-1} recursively, which satisfies that $|c_r(V(H_i))| = |c_{r-1}(V(H_i))| + 1$, for every $i \in \{1, 2, \ldots, n\}$, as follows.
Observe that, for each $i \in \{1, 2, \ldots, n\}$, $|c_{\chi_{\Delta(H)}(H)}(V(H_i))| = \chi_{\Delta(H)}(H)$. Hence, $|c_{r-1}(V(H_i))| = |c_{r-2}(V(H_i))| + 1 = |c_{r-3}(V(H_i))| + 2 = \ldots = |c_{\chi_{\Delta(H)}(H)}(V(H_i))| + r - \chi_{\Delta(H)}(H) - 1 = r - 1 < |V(H)|$. Therefore, for each $i \in \{1, 2, \ldots, n\}$, in H_i,

there exist vertices x_i and y_i such that $c_{r-1}(x_i) = c_{r-1}(y_i)$. Modify the onto mapping $c_{r-1} : V(G \odot H) \to S$ as follows:
• If $S \setminus c_{r-1}(V(H_i)) \neq \emptyset$, then take any $\alpha \in S \setminus c_{r-1}(V(H_i))$ and assign α to y_i.
• If $S \setminus c_{r-1}(V(H_i)) = \emptyset$, then assign a new color β to y_i.
Call this modified coloring as c_r.
Let $i \in \{1, 2, \ldots, n\}$. Then $|c_r(N_{G\odot H}(v_i))| \geq |c_r(V(H_i))| = |c_{r-1}(V(H_i))| + 1 = r$. Since $r \leq |V(H)|$, we have $\min\{r, d_{G\odot H}(v_i)\} = \min\{r,\ d_G(v_i) + |V(H)|\} = r$. Hence, $|c_r(N_{G\odot H}(v_i))| \geq \min\{r, d_{G\odot H}(v_i)\}$. Also, let $u \in V(H_i)$. Then $|c_r(N_{G\odot H}(u))| \geq |\{c_r(v_i)\}| + |c_r(N_{H_i}(u))| = 1 + d_H(u) = \min\{r, 1 + d_H(u)\} = \min\{r, d_{G\odot H}(u)\}$, since $1 + d_H(u) \leq 1 + \Delta(H) < \Delta(H) + 2 \leq r$.
Hence, c_r is an r-dynamic k-coloring, where $k = \max\{\chi(G), r + 1\}$.
Proof of 6. Let c be an arbitrary r-dynamic $\chi_r(G \odot H)$-coloring of $G \odot H$. Then $|c(N_G(v_i))| + |V(H)| \geq |c(N_{G\odot H}(v_i))| \geq \min\{r, d_{G\odot H}(v_i)\} = \min\{r, d_G(v_i) + |V(H)|\}$ $= |V(H)| + \min\{r - |V(H)|, d_G(v_i)\}$ implies $|c(N_G(v_i))| \geq \min\{r - |V(H)|, d_G(v_i)\}$. So, c yields an $(r - |V(H)|)$-dynamic $\chi_r(G \odot H)$-coloring for G. Hence, $\chi_r(G \odot H)$ $\geq \chi_{r-|V(H)|}(G)$. This together with $\chi_r(G \odot H) \geq r + 1$ imply that $\chi_r(G \odot H) \geq \max\{\chi_{r-|V(H)|}(G), r + 1\}$.
Let $r = |V(H)| + s$, $1 \leq s \leq \Delta(G)$. Let $c' : V(G) \to \{1, 2, \ldots, \chi_s(G)\}$ be an s-dynamic $\chi_s(G)$-coloring of G. For $i \in \{1, 2, \ldots, n\}$, let $A_i = c'(N_G(v_i)) = \{\alpha_1, \alpha_2, \ldots, \alpha_{d_G(v_i)}\}$ if $\min\{s, d_G(v_i)\} = d_G(v_i)$ and $A_i = \{\alpha_1, \alpha_2, \ldots, \alpha_s\} \subseteq c'(N_G(v_i))$ if $\min\{s, d_G(v_i)\} = s$, and let $B_i = A_i \cup \{c'(v_i)\}$. Let $k = \max\{\chi_s(G), r + 1\}$. Define $c_r : V(G \odot H) \to \{1, 2, \ldots, k\}$ as follows: For $i \in \{1, 2, \ldots, n\}$, $c_r(v_i) = c'(v_i)$ and assign colors to the vertices of H_i so that c_r restricted to $V(H_i)$ is a one-to-one function from $V(H_i)$ to $\{1, 2, \ldots, k\} \setminus B_i$.
Let $i \in \{1, 2, \ldots, n\}$. Then $|c_r(N_{G\odot H}(v_i))| \geq \min\{s, d_G(v_i)\} + |V(H)| = \min\{s + |V(H)|, d_G(v_i) + |V(H)|\} = \min\{r, d_{G\odot H}(v_i)\}$. Also, let $u \in V(H_i)$. Then $|c_r(N_{G\odot H}(u))| = 1 + d_H(u) = \min\{r, 1 + d_H(u)\} = \min\{r, d_{G\odot H}(u)\}$. Hence, c_r is an r-dynamic k-coloring of $G \odot H$, and so $\chi_r(G \odot H) \leq k$.
This completes the proof of 6 and also the proof of the theorem. ■

In Section 3, we deduce the r-dynamic chromatic number of the corona products $(G_1 \vee K_1) \odot H$ and $G \odot (H_1 \vee K_1)$.

3. Corollaries

It is easy to observe that, for any graph F, if $E(F) \neq \emptyset$, then

$$\chi_s(F \vee K_1) = \begin{cases} \chi(F) + 1 & \text{for } s \in \{1,\ 2\}, \\ \chi_{s-1}(F) + 1 & \text{for } 3 \leq s \leq \Delta(F) + 1, \\ \chi_{\Delta(F)}(F) + 1 & \text{for } \Delta(F) + 2 \leq s \leq \chi_{\Delta(F)}(F), \\ s + 1 & \text{for } \chi_{\Delta(F)}(F) + 1 \leq s \leq |V(F)|; \end{cases}$$

if $E(F)=\emptyset$, then

$$\chi_s(F\vee K_1) \;=\; s+1 \text{ for } 1\le s\le |V(F)|.$$

Hence, from Theorem 1, we have the following corollaries 1, 2, 3 and 4.

Corollary 1. *Let $G=G_1\vee K_1$ and $E(G_1)\neq\emptyset$.*
1. $\chi_2(G\odot H)=\chi_1(G\odot H)=1+\max\{\chi(G_1),\chi(H)\}$.
2. *For* $3\le r\le \Delta(H)+1$,
$\chi_r(G\odot H)=1+\max\{\chi(G_1),\chi_{r-1}(H)\}$.
3. *For* $\Delta(H)+2\le r\le \chi_{\Delta(H)}(H)$,
$\chi_r(G\odot H)=1+\max\{\chi(G_1),\chi_{\Delta(H)}(H)\}$.
4. *For* $\chi_{\Delta(H)}(H)+1\le r\le |V(H)|+2$,
$\chi_r(G\odot H)=1+\max\{\chi(G_1),r\}$.
5. *For* $|V(H)|+3\le r\le |V(H)|+\Delta(G_1)+1$,
$\chi_r(G\odot H)=1+\max\{\chi_{r-|V(H)|-1}(G_1),r\}$.
6. *For* $|V(H)|+\Delta(G_1)+2\le r\le |V(H)|+\chi_{\Delta(G_1)}(G_1)$,
$\chi_r(G\odot H)=1+\max\{\chi_{\Delta(G_1)}(G_1),r\}$.
7. *For* $|V(H)|+\chi_{\Delta(G_1)}(G_1)+1\le r\le |V(H)|+|V(G_1)|$,
$\chi_r(G\odot H)=\max\{r-|V(H)|+1,r+1\}=r+1$.

Corollary 2. *Let $H=H_1\vee K_1$ and $E(H_1)\neq\emptyset$.*
1. *For* $r\in\{1,2,3\}$, $\chi_r(G\odot H)=\max\{\chi(G),\chi(H_1)+2\}$.
2. *For* $4\le r\le\Delta(H_1)+2$,
$\chi_r(G\odot H)=\max\{\chi(G),\chi_{r-2}(H_1)+2\}$.
3. *For* $\Delta(H_1)+3\le r\le \chi_{\Delta(H_1)}(H_1)+1$,
$\chi_r(G\odot H)=\max\{\chi(G),\chi_{\Delta(H_1)}(H_1)+2\}$.
4. *For* $\chi_{\Delta(H_1)}(H_1)+2\le r\le |V(H_1)|+2$,
$\chi_r(G\odot H)=\max\{\chi(G),r+1\}$.
5. *For* $|V(H_1)|+3\le r\le |V(H_1)|+\Delta(G)+1$,
$\chi_r(G\odot H)=\max\{\chi_{r-|V(H_1)|-1}(G),r+1\}$.

Let $G=G_1\vee K_1$ and $E(G_1)=\emptyset$. Then, G is the star $S_q=K_{1,q}$ with $q+1$ vertices, where $q=|V(G_1)|$. For $q=1$, $S_1=K_2$. For $q\ge 2$, $\Delta(S_q)=q$ and, for $1\le s\le q$, $\chi_s(S_q)=s+1$.

Corollary 3. *Let $n\ge 1$.*
1. $\chi_1(S_n\odot H)=1+\chi(H)$.
2. $\chi_2(S_n\odot H)=\begin{cases}3 & \text{if } E(H)=\emptyset,\\ \chi_1(S_n\odot H) & \text{otherwise.}\end{cases}$
3. *For* $3\le r\le\Delta(H)+1$, $\chi_r(S_n\odot H)=1+\chi_{r-1}(H)$.
4. *For* $\Delta(H)+2\le r\le\chi_{\Delta(H)}(H)$, $\chi_r(S_n\odot H)=1+\chi_{\Delta(H)}(H)$.
5. *For* $\chi_{\Delta(H)}(H)+1\le r\le |V(H)|+n$, $\chi_r(S_n\odot H)=r+1$.

Corollary 4. *Let $m\ge 1$.*
1. $\chi_1(G\odot S_m)=\max\{\chi(G),3\}$.
2. *For* $2\le r\le m+2$, $\chi_r(G\odot S_m)=\max\{\chi(G),r+1\}$.
3. *For* $m+3\le r\le m+1+\Delta(G)$, $\chi_r(G\odot S_m)=\max\{\chi_{r-m-1}(G),r+1\}$.

4. Associativity

In this section, we obtain the r-dynamic chromatic number of the corona product of three graphs.

Let G_1, G_2 and G_3 be graphs with n_1, n_2 and n_3 vertices, respectively. Then the graphs $G_1 \odot G_2$, $G_2 \odot G_3$, $(G_1 \odot G_2) \odot G_3$ and $G_1 \odot (G_2 \odot G_3)$, respectively, have $n_1(1+n_2)$, $n_2(1+n_3)$, $n_1(1+n_2)(1+n_3)$ and $n_1(1+n_2(1+n_3))$ vertices. It follows that, we consider at-least two different graphs, then $(G_1 \odot G_2) \odot G_3$ and $G_1 \odot (G_2 \odot G_3)$ are non-isomorphic and so the operation $\odot$ is not associative. Suppose that $G_1 \equiv G_2 \equiv G_3$, then the operation $\odot$ is associative.

From Theorem 1, we have the following corollaries.

Corollary 5. *Let* $G' = (G_1 \odot G_2) \odot G_3$.

1. $\chi_1(G') = \max\{\chi(G_1), \chi(G_2)+1, \chi(G_3)+1\}$.
2. $\chi_2(G') = \begin{cases} 3 & \text{if } \chi(G_1) \leq 2,\ E(G_2) = \emptyset \text{ and } E(G_3) = \emptyset, \\ \chi_1(G') & \text{otherwise.} \end{cases}$
3. *For* $3 \leq r \leq \Delta(G_3)+1$,
$$\chi_r(G') = \max\{\chi(G_1), \chi(G_2)+1, \chi_{r-1}(G_3)+1\}.$$
4. *For* $\Delta(G_3)+2 \leq r \leq \chi_{\Delta(G_3)}(G_3)$,
$$\chi_r(G') = \max\{\chi(G_1), \chi(G_2)+1, \chi_{\Delta(G_3)}(G_3)+1\}.$$
5. *For* $\chi_{\Delta(G_3)}(G_3)+1 \leq r \leq |V(G_3)|+1$,
$$\chi_r(G') = \max\{\chi(G_1), \chi(G_2)+1, r+1\}.$$
6. $\chi_{|V(G_3)|+2}(G')$
$$= \begin{cases} |V(G_3)|+3 & \text{if } \chi(G_1) = 2 \text{ and } E(G_2) = \emptyset, \\ \max\{\chi(G_1), \chi(G_2)+1, |V(G_3)|+3\} & \text{otherwise.} \end{cases}$$
7. *For* $|V(G_3)|+3 \leq r \leq |V(G_3)|+\Delta(G_2)+1$,
$$\chi_r(G') = \max\{\chi(G_1), \chi_{r-|V(G_3)|-1}(G_2)+1, r+1\}.$$
8. *For* $|V(G_3)|+\Delta(G_2)+2 \leq r \leq |V(G_3)|+\chi_{\Delta(G_2)}(G_2)$,
$$\chi_r(G') = \max\{\chi(G_1), \chi_{\Delta(G_2)}(G_2)+1, r+1\}.$$
9. *For* $|V(G_3)|+\chi_{\Delta(G_2)}(G_2)+1 \leq r \leq |V(G_3)|+|V(G_2)|$,
$$\chi_r(G') = \max\{\chi(G_1), r+1\}.$$
10. *For* $|V(G_3)|+|V(G_2)|+1 \leq r \leq |V(G_3)|+|V(G_2)|+\Delta(G_1)$,
$$\chi_r(G') = \max\{\chi_{r-|V(G_3)|-|V(G_2)|}(G_1), r+1\}.$$

Corollary 6. *Let* $H' = G_1 \odot (G_2 \odot G_3)$.

1. $\chi_2(H') = \chi_1(H') = \max\{\chi(G_1), \chi(G_2)+1, \chi(G_3)+2\}$.
2. $\chi_3(H')$
$$= \begin{cases} \max\{\chi(G_1), 4\} & \text{if } \chi(G_2) = 2 \text{ and } E(G_3) = \emptyset, \\ \max\{\chi(G_1), \chi(G_2)+1, \chi(G_3)+2\} & \text{otherwise.} \end{cases}$$
3. *For* $4 \leq r \leq \Delta(G_3)+2$,
$$\chi_r(H') = \max\{\chi(G_1), \chi(G_2)+1, \chi_{r-2}(G_3)+2\}.$$
4. *For* $\Delta(G_3)+3 \leq r \leq \chi_{\Delta(G_3)}(G_3)+1$,
$$\chi_r(H') = \max\{\chi(G_1), \chi(G_2)+1, \chi_{\Delta(G_3)}(G_3)+2\}.$$
5. *For* $\chi_{\Delta(G_3)}(G_3)+2 \leq r \leq |V(G_3)|+1$,
$$\chi_r(H') = \max\{\chi(G_1), \chi(G_2)+1, r+1\}.$$
6. *For* $|V(G_3)|+2 \leq r \leq |V(G_3)|+\Delta(G_2)+1$,
$$\chi_r(H') = \max\{\chi(G_1), \chi_{r-1-|V(G_3)|}(G_2)+1, r+1\}.$$

7. For $|V(G_3)| + \Delta(G_2) + 2 \leq r \leq \max\{\chi_{\Delta(G_2)}(G_2), |V(G_3)| + \Delta(G_2) + 1\}$,
$\chi_r(H') = \max\{\chi(G_1), \chi_{\Delta(G_2)}(G_2) + 1, |V(G_3)| + \Delta(G_2) + 2\}$.
8. For $\max\{\chi_{\Delta(G_2)}(G_2), |V(G_3)| + \Delta(G_2) + 1\} + 1 \leq r \leq (1 + |V(G_3)|)|V(G_2)|$,
$\chi_r(H') = \max\{\chi(G_1), r + 1\}$.
9. For $(1 + |V(G_3)|)|V(G_2)| + 1 \leq r \leq (1 + |V(G_3)|)|V(G_2)| + \Delta(G_1)$,
$\chi_r(H') = \max\{\chi_{r-(1+|V(G_3)|)|V(G_2)|}(G_1), r + 1\}$.

Conflict of Interest: The authors declare that they have no conflict of interest.

Data Availability: Data sharing is not applicable to this article as no datasets were generated or analyzed during the current study.

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