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Research Article

Some results on the strongly annihilator ideal graph of a lattice

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Abstract: For a lattice L, the strongly annihilator ideal graph of L is denoted by $SAnnIG(L)$. It is a graph with the vertex set, which consists of all ideals in L that have nontrivial annihilators such that any two distinct vertices I and J are adjacent in $SAnnIG(L)$ if and only if the annihilator of I contains a nonzero element of J and the annihilator of J contains a nonzero element of I. In this paper, we determine the radius, circumference, and domination number of $SAnnIG(L)$. We obtain necessary and sufficient conditions for $SAnnIG(L)$ to be in the class of paths, cycles, unicyclic, triangle-free, trees, complete multipartite, split or claw-free graphs. Among other results, we study the affinity between the strongly annihilator ideal and the annihilator ideal graph of a lattice.

Keywords: lattice, ideal, domination number and strongly annihilator ideal graph.

AMS Subject classification: 06D50, 06B99

1. Introduction

The concept of zero divisor graphs from algebraic structures is initiated by Beck in [7]. For a commutative ring R, he let all the elements of R as the vertex set such that any two distinct vertices x and y are adjacent if and only if $xy = 0$. In the said work, his complete focus was on coloring of the graph. Later, Anderson and Livingston studied the zero divisor graph of a commutative ring R in [5]. They considered the set of all non-zero zero-divisors in R as the vertex set and studied various graph-theoretic properties. For a lattice L with bottom element 0, the zero divisor graph of L is introduced by Estaji *et al*. in [9], denoted by $ZG(L)$. The $ZG(L)$ is an undirected

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graph with vertex set as $Z^*(L) = \{a \in L \mid a \neq 0,\ \exists\, b \neq 0$ such that $a \wedge b = 0\}$ and for any $u, v \in Z^*(L)$, $u - v$ is an edge in $ZG(L)$ if and only if $u \wedge v = 0$. In [14], Kulal *et al*. studied the annihilator ideal graph of a lattice L, denoted by $AnnIG(L)$. It is a graph with the vertex set $N(L) = \{I$ is a non trivial ideal in $L \mid Ann(I) \neq \{0\}\}$ and two distinct vertices I and J are adjacent if and only if $I \cap Ann(J) \neq \{0\}$ or $J \cap Ann(I) \neq \{0\}$. To get more work related to the zero divisor graphs from algebraic structures, see [1–4, 6, 8, 11, 13, 15].

In [16], Tohidi *et al*. studied the strongly annihilating-ideal graph of a commutative ring R with the vertex set $A(R)^* = A(R) \setminus \{0\}$, where $A(R)$ is the set consists of all ideals with nonzero annihilator and any two distinct vertices I and J are adjacent if and only if $I \cap Ann(J) \neq 0$ and $J \cap Ann(I) \neq 0$. Motivated by this work, in [12], we have defined the strongly annihilator ideal graph of a lattice L, denoted by $SAnnIG(L)$. The vertex set of $SAnnIG(L)$ is a set consists of all ideals in L whose annihilator is nontrivial and any two distinct vertices I and J are adjacent if and only if $I \cap Ann(J) \neq 0$ and $J \cap Ann(I) \neq 0$.

Throughout this paper, $L = \langle L, \wedge, \vee \rangle$ is an atomic lattice with the least element 0 and greatest element 1. A sublattice I of L is said to be an ideal of L if $u \wedge i \in I$ for all $u \in L$ and $i \in I$. For two elements $u, v \in L$ we have $v < u$, if $v \wedge u = v$ and $v \vee u = u$. An element $u \in L$ is said to be least element if $u < v$ for every element $v \in L \setminus \{u\}$. An element $a \in L$ is said to be an atom if $0 < a$ and there is no u such that $0 < u < a$, where 0 is the least element in L. Dually, we have the concept of coatom. For any $u \in L$, the set $(u] = \{v \in L \mid v \leq u\}$ is the principal ideal in L generated by u and $Ann(u)$=$\{v \in L \mid u \wedge v = 0\}$ is the annihilator of u. Denote the set $[u)_z = \{v \in L \mid v \geq u$ and $v \in Z^*(L)\}$. We denote the set of all atoms in L by $A(L)$. Any two elements u and v in L are said to be incomparable if $u \nleq v$ and $v \nleq u$, we denote the same by $u||v$. A lattice L is called atomic if, for each element $v \in L \setminus \{0\}$, there exists an atom $a_v \in A(L)$ such that $a_v \leq v$. If $p \in L$, then $\mathcal{B}(p) = \{a_p \in A(L) | a_p \leq p\}$ is defined as the base of p in L. For more concepts in lattice theory, we refer [10].

If V is a nonempty set (known as vertices) and E is a set of two-subsets of V (known as edges), then $G = (V, E)$ is known as a graph on the vertex set V and the edge set E. The two-subset $e = \{u, v\} \in E$ is called an edge between u and v, and in this case, we say that u and v are adjacent in G and this edge is denoted by $u - v$. If there is a path joining any two distinct vertices of G, then the graph G is said to be connected; otherwise, we say that G is disconnected. For two vertices, a and $b \in V(G)$, the length of a shortest path from a to b is denoted by $d(a, b)$, called the distance between a and b, and $diam(G)$=$sup\{d(a, b) \mid a, b \in V(G)\}$ is called the diameter of G. The length of a shortest cycle in G is called the girth $gr(G)$ of G. If there are no cycles in G, we use $gr(G) = \infty$. If the set of vertices in a graph is empty, it is said to be empty. A connected acyclic graph is called a tree. A graph is planar if it can be drawn in a plane so that edges intersect at only vertices. For any vertex u, $nbd(u) = \{v \in V(G) \mid u$ and v are adjacent in $G\}$ is called the neighbourhood of u. The degree of a vertex u in a graph, denoted by $deg(u)$, is the number of edges that are incident to vertex u. In other words, it is the count of vertices that are adjacent

to u. If the degree of all vertices in a graph G are the same, then G is called a regular graph. The circumference of a graph G, denoted by $\mathcal{C}(G)$, is defined as the length of longest cycle in G. The eccentricity $e(v)$ of a vertex v in a connected graph G is defined as the maximum distance from vertex v to any other vertex u in G. The radius of a graph G, denoted by $rad(G)$, is defined as the minimum eccentricity of G; that is, $rad(G) = min\{e(v) \mid v \in V(G)\}$. If every pair of distinct vertices in G is connected by an edge, then G is called a complete graph, and the complete graph with n vertices is denoted by K_n. If a graph G is the union of two disjoint sets of vertices, one of size m and the other of size n such that every vertex in the first set is connected to every vertex in the other set, and there are no edges within each set, then G is called a complete bipartite graph, denoted by $K_{m,n}$. The star graph is a complete bipartite graph $K_{1,n}$. For more concepts in graph theory, refer [17].
We recall the following results from [12].

Lemma 1 ([12], Lemma 2.1). *Let L be a lattice and $I, J \in N(L)$. Then I and J are adjacent in $SAnnIG(L)$ if and only if $Ann(I) \nsubseteq Ann(J)$ and $Ann(J) \nsubseteq Ann(I)$.*

Lemma 2 ([12], Lemma 2.3). *Let L be a lattice and $I, J \in N(L)$. Then I and J are adjacent in $SAnnIG(L)$ if and only if $A(I) \nsubseteq A(J)$ and $A(J) \nsubseteq A(I)$.*

Theorem 1 ([12], Theorem 3.2). *Let L be a lattice. Then $SAnnIG(L)$ is a complete bipartite or a star graph if and only if $|A(L)| = 2$.*

Theorem 2 ([12], Theorem 3.1 (iii)). *Let L be a lattice. If $S = \{(a] \mid a \in A(L)\}$, then S forms an induced complete subgraph in $SAnnIG(L)$.*

Corollary 1 ([12], Corollary 3.3). *Let L be a lattice. Then $SAnnIG(L)$ is connected and $diam(SAnnIG(L)) \leq 2$.*

Theorem 3 ([12], Theorem 4.1(i)). *Let L be a lattice. Then $SAnnIG(L)$ is complete if and only if $N(L) = \{(a] \mid a \in A(L)\}$.*

In the section 2 of this paper, we determine the radius, circumference, and domination number of $SAnnIG(L)$. We obtain necessary and sufficient conditions for $SAnnIG(L)$ to be in the class of paths, cycles, unicyclic, triangle-free, trees, complete multipartite, split or claw-free graphs. In the section 3, we study the relationship between strongly annihilator ideal and the annihilator ideal graph of a lattice.

2. Characterizations of $SAnnIG(L)$

In the following theorem, we determine a necessary and sufficient condition such that $rad(SAnnIG(L)) = 1$.

Theorem 4. *Let L be a lattice. Then $rad(SAnnIG(L)) = 1$ if and only if $|[a)_z| = 1$ for some atom $a \in A(L)$.*

Proof. Let $rad(SAnnIG(L)) = 1$. On the contrary, suppose that $|[a)_z| > 1$ for all atoms $a \in A(L)$. If $(x] \in N(L)$ with $x \notin A(L)$, then there exists $a \in A(L)$ such that $a < x$. Therefore, $A((a]) \subseteq A((x])$ and hence by Lemma 2, vertex $(a]$ is not adjacent to $(x]$. By Corollary 1, we have $d((x], (a]) = 2$. Similarly, for every $(a] \in N(L)$ with $a \in A(L)$, there exists $(y] \in N(L)$ with $y > a$ such that $d((a], (y]) = 2$. Therefore, eccentricity of every vertex in $SAnnIG(L)$ is two and hence $rad(SAnnIG(L)) = 2$, a contradiction. Conversely, suppose $|[a)_z| = 1$ for some atom $a \in A(L)$. Then, for every $(x] \in N(L) \setminus \{(a]\}$, we have, $A((a]) \nsubseteq A((x])$ and $A((x]) \nsubseteq A((a])$. Hence, by Lemma 2, vertices $(a]$ and $(x]$ are adjacent. That is, $d((a], (x]) = 1$. Therefore, for every $(x] \in N(L) \setminus \{(a]\}$, we have, $e((x]) = 1$. Hence, $min\{e(I) \mid I \in N(L)\} = 1$. Thus, $rad(SAnnIG(L)) = 1$. □

By Corollary 1 and Theorem 4, we have the following immediate consequence.

Corollary 2. *Let L be a lattice. Then $rad(SAnnIG(L)) = 2$ if and only if $|[a)_z| > 1$ for all atoms $a \in A(L)$.*

Example 1. Let L be a lattice depicted in Figure 1. Observe that, $A(L) = \{a_1, a_2, a_3\}$ and $|[a_i)_z| > 1$ for every $i = 1, 2, 3$. Therefore, by Corollary 2, $rad(SAnnIG(L)) = 2$.

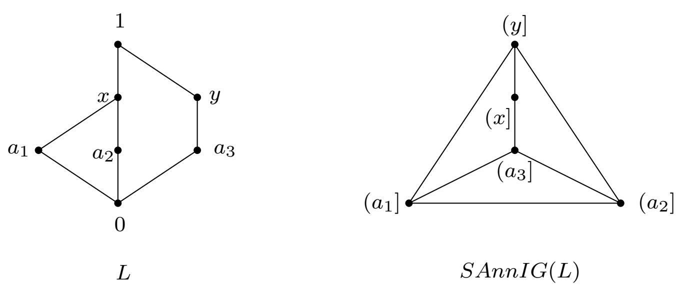

Figure 1. **Some lattice L for which $rad(SAnnIG(L)) = 2$.**

In the following result, we show that the circumference of $SAnnIG(L)$ does not exceed four.

Theorem 5. *Let L be a lattice. Then $\mathcal{C}(SAnnIG(L)) \leq 4$.*

Proof. Assume that there is an induced cycle $I_1 - I_2 - \cdots - I_n - I_1$ in $SAnnIG(L)$ with length $n \geq 5$. Since there is no edge between I_1 and I_3, by Lemma 2, suppose that $A(I_1) \subseteq A(I_3)$. If $A(I_4) \subseteq A(I_1)$, then $A(I_4) \subseteq A(I_3)$ and hence I_3 is not adjacent to

I_4, which is not true. Therefore, $A(I_1) \subseteq A(I_4)$. Now, we show that $A(I_2) \subseteq A(I_n)$. As I_2 is not adjacent to I_4, by Lemma 2, we have $A(I_2) \subseteq A(I_4)$ or $A(I_4) \subseteq A(I_2)$. If $A(I_4) \subseteq A(I_2)$, then as $A(I_1) \subseteq A(I_4)$, we have $A(I_1) \subseteq A(I_2)$ and hence I_1 is not adjacent to I_2, which is incorrect. Therefore $A(I_2) \subseteq A(I_4)$. This process is therefore carried out for $I_5, \ldots, I_n$. Then, we obtain $A(I_2) \subseteq A(I_5), \ldots, A(I_2) \subseteq A(I_n)$.
Since $A(I_2) \subseteq A(I_5)$, hence from I_3, we do the same argument on $I_5, \ldots, I_n, I_1$. Then, we get $A(I_3) \subseteq A(I_5), \ldots, A(I_3) \subseteq A(I_n), A(I_3) \subseteq A(I_1)$. Therefore $A(I_1) \subseteq A(I_3)$ and $A(I_3) \subseteq A(I_1)$ implies that $A(I_1) = A(I_3)$. But, since $A(I_3) \nsubseteq A(I_4)$ and $A(I_4) \nsubseteq A(I_3)$, we have $A(I_1) \nsubseteq A(I_4)$ and $A(I_4) \nsubseteq A(I_1)$, and hence I_1 is adjacent to I_4, a contradiction. Similarly, if $A(I_3) \subseteq A(I_1)$, we have a contradiction. Thus $n \leq 4$. That is $\mathcal{C}(SAnnIG(L)) \leq 4$. □

A subset D of the vertex set V of a graph G is called a dominating set if every vertex in the graph that is not in D is adjacent to at least one vertex in D. The domination number of a graph G, denoted by $\gamma(G)$, is defined as the size of the smallest dominating set of G. In the following result, we study the domination number of $SAnnIG(L)$.

Theorem 6. *Let L be a lattice. Then, the following statements hold.*

(1) $\gamma(SAnnIG(L)) = 1$ *if and only if* $|[a)_z| = 1$ *for some atom* $a \in A(L)$.

(2) $\gamma(SAnnIG(L)) \leq 2$.

Proof. (1) For a graph $G = (V, E)$ with $|V| \geq 2$, it is well known that $\gamma(G) = 1$ if and only if $rad(G) = 1$. Therefore, by Theorem 4, $\gamma(SAnnIG(L)) = 1$ if and only if $|[a)_z| = 1$ for some atom $a \in A(L)$.
(2) Let $b \in Z^*(L)$ be the element such that $|\mathcal{B}(a)| \leq |\mathcal{B}(b)|$, for all $a \in Z^*(L)$ and $\mathcal{B}(b) = S$. Consider $B = \{I \in N(L) \mid A(I) \subseteq S\}$. Then $B^c = \{J \in N(L) \mid A(J) \nsubseteq S\}$. Hence $N(L) = B \cup B^c$. Then there exists an ideal $(a_i]$ with $a_i \in A(L) \setminus S$. We show that $D = \{(b], (a_i]\}$ is a dominating set. Note that, $A((a_i]) \nsubseteq A(I)$ and $A(I) \nsubseteq A((a_i])$ for every $I \in B$. Therefore, $(a_i] - I$ is an edge for every $I \in B$. Also, observe that $A(J) \nsubseteq A((b])$ and $A((b]) \nsubseteq A(J)$ for every $J \in B^c$. Therefore, $(b] - J$ is an edge for every $J \in B^c$. Therefore, D is a dominating set. Hence, we have, $\gamma(SAnnIG(L)) \leq 2$. □

Example 2. Let L be a lattice depicted in Figure 2. Observe that, $A(L) = \{a_1, a_2, a_3\}$ and $|[a_3)_z| = 1$. Therefore, by Theorem 6, $\gamma(SAnnIG(L)) = 1$. Note that, $D = \{(a_3]\}$ is a dominating set.

Corollary 3. *Let L be a Boolean lattice such that $|L| = 2^n, n \in \mathbb{N}$ with $n \geq 3$. Then* $\gamma(SAnnIG(L)) = 2$.

Proof. Since $|[a)_z| > 1$ for every atom $a \in A(L)$, by Theorem 6(1), $\gamma(SAnnIG(L)) \neq 1$. By Theorem 6(2), we have, $\gamma(SAnnIG(L)) = 2$. □

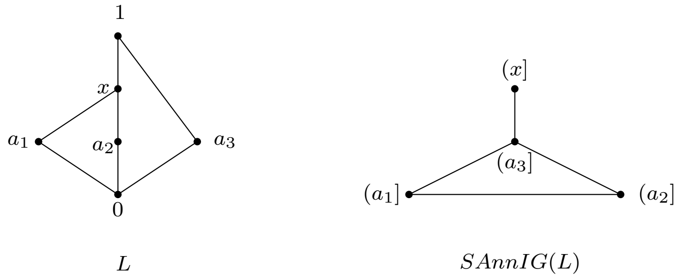

Figure 2. **Some lattice L for which $\gamma(SAnnIG(L)) = 1$.**

In the following result, we give the necessary and sufficient condition for $SAnnIG(L)$ to be a path or cycle.

Theorem 7. *Let L be a lattice. Then, the following statements hold.*

(1) *$SAnnIG(L)$ is a path if and only if $A(L) = \{a, b\}$ with $|[a)_z| \leq 2$ and $|[b)_z| = 1$.*

(2) *$SAnnIG(L)$ is a cycle if and only if any of the following statement holds.*

  (a) $A(L) = \{a, b\}$ with $|[a)_z| = 2$ and $|[b)_z| = 2$.
  (b) $A(L) = Z^*(L)$ with $|A(L)| = 3$.

Proof. (1) Suppose $SAnnIG(L)$ is a path. If $|A(L)| \geq 3$, then the ideals generated by three distinct atoms form a cycle of length three, a contradiction. Therefore $|A(L)| \leq 2$. If $|A(L)| = 1$, then $SAnnIG(L)$ is empty. Thus $|A(L)| = 2$. By Theorem 1, we have, $SAnnIG(L) = K_{|[a)_z|,|[b)_z|}$, where $A(L) = \{a, b\}$. Thus $|[a)_z| \leq 2$ and $|[b)_z| = 1$.
Conversely, suppose $A(L) = \{a, b\}$ such that $|[a)_z| \leq 2$ and $|[b)_z| = 1$. Then by Theorem 1, we have $SAnnIG(L) = K_{1,1}$ or $K_{2,1}$. Thus $SAnnIG(L)$ is a path.
(2) Suppose $SAnnIG(L)$ is a cycle. If $|A(L)| \geq 4$, then by Theorem 3, we have $SAnnIG(L)$ has a subgraph isomorphic to K_4, which is not true. Therefore $|A(L)| \leq 3$. If $|A(L)| = 1$, then $SAnnIG(L)$ is empty, a contradiction. Thus $|A(L)| = 2$ or 3. Suppose $A(L) = \{a, b\}$. Then by Theorem 1, we have, $SAnnIG(L) = K_{|[a)_z|,|[b)_z|}$. Thus $|[a)_z| = 2$ and $|[b)_z| = 2$. Now, suppose $A(L) = \{a_1, a_2, a_3\}$. By Theorem 2, for every distinct i, j and k, we have $(a_j], (a_k] \in nbd((a_i])$ and hence $deg((a_i]) \geq 2$. If there exists $x \in Z^*(L)$ such that $x > a_j$, then there exists $a_i \in A(L)$ such that $a_i \notin (x]$. Then $A((a_i]) \nsubseteq A((x])$ and $A((x]) \nsubseteq A((a_i])$. Hence $(a_i] - (x]$ is an edge. Therefore $\{(a_j], (a_k], (x]\} \subseteq nbd((a_i])$ and hence $deg((a_i]) \geq 3$, a contradiction. Thus $Z^*(L) = A(L) = \{a_1, a_2, a_3\}$.
Conversely, suppose statement (a) holds. Then by Theorem 1, we have, $SAnnIG(L) = K_{2,2}$ and hence the result follows. Now, suppose statement (b) holds. Then by Theorem 2, we have, $SAnnIG(L) = K_3$ and hence $SAnnIG(L)$ is a cycle. □

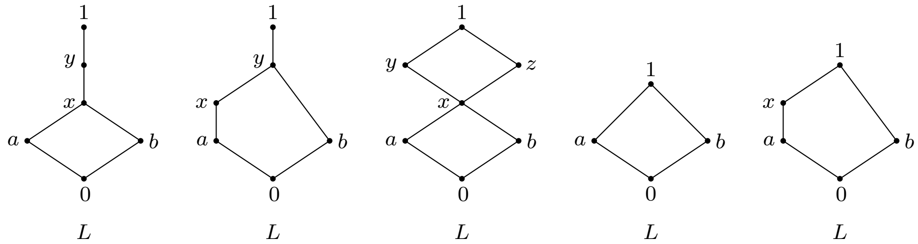

Figure 3. **Some lattices L for which $SAnnIG(L)$ is a path.**

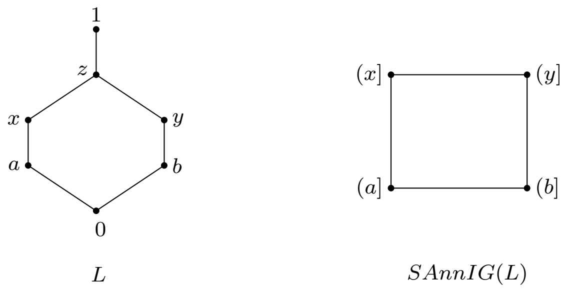

Figure 4. **Some lattice L for which $SAnnIG(L)$ is a cycle.**

A graph G is said to be unicyclic if there is only one cycle in G. Let $A_k = \{I \in N(L) | |A(I)| = k\}$. In the following result, we give the necessary and sufficient conditions for $SAnnIG(L)$ to be unicyclic.

Theorem 8. *Let L be a lattice. Then $SAnnIG(L)$ is unicyclic if and only if any of the following statement holds.*

(1) $A(L) = \{a, b\}$ *with* $|[a)_z| = 2$ *and* $|[b)_z| = 2$.

(2) $A(L) = \{a, b, c\}$ *with* $|A_1| = 3$ *and no ideals* $I, J \in A_2$ *exist such that* $A(I) \neq A(J)$.

Proof. Suppose $SAnnIG(L)$ is unicyclic. By Theorem 2, we have, $|A(L)| \leq 3$. If $|A(L)| = 1$, then $SAnnIG(L)$ is empty and hence it has no cycle, a contradiction. Suppose $A(L) = \{a, b\}$. Then $SAnnIG(L) = K_{|[a)_z|,|[b)_z|}$ and hence $|[a)_z| = 2$ and $|[b)_z| = 2$. Suppose $A(L) = \{a, b, c\}$. If $|A_1| \geq 4$, then there exists $(d] \in A_1$ with $d > a$. Since $A((d]) = \{a\}$, we have, $A((d]) \not\subseteq A((b])$ and $A((b]) \not\subseteq A((d])$. Also, $A((d]) \not\subseteq A((c])$ and $A((c]) \not\subseteq A((d])$. Therefore $(d] - (b]$ and $(d] - (c]$ are the edges. Hence $(d] - (b] - (c] - (d]$ is a cycle different from $(a] - (b] - (c] - (a]$. This contradicts the assumption that $SAnnIG(L)$ is unicyclic. Thus $|A_1| = 3$. Now, suppose there exist ideals I and J such that $A(I) = \{a, b\}$ and $A(J) = \{a, c\}$. Then $(a] - (b] - (c] - (a]$ and $I - (c] - (b] - J - I$ are the two distinct cycles in $SAnnIG(L)$ and hence $SAnnIG(L)$

is not unicyclic, a contradiction. Thus, there does not exist $I, J \in A_2$ such that $A(I) \neq A(J)$.
Conversely, suppose the statement (1) holds. Then $SAnnIG(L) = K_{2,2}$ and hence $SAnnIG(L)$ is unicyclic. Now, suppose the statement (2) holds. Then $(a] - (b] - (c] - (a]$ is the only cycle formed by elements in A_1. If $I \in A_3$ then $I \notin N(L)$ and hence $N(L) = A_1 \cup A_2$. Since no any two elements in A_2 are adjacent to each other, they never form a cycle of length 3 or 4. Also, since any element of A_1 is adjacent to at most one element of A_2 and vice versa, and no two elements of A_2 are adjacent, we have elements in A_1 and A_2 do not form a cycle of length 3 or 4. Thus, by Theorem 5, we have $SAnnIG(L)$ is unicyclic. □

Example 3. Consider a lattice L depicted in Figure 5. We have, $A(L) = \{a, b, c\}$, $|A_1| = |\{(a], (b], (c]\}| = 3$ and $A_2 = \{(x], (y], (z]\}$. Observe that, $A(I) = \{a, b\}$ for every $I \in A_2$. Thus, by Theorem 8(2), $SAnnIG(L)$ is unicyclic.

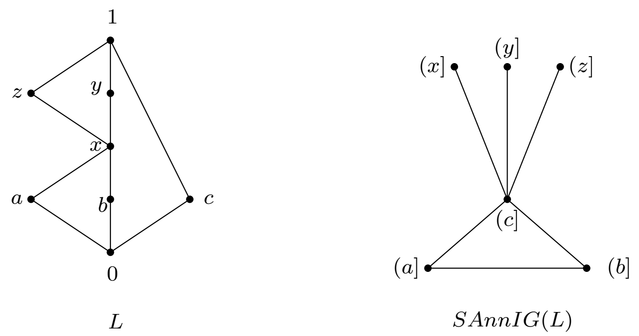

Figure 5. **Some lattice L for which $SAnnIG(L)$ is unicyclic.**

The following corollary is an immediate consequence of Theorem 7(2) and 8.

Corollary 4. *Let L be a lattice such that $|A(L)| = 2$. Then $SAnnIG(L)$ is a cycle if and only if $SAnnIG(L)$ is unicyclic.*

An undirected graph is a triangle-free graph if none of its three vertices combine to form a triangle with edges. An undirected graph is a tree if there is only one path connecting any two vertices. In the following result, we give the characterization for $SAnnIG(L)$ to be in the class of triangle-free or tree.

Proposition 1. *Let L be a lattice such that $SAnnIG(L)$ is nonempty. Then, the following statements hold.*

(1) *$SAnnIG(L)$ is a triangle-free graph if and only if $|A(L)| = 2$.*

(2) *$SAnnIG(L)$ is a tree if and only if $A(L) = \{a, b\}$ with $|[a)_z| = 1$ or $|[b)_z| = 1$.*

Proof. Follows by Theorem 1 and 2 □

In general, the triangle-free graph is not necessarily a bipartite graph. For example, a cycle C_n with n is odd, $n \geq 5$. But, in the following corollary, we show that it holds in $SAnnIG(L)$,

Corollary 5. *Let L be a lattice such that $SAnnIG(L)$ is nonempty. Then $SAnnIG(L)$ is a triangle-free graph if and only if it is a bipartite graph.*

Proof. By Theorem 1 and Proposition 1(1), it is trivial. □

In the following result, we give a characterization for $SAnnIG(L)$ to be a complete multipartite graph.

Theorem 9. *Let L be a lattice. Then the following statements are equivalent:*

1. *$SAnnIG(L)$ is a complete multipartite graph.*
2. *$|A(I)| = 1$ for each $I \in N(L)$.*
3. *$AnnIG(L) = SAnnIG(L)$ is a complete multipartite graph.*

Proof. (1) ⇒ (2) Assume that $SAnnIG(L)$ is a complete multipartite graph. Hence, there exists a partition $N(L) = \bigcup_{\lambda \in \Lambda} V_\lambda$ such that no two vertices of V_λ are adjacent in $SAnnIG(L)$ for each $\lambda \in \Lambda$ and for any distinct $\lambda, \lambda' \in \Lambda$, any $I \in V_\lambda, J \in V_{\lambda'}$, I and J are adjacent in $SAnnIG(L)$. Let $I \in N(L)$. Suppose that $|A(I)| > 1$. Let $a, b \in A(I)$ with $a \neq b$. Observe that $A((a]) \subset A(I)$. Therefore, $(a]$ and I are not adjacent in $SAnnIG(L)$ by Lemma 2. Therefore, both $(a]$ and I must belong to V_λ for some $\lambda \in \Lambda$. Similarly, as $A((b]) \subset A(I)$, it follows that $(b]$ and I are not adjacent in $SAnnIG(L)$. Hence, $(b]$ must be in V_λ. This is impossible, since $(a]$ and $(b]$ are adjacent in $SAnnIG(L)$. Therefore, $|A(I)| = 1$ for each $I \in N(L)$.
(2) ⇒ (3) Assume that $|A(I)| = 1$ for each $I \in N(L)$. Let $A(L) = \{a_\lambda \mid \lambda \in \Lambda\}$. The relation $\sim$ defined on $N(L)$ by for any $I, J \in N(L), I \sim J$ if and only if $A(I) = A(J)$ is an equivalence relation. For any $\lambda \in \Lambda$, let us denote the equivalence class containing $(a_\lambda]$ determined by $\sim$ by $[(a_\lambda]]$. Since $|A(I)| = 1$ for each $I \in N(L)$, it follows that $\{[(a_\lambda]] \mid \lambda \in \Lambda\}$ is the collection of all equivalence classes determined by $\sim$. Let $\lambda \in \Lambda$. If $I, J \in [(a_\lambda]]$, then $A(I) = A(J)$ and so, I and J are not adjacent in $SAnnIG(L)$. Let $\lambda' \in \Lambda$ be such that $\lambda \neq \lambda'$. Let $I \in [(a_\lambda]]$ and $J \in [(a_{\lambda'}]]$. Then $A(I) = \{a_\lambda\} \nsubseteq A(J) = \{a_{\lambda'}\}$ and similarly $A(J) \nsubseteq A(I)$. Therefore, I and J are adjacent in $SAnnIG(L)$. This shows that $SAnnIG(L)$ is a complete multipartite graph whose parts are $\{[(a_\lambda]] \mid \lambda \in \Lambda\}$. Observe that $V(AnnIG(L)) = N(L) = V(SAnnIG(L))$. By assumption, $|A(I)| = 1$ for each $I \in N(L)$. Let $I, J \in N(L)$ be distinct. Notice that I and J are adjacent in $SAnnIG(L)$ if and only if $A(I) \nsubseteq A(J)$ and $A(J) \nsubseteq A(I)$ by Lemma 2, if and only if $A(I) \neq A(J)$, if and only if I and J are adjacent in $AnnIG(L)$ by [[14], Lemma 2.2]. Therefore, $AnnIG(L) = SAnnIG(L)$ is a complete multipartite graph.

(3) $\Rightarrow$ (1) This is clear. $\square$

Example 4. Let $L = \{\phi\} \cup \{\{2\}, \{4\}, \{6\}, \dots\} \cup \{\{1,2\}, \{3,4\}, \{5,6\}, \dots\} \cup \mathbb{N}$, where $\mathbb{N}$ = set of all natural numbers. Define a partial order relation on L such that for any two sets $A, B \in L$, we have, $A \le B$ if and only if $A \subseteq B$. Then, $(L, \le)$ is a lattice as shown in Figure 6 in which ϕ is a least element and $\mathbb{N}$ is a greatest element, and $A(L) = \{\{2\}, \{4\}, \{6\}, \dots\}$ and $C(L) = \{\{1,2\}, \{3,4\}, \{5,6\}, \dots\}$, where $C(L)$ be the set containing all coatoms in L. Observe that, $L = \phi \cup A(L) \cup C(L) \cup \mathbb{N}$ and $|A(I)| = 1$ for every $I \in N(L)$. Therefore, by Theorem 9, $SAnnIG(L)$ is a complete multipartite graph.

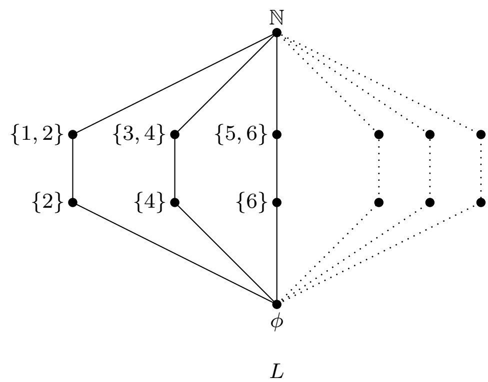

Figure 6. Atomic lattice L with infinite number of atoms satisfying the condition (2) of the statement in Theorem 9.

Example 5. Let L be a lattice as shown in Figure 7. Observe that the vertex $(b]$ is not adjacent to $(e]$ as well as to $(d]$. Hence, vertices $(b]$, $(e]$ and $(d]$ are the members of same partition. Since, vertex $(e]$ is adjacent to $(d]$, we have, $SAnnIG(L)$ is not a complete multipartite graph. Since, $A(I) \neq A(J)$ for every $I, J \in N(L)$, we have, $AnnIG(L)$ is a complete multipartite graph. Note that, there exists ideal $(d] \in N(L)$ such that $|A((d])| > 1$.

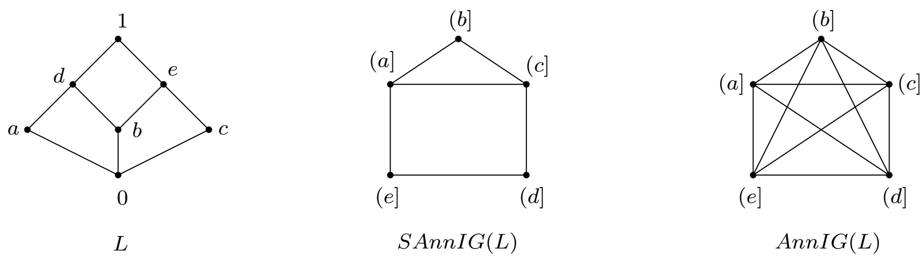

Figure 7. Some lattice L for which $SAnnIG(L)$ is not complete multipartite, while $AnnIG(L)$ is complete multipartite.

Corollary 6. *Let L be a lattice such that $|A(L)| = n$. Then $SAnnIG(L)$ is a complete multipartite graph if and only if $SAnnIG(L)$ is a complete n-partite graph.*

Example 6. Let L be a lattice as shown in Figure 8. Observe that, $|A(L)| = 3$, $|A(I)| = 1$ for every $I \in N(L)$ and $SAnnIG(L) = AnnIG(L)$ is a complete 3-partite graph.

Figure 8. Some lattice L for which $SAnnIG(L) = AnnIG(L)$ is a complete multipartite graph.

A graph with its vertices divided into an independent set and a clique is called a split graph. If L is a lattice such that $A(L) = \{a, b\}$, then $SAnnIG(L) = K_{|[a)_z|,|[b)_z|}$. Hence, if $A(L) = \{a, b\}$, then $SAnnIG(L)$ is a split graph if and only if $|[a)_z| = 1$ or $|[b)_z| = 1$.
Let L be a lattice with $A(L) = \{a_1, a_2, \ldots, a_n\}$. Also, let $S_i = \{I \in N(L) \mid A(I) = \{a_i\}\}$. For distinct i and j, let $S_{i,j} = \{I \in N(L) \mid A(I) = \{a_i, a_j\}\}$. For distinct i, j and k, let $S_{i,j,k} = \{I \in N(L) \mid A(I) = \{a_i, a_j, a_k\}\}$ and so on. In the following result, we give the necessary and sufficient condition for $SAnnIG(L)$ to be a split graph when L is a lattice with $|A(L)| \geq 3$.

Theorem 10. *Let L be a lattice such that $A(L) = \{a_1, a_2, \ldots, a_n\}, n \geq 3$. Then $SAnnIG(L)$ is a split graph if and only if $N(L) = S_1 \cup S_{1,2} \cup S_{1,2,3} \cup \cdots \cup S_{1,2,3,\ldots,n-1} \cup (\bigcup_{a_i \in A(L) \setminus \{a_1\}} (a_i])$.*

Proof. Let $SAnnIG(L)$ be a split graph with Q as a clique and U as an independent set. Let $B_k = \{I \in N(L) \setminus V(Q) \mid |A(I)| = k\}$. Observe that, $N(L) = V(Q) \cup V(U)$, where $V(U) = \bigcup_{i=1}^{n-1} B_i$. If there exist ideals $(a_i], (a_j] \in V(U)$ with $i \neq j$, then $A((a_i]) \nsubseteq A((a_j])$ and $A((a_j]) \nsubseteq A((a_i])$ and hence $(a_i]$ and $(a_j]$ are adjacent vertices, a contradiction. Thus, we have the following two cases.
Case (i) $(a_i] \notin V(U)$ for all $a_i \in A(L)$.
Then, by Theorem 2, we have $(a_i] \in V(Q)$ for all $a_i \in A(L)$. Now, for every $(x] \in N(L)$ with $x \notin A(L)$, if $(x] \in V(Q)$ then for $a_j < x$, we have $A((a_j]) \subseteq A((x])$ and hence $(a_j]$ and $(x]$ are non adjacent vertices, a contradiction. Thus, $V(Q) =$

$\{(a_1], (a_2], \ldots, (a_n]\}$. If $V(U) = \phi$, then $S_{1,2} = \phi = S_{1,2,3} =, \ldots, = S_{1,2,\ldots,n-1}$. Hence, $N(L) = S_1 \cup S_{1,2} \cup S_{1,2,3} \cup \cdots \cup S_{1,2,3,\ldots,n-1} \cup \big(\bigcup\limits_{a_i \in A(L) \setminus \{a_1\}} (a_i] \big)$, where $S_1 = \{(a_1]\}$. Suppose $I_1 \in V(U)$. Then, $I_1 \in B_k$ for some k. If there exists $I_2 \in B_k$ such that $A(I_1) \neq A(I_2)$, then by Lemma 2, vertices I_1 and I_2 are adjacent, which is impossible as $I_2 \in V(U)$. Therefore, for every $I \in B_k$, we have $A(I) = \{a_1, a_2, \ldots, a_k\}$. That is, for every $I \in B_k$, we have $I \in S_{1,2,\ldots,k}$. If there exists $I_3 \in B_{k-1}$, then since $I_3 \in V(U)$, by Lemma 2, $A(I_3) \subset A(I)$ for every $I \in B_k$, and for any $I_4 \in B_{k-1}$ if $A(I_3) \neq A(I_4)$, then by Lemma 2, vertices I_3 and I_4 are adjacent, which is impossible as $I_4 \in V(U)$. Thus, for every $J \in B_{k-1}$, we have $A(J) = \{a_1, a_2, \ldots, a_{k-1}\}$. That is, for every $J \in B_{k-1}$, we have $J \in S_{1,2,\ldots,k-1}$. Continuing in this way, we have, if $K \in B_1$, then $K \in S_1$. Now, if there exists $I_5 \in B_{k+1}$, then since $I_5 \in V(U)$, by Lemma 2, $A(I) \subset A(I_5)$ for every $I \in B_k$, and for any $I_6 \in B_{k+1}$ if $A(I_5) \neq A(I_6)$, then by Lemma 2, vertices I_5 and I_6 are adjacent, which is impossible as $I_6 \in V(U)$. Thus, for every $M \in B_{k+1}$, we have $A(M) = \{a_1, a_2, \ldots, a_k, a_{k+1}\}$. That is, for every $M \in B_{k+1}$, we have $M \in S_{1,2,\ldots,k+1}$. Continuing in this way, we have, if $N \in B_{n-1}$, then $N \in S_{1,2,\ldots,n-1}$. This shows that $V(U) = S_1 \setminus \{(a_1]\} \cup S_{1,2} \cup S_{1,2,3} \cup \cdots \cup S_{1,2,3,\ldots,n-1}$. Therefore, $N(L) = V(Q) \cup V(U) = \{(a_1], (a_2], \ldots, (a_n]\} \cup \{S_1 \setminus \{(a_1]\} \cup S_{1,2} \cup S_{1,2,3} \cup \cdots \cup S_{1,2,3,\ldots,n-1}\} = S_1 \cup S_{1,2} \cup S_{1,2,3} \cup \cdots \cup S_{1,2,3,\ldots,n-1} \cup \big(\bigcup\limits_{a_i \in A(L) \setminus \{a_1\}} (a_i] \big)$.

Case (ii) $(a_i] \in V(U)$ for some $a_i \in A(L)$.
Suppose $(a_1] \in V(U)$. By Theorem 2, for every $2 \leq i \leq n$, we have $(a_i] \notin V(U)$ and hence $(a_i] \in V(Q)$. Also, if there exists $J \in V(Q)$ such that $J \neq (a_i]$, $\forall\ i$ and $a_i \in A(J)$ with $a_i \neq a_1$, then $A((a_i]) \subseteq A(J)$ and hence $(a_i] - J$ is not an edge, which is not true. Thus, $V(Q) = \{I, (a_2], \ldots, (a_n]\}$ with $A(I) = \{a_1\}$ or $V(Q) = \{(a_2], \ldots, (a_n]\}$. Observe that, $(a_1] \in S_1 \cap B_1$. By Lemma 2, for every $J \in B_1$, we have $A(J) = \{a_1\}$. That is, for every $J \in B_1$, we have $J \in S_1$. From B_2 onward, we give the similar proof as in the Case (i) and conclude that $N(L) = S_1 \cup S_{1,2} \cup S_{1,2,3} \cup \cdots \cup S_{1,2,3,\ldots,n-1} \cup \big(\bigcup\limits_{a_i \in A(L) \setminus \{a_1\}} (a_i] \big)$.

Conversely, let $N(L) = S_1 \cup S_{1,2} \cup S_{1,2,3} \cup \cdots \cup S_{1,2,3,\ldots,n-1} \cup \big(\bigcup\limits_{a_i \in A(L) \setminus \{a_1\}} (a_i] \big)$. Then, for any $I, J \in \bigcup\limits_{a_i \in A(L) \setminus \{a_1\}} (a_i]$, we have, $A(I) \nsubseteq A(J)$ and $A(J) \nsubseteq A(I)$ and hence $I - J$ is an edge. Therefore, elements in $\bigcup\limits_{a_i \in A(L) \setminus \{a_1\}} (a_i]$ forms a clique. Now, for any $I, J \in N(L) \setminus \big(\bigcup\limits_{a_i \in A(L) \setminus \{a_1\}} (a_i] \big)$, we have, $A(I) \subseteq A(J)$ or $A(J) \subseteq A(I)$ and hence I and J are non adjacent vertices. Therefore, $N(L) \setminus \big(\bigcup\limits_{a_i \in A(L) \setminus \{a_1\}} (a_i] \big)$ forms an independent set. Thus, $SAnnIG(L)$ is split. □

Example 7. Let L be a lattice as shown in Figure 9. We have, $A(L) = \{a_1, a_2, a_3, a_4\}$, $S_1 = \{(a_1], (a_{1'}], (a_{1''}]\}$, $S_{1,2} = \{(a_{12}], (a_{12'}]\}$, $S_{1,2,3} = \{(a_{123}], (a_{123'}]\}$ and $N(L) = S_1 \cup$

$S_{1,2} \cup S_{1,2,3} \cup \big(\bigcup\limits_{a_i \in A(L)\setminus\{a_1\}} (a_i] \big)$. Observe that, $SAnnIG(L)$ is a split graph.

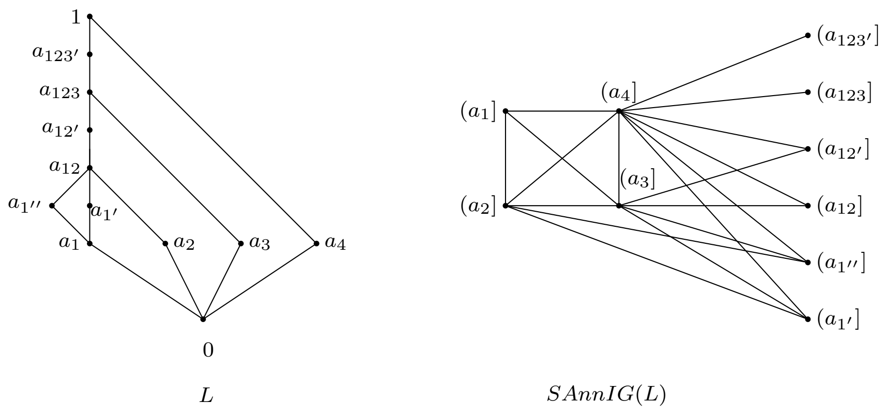

Figure 9. **Some lattice L for which $SAnnIG(L)$ is a split graph.**

Example 8. Let L be a lattice as shown in Figure 10. We have, $A(L) = \{a_1, a_2, a_3, a_4\}$, $S_1 = \{(a_1], (a_{1'}], (a_{1''}], (a_{1'''}]\}$, $S_{1,2} = \phi$, $S_{1,2,3} = \{(a_{123}], (a_{123'}], (a_{123''}]\}$ and $N(L) = S_1 \cup S_{1,2} \cup S_{1,2,3} \cup \big(\bigcup\limits_{a_i \in A(L)\setminus\{a_1\}} (a_i] \big)$. Observe that, $SAnnIG(L)$ is a split graph.

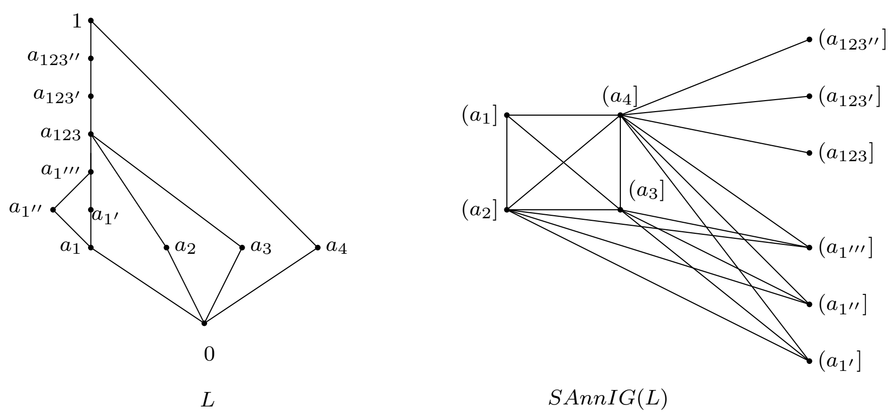

Figure 10. **Some lattice L for which $SAnnIG(L)$ is a split graph.**

The complete bipartite graph $K_{1,3}$ is a tree called claw. A claw-free graph is a graph that does not have a claw as an induced subgraph. In the following result, we give the characterization for $SAnnIG(L)$ to be a claw-free graph.

Theorem 11. *Let L be a lattice such that $|A(L)| \geq 2$. Then $SAnnIG(L)$ is a claw-free graph if and only if for every $a \in A(L)$ there does not exist distinct $b_i, b_j \in [a)_z \setminus \{a\}$ such that $A((b_i]) \subseteq A((b_j])$.*

Proof. Suppose $SAnnIG(L)$ is a claw-free graph. For some $a_i \in A(L)$, if there exists distinct $b_i, b_j \in [a_i)_z \setminus \{a_i\}$ such that $A((b_i]) \subseteq A((b_j])$, then $\{(a_i], (b_i], (b_j]\}$ forms an independent set. Since $(x] \in N(L)$ for every $x \in [a_i)_z$, there exists $a_j \in A(L) \setminus \{a_i\}$ such that $A((a_j]) \nsubseteq A((x])$ and $A((x]) \nsubseteq A((a_j])$. Therefore ideal $(a_j]$ is adjacent to $(x]$ for every $x \in [a_i)_z$. Thus, the set $\{(a_i], (b_i], (b_j], (a_j]\}$ forms a claw, a contradiction. Conversely, suppose for every $a \in A(L)$ there does not exist distinct $b_i, b_j \in [a)_z \setminus \{a\}$ such that $A((b_i]) \subseteq A((b_j])$. On the contrary, assume that $SAnnIG(L)$ is not a claw-free graph. Therefore, there exist an independent set, say $S = \{I_1, I_2, I_3\}$. By Lemma 2, suppose that $A(I_1) \subseteq A(I_2)$, $A(I_1) \subseteq A(I_3)$ and $A(I_2) \subseteq A(I_3)$. Assume that $a \in A(L) \cap A(I_1), I_2 = (b_i]$ and $I_3 = (b_j]$, with $b_i \neq b_j$. Clearly, there does exist distinct $b_i, b_j \in [a)_z \setminus \{a\}$ such that $A((b_i]) \subseteq A((b_j])$, a contradiction. Similarly, in all the other possibilities, we get a contradiction. Thus $SAnnIG(L)$ is a claw-free graph. □

Example 9. Consider a lattice as shown in Figure 8. Since, there exists $a \in A(L)$ and $d, e \in [a)_z \setminus \{a\}$ such that $A((d]) = \{a\} = A((e])$, we have, $SAnnIG(L)$ is not a claw-free graph. Note that, the elements in the set $\{(a], (d], (e], (b]\}$ forms a claw in $SAnnIG(L)$.

In the following result, we study $SAnnIG(L)$ when L is a lattice of positive divisors.

Theorem 12. *Let n be a natural number, and $L = D(n)$ be the lattice of all divisors of n, and $n = p_1^{m_1} p_2^{m_2} \dots p_k^{m_k}$, $1 \leq m_1 \leq m_2 \leq \cdots \leq m_k$ be the prime factorization of n, where $n > 1$, $k \geq 2$,*

(1) *Then $\mathcal{C}(SAnnIG(L)) = 4$ provided $SAnnIG(L)$ admits a cycle.*

(2) *$\gamma(SAnnIG(L)) = 1$ if and only if $n = p_1 p_2^{m_2}$.*

(3) *$\gamma(SAnnIG(L)) = 2$ if and only if $n \neq p_1 p_2^{m_2}$.*

(4) *$SAnnIG(L)$ is a path if and only if $k = 2$ with $m_1 = 1, m_2 \leq 2$.*

(5) *$SAnnIG(L)$ is a cycle if and only if $k = 2$ with $m_1 = 2 = m_2$ if and only if $SAnnIG(L)$ is unicyclic.*

(6) *$SAnnIG(L)$ is a tree if and only if $k = 2$ with $m_1 = 1$.*

(7) *$SAnnIG(L)$ is a triangle-free graph if and only if $k = 2$ if and only if $SAnnIG(L)$ is a complete multipartite graph.*

(8) *$SAnnIG(L)$ is a split graph if and only if $k = 2$ with $m_1 = 1$.*

(9) *$SAnnIG(L)$ is a split graph if and only if $SAnnIG(L)$ is a tree.*

(10) *$SAnnIG(L)$ is a claw-free graph if and only if $k = 2$ with $m_2 \leq 2$ or $k = 3$ with $m_1 = m_2 = m_3 = 1$.*

Proof. (1) If $k = 2$, then $SAnnIG(L) = K_{m_1,m_2}$ and hence no cycle of length 3 exists. If $k \geq 3$, then $SAnnIG(L)$ contains a cycle $(p_1] - (p_2] - (p_1p_3] - (p_2p_3] - (p_1]$ of length 4. Thus by Theorem 5, we have $\mathcal{C}(SAnnIG(L)) = 4$.
(2) Here $A(D(n)) = \{p_1, \ldots, p_n\}$. Now, $|[p_i)_z| = 1$ for some i if and only if $n = p_1p_2^{m_2}$. Thus by Theorem 6, we have $\gamma(SAnnIG(L)) = 1$.
(3) By (2) and Theorem 6, it is obvious.
(4) Suppose $SAnnIG(L)$ is a path. By Theorem 2, $k = 2$ and hence $SAnnIG(L) = K_{m_1,m_2}$. Thus $m_1 = 1, m_2 \leq 2$. The converse of the statement is trivial.
(5) Suppose $SAnnIG(L)$ is a cycle. If $k \geq 3$, then $|Z^*(L)| \geq 6$ and by Theorem 7, we have, $k = 2$. Therefore $A(L) = \{p_1, p_2\}$. Hence by Theorem 7, we have $|[p_1)_z| = m_1 = 2 = m_2 = |[p_2)_z|$ and then by Theorem 8, $SAnnIG(L)$ is unicyclic. Conversely, suppose $SAnnIG(L)$ is unicyclic. By Theorem 8, we have $k \leq 3$. Suppose $k = 3$. Then $|A_1| = 3$ if and only if $m_i = 1, \forall\ i$. But, then there exists ideals $I = (p_ip_j]$ and $J = (p_jp_k]$ such that $A(I) \neq A(J)$ and hence by Theorem 8, we have, $SAnnIG(L)$ is not unicyclic, a contradiction. Thus $k = 2$. Therefore by Theorem 8, we have $|[p_1)_z| = m_1 = 2 = m_2 = |[p_2)_z|$. Thus by Theorem 7(2), we have, $SAnnIG(L)$ is a cycle.
(6) Suppose $SAnnIG(L)$ is a tree. Then $SAnnIG(L) = K_{|[p_1)_z|,|[p_2)_z|} = K_{m_1,m_2}$ if and only if $k = 2$. Thus by Proposition 1(2), we have $m_1 = 1$. The converse of the statement is trivial.
(7) Suppose $SAnnIG(L)$ is a triangle-free graph. By Theorem 2 and Corollary 5, we have $k = 2$. Therefore $SAnnIG(L) = K_{m_1,m_2}$. Conversely, suppose $SAnnIG(L)$ is a complete multipartite graph. If $k \geq 3$, then there exists ideal $I = (p_ip_j]$ such that $|A(I)| = 2 > 1$ and a contradiction to Theorem 9. Therefore $k = 2$ and hence $|A(L)| = 2$. Thus, by Proposition 1(1), we have, $SAnnIG(L)$ is a triangle-free graph.
(8) Suppose $SAnnIG(L)$ is a split graph. If $k \geq 3$, then $S_1 = \{I \in N(L) \mid A(I) = \{p_1\}\} \neq \phi$ and $S_2 = \{I \in N(L) \mid A(I) = \{p_2\}\} \neq \phi$. Therefore, by Theorem 10, we have, $SAnnIG(L)$ is not split, a contradiction. Thus $k = 2$. Hence $SAnnIG(L) = K_{m_1,m_2}$. Thus $m_1 = 1$. The converse of the statement is trivial.
(9) Result follows by statements (6) and (8).
(10) Suppose $SAnnIG(L)$ is a claw-free graph. If $k \geq 4$, then there exists $p_1p_2, p_1p_2p_3 \in [p_1)_z$ such that $A((p_1p_2]) \subset A((p_1p_2p_3])$ and hence by Theorem 11, we have, $SAnnIG(L)$ is not claw-free, a contradiction. Thus $k \leq 3$. Suppose $k = 2$. Then $SAnnIG(L) = K_{m_1,m_2}$. Thus $m_2 \leq 2$. Now, suppose $k = 3$. If $m_3 > 1$, then there exists $p_1p_3, p_1p_3^2 \in [p_3)_z$ such that $A((p_1p_3]) = A((p_1p_3^2])$ and hence by Theorem 11, we have, $SAnnIG(L)$ is not claw-free, a contradiction. Therefore $m_3 = 1$. Thus $m_1 = m_2 = m_3 = 1$. The converse of the statement is trivial. □

3. Comparison between $SAnnIG(L)$ and $AnnIG(L)$

In this section, we determine the condition on lattice L so that $SAnnIG(L)$ is identical to $AnnIG(L)$. Also, we study the relationship between the diameters of $SAnnIG(L)$ and $AnnIG(L)$ and the relationship between their girths.

Recall the following result from [12].

Theorem 13 ([12], Theorem 2.1). *Let L be a lattice. Then $SAnnIG(L)$ is a subgraph of $AnnIG(L)$.*

Recall the following result from [14].

Theorem 14 ([14], Theorem 2.3). *Let L be a lattice. Then $AnnIG(L)$ is a complete multipartite graph.*

The following result gives the characterization for $SAnnIG(L)$ and $AnnIG(L)$ to be identical.

Theorem 15. *Let L be a lattice. Then $SAnnIG(L)$ and $AnnIG(L)$ are identical if and only if $|A(I)| = 1$, for all $I \in N(L)$.*

Proof. Let $SAnnIG(L)$ and $AnnIG(L)$ be the identical graphs. Hence, by Theorem 14, $SAnnIG(L)$ is a complete multipartite graph. Therefore, according to Theorem 9, we have $|A(I)| = 1$, for all $I \in N(L)$. The converse of the statement follows from Theorem 9. □

The following corollary is an immediate consequence of Theorem 15.

Corollary 7. *Let L be a lattice with $|A(L)| \leq 2$. Then $SAnnIG(L)$ and $AnnIG(L)$ are identical.*

Theorem 16. *Let L be a lattice such that $SAnnIG(L)$ is nonempty. Then the following statements are equivalent:*

(1) $gr(SAnnIG(L)) = 4$

(2) $SAnnIG(L) = AnnIG(L)$ *and* $gr(AnnIG(L)) = 4$.

(3) $gr(AnnIG(L)) = 4$.

(4) $|A(L)| = 2$ *and there exist elements* x_a *and* x_b *such that* $a \prec x_a$, $b||x_a$ *and* $b \prec x_b$, $a||x_b$, *where* $A(L) = \{a, b\}$.

(5) $AnnIG(L) = K_{m,n},\ m, n \geq 2$;

(6) $SAnnIG(L) = K_{m,n},\ m, n \geq 2$;

Proof. (1) ⇒ (2) By [[12], Corollary 3.2(2)], we have, $|A(L)| = 2$. Then by Corollary 7, we have, $SAnnIG(L) = AnnIG(L)$. Thus, the statement is clear.
(2) ⇒ (3) It is trivial.
(3) ⇒ (4) Let $gr(AnnIG(L)) = 4$. By [[14], Corollary 2.5(ii)], the statement is clear.

(4) ⇒ (5) Let $|A(L)| = 2$. By Corollary 7, $AnnIG(L) = K_{|[a)_z|,|[b)_z|}$. Then by [[14], Corollary 2.5(ii)], $AnnIG(L) = K_{m,n},\ m, n \geq 2$;.
(5) ⇒ (6) This is clear by Corollary 7.
(6) ⇒ (1) The graph $SAnnIG(L)$ is complete bipartite that is not a star. Thus, the statement is clear. □

Theorem 17. *Let L be a lattice such that $SAnnIG(L)$ is nonempty. Then the following statements are equivalent:*

(1) $gr(SAnnIG(L)) = \infty$

(2) $SAnnIG(L) = AnnIG(L)$, *and* $gr(AnnIG(L)) = \infty$.

(3) $gr(AnnIG(L)) = \infty$.

(4) $|A(L)| = 2$ *and there doesn't exist elements* x_a *and* x_b *such that* $a \prec x_a$, $b||x_a$ *and* $b \prec x_b$, $a||x_b$, *where* $A(L) = \{a, b\}$.

(5) $AnnIG(L) = K_{1,n}, n \geq 1$;

(6) $SAnnIG(L) = K_{1,n}, n \geq 1$;

Proof. (1) ⇒ (2) By [[12], Corollary 3.2(2)], we have $|A(L)| = 2$. Then by Corollary 7, we have $SAnnIG(L) = AnnIG(L)$. Thus, the statement is obvious.
(2) ⇒ (3) It is trivial.
(3) ⇒ (4) Let $gr(AnnIG(L)) = \infty$. By [[14], Corollary 2.5(iii)], the statement is clear.
(4) ⇒ (5) Let $|A(L)| = 2$. By Corollary 7, $AnnIG(L) = K_{|[a)_z|,|[b)_z|}$. Then by [[14], Corollary 2.5(iii)], $AnnIG(L) = K_{1,n}, n \geq 1$;.
(5) ⇒ (6) and (6) ⇒ (1). Similar to the proof of (5) ⇒ (6) and (6) ⇒ (1) of Theorem 16. □

In the following result, for the graphs $SAnnIG(L)$ and $AnnIG(L)$, we give the relationship between their diameters.

Theorem 18. *Let L be a lattice such that $|A(L)| \geq 2$. Then the following statements hold:*

(1) *If* $diam(SAnnIG(L)) = 1$, *then* $diam(AnnIG(L)) = 1$.

(2) *If* $diam(SAnnIG(L)) = 2$, *then* $diam(AnnIG(L)) = 1$ *or* 2.

(3) *If* $diam(AnnIG(L)) = 1$, *then* $diam(SAnnIG(L)) = 1$ *or* 2.

(4) *If* $diam(AnnIG(L)) = 2$, *then* $diam(SAnnIG(L)) = 2$.

Proof. (1) By Theorem 3, we have, $N(L) = \{(a] \mid a \in A(L)\}$. Thus, $A(I) \neq A(J)$ for every distinct $I, J \in N(L)$. By [[14], Lemma 2.2], we have, $diam(AnnIG(L)) = 1$.

(2) Let $diam(SAnnIG(L)) = 2$. If $A(I) \neq A(J)$ for every distinct $I, J \in N(L)$, then by [[14], Lemma 2.2], we have $diam(AnnIG(L)) = 1$, otherwise $diam(AnnIG(L)) = 2$.
(3) Suppose $diam(AnnIG(L)) = 1$. If $N(L) = \{(a] \mid a \in A(L)\}$, then we have, $diam(SAnnIG(L)) = 1$; otherwise $diam(SAnnIG(L)) = 2$ by Corollary 1.
(4) It is obvious by Theorem 13. □

By [[12], Corollary 3.2] and [[14], Corollary 2.5], we have the following immediate result.

Corollary 8. *Let L be a lattice. Then $gr(SAnnIG(L)) = gr(AnnIG(L))$.*

Corollary 9. *Let L be a lattice. If $SAnnIG(L) \neq AnnIG(L)$, then $gr(SAnnIG(L)) = gr(AnnIG(L)) = 3$.*

The converse of Corollary 9 is not true. To observe this, refer to the following example.

Example 10. For a diamond lattice $L = D_3$, we have, $AnnIG(L) = K_3 = SAnnIG(L)$. Therefore, we have $diam(SAnnIG(L)) = 1 = diam(AnnIG(L))$ and $gr(SAnnIG(L)) = 3 = gr(AnnIG(L))$.

Example 11. Let $L = (D(30), |)$. Then $AnnIG(L) = K_6$ and $SAnnIG(L)$ is a 3-regular graph. Therefore $diam(AnnIG(L)) = 1$ and $diam(SAnnIG(L)) = 2$.

Example 12. Let L be a lattice as shown in Figure 11. Observe that, $diam(SAnnIG(L)) = 2 = diam(AnnIG(L))$.

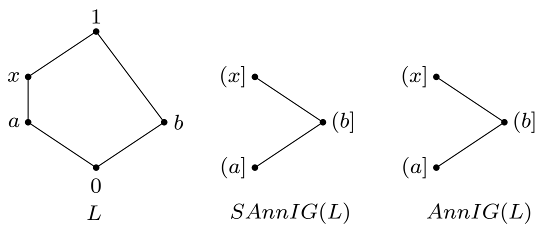

Figure 11. **Some lattice L for which** $diam(SAnnIG(L)) = 2 = diam(AnnIG(L))$.

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References

[1] M. Afkhami and K. Khashyarmanesh, *The comaximal graph of a lattice*, Bull. Malays. Math. Sci. Soc. **37** (2014), no. 1, 261–269.

[2] M. Afkhami, K. Khashyarmanesh, and F. Shahsavar, *On the properties of zero-divisor graphs of posets*, Quasigr. Relat. Syst. **43** (2020), no. 1, 1–16.

[3] J. Amjadi, R. Khoeilar, and A. Alilou, *The annihilator-inclusion ideal graph of a commutative ring*, Commun. Comb. Optim. **6** (2021), no. 2, 231–248. https://doi.org/10.22049/cco.2020.26752.1139.

[4] D.F. Anderson, T. Asir, A. Badawi, and T.T. Chelvam, *Graphs from rings*, Springer Nature Switzerland AG, 2021.

[5] D.F. Anderson and P.S. Livingston, *The zero-divisor graph of a commutative ring*, J. Algebra **217** (1999), 434–447. https://doi.org/10.1006/jabr.1998.7840.

[6] S. Bagheri and M.K. Kerahroodi, *The annihilator graph of a* 0*-distributive lattice*, Trans. Comb. **7** (2018), no. 3, 1–18. https://doi.org/10.22108/toc.2017.104919.1507.

[7] I. Beck, *Coloring of commutative rings*, J. Algebra **116** (1988), no. 1, 208–226. https://doi.org/10.1016/0021-8693(88)90202-5.

[8] T.T. Chelvam and S. Nithya, *A note on the zero divisor graph of a lattice*, Trans. Comb. **3** (2014), no. 3, 51–59. https://doi.org/10.22108/toc.2014.5626.

[9] E. Estaji and K. Khashyarmanesh, *The zero-divisor graph of a lattice*, Results Math. **61** (2012), no. 1, 1–11.

[10] G.A. Grätzer, *Lattice Theory: Foundation*, Springer, 2011.

[11] A. Khairnar and B.N. Waphare, *Zero-divisor graphs of laurent polynomials and laurent power series*, Algebra and its Applications (Singapore) (S.T. Rizvi, A. Ali, and V.D. Filippis, eds.), Springer Singapore, 2016, pp. 345–349.

[12] V. Kulal and A. Khairnar, *Strongly annihilator ideal graph of a lattice*, Asian-Eur. J. Math. **17** (2024), no. 7, Article ID: 2450050. https://doi.org/10.1142/S1793557124500505.

[13] V. Kulal, A. Khairnar, and T.T. Chelvam, *Annihilator intersection graph of a lattice*, Discrete Math. Algorithms Appl. (2024), In press. https://doi.org/10.1142/S1793830924500344.

[14] V. Kulal, A. Khairnar, and K. Masalkar, *Annihilator ideal graph of a lattice*, Palest. J. Math. **11** (2022), no. 4, 195–204.

[15] A. Patil, A. Khairnar, and B.N. Waphare, *Zero-divisor graph of a poset with*

respect to an automorphism, Discrete Appl. Math. **283** (2020), 604–612. https://doi.org/10.1016/j.dam.2020.02.015.

[16] N.K. Tohidi, M.J. Nikmehr, and R. Nikandish, *On the strongly annihilating-ideal graph of a commutative ring*, Discrete Math. Algorithms Appl. **9** (2017), no. 2, Article ID: 1750028. https://doi.org/10.1142/S1793830917500288.

[17] D.B. West, *Introduction to Graph Theory*, Prentice hall Upper Saddle River, 2001.