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Research Article

On metric dimension of cube of trees

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Abstract: Let $G = (V, E)$ be a connected graph and $d_G(u, v)$ be the shortest distance between the vertices u and v in G. A set $S = \{s_1, s_2, \ldots, s_n\} \subset V(G)$ is said to be a *resolving set* if for all distinct vertices u, v of G, there exists an element $s \in S$ such that $d_G(s, u) \neq d_G(s, v)$. The minimum cardinality of a resolving set for a graph G is called the metric dimension of G, and it is denoted by $\beta(G)$. A resolving set having $\beta(G)$ number of vertices is named as metric basis of G. The metric dimension problem is to find a metric basis in a graph G, and it has several real-life applications in network theory, telecommunication, image processing, pattern recognition, and many other fields. In this article, we consider cube of trees $T^3 = (V, E)$, where any two vertices u, v are adjacent if and only if the distance between them is less than or equal to three in T. We establish the necessary and sufficient conditions for a vertex subset of V to become a resolving set for T^3. This helps to determine the tight bounds (upper and lower) on the metric dimension of T^3. Then, for certain well-known cube of trees, such as caterpillars, lobsters, spiders, and d-regular trees, we establish the boundaries for the metric dimension. Also, for every positive integer, we provide a construction showing the existence of a cube of a tree satisfying its metric dimension as the given integer. Further, we characterize some restricted families of cube of trees satisfying $\beta(T^3) = \beta(T)$.

Keywords: resolving set, metric basis, metric dimension, trees.

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1. Introduction

For a simple undirected connected graph $G = (V, E)$, the length of the shortest distance between the vertices u and v in G is denoted by $d_G(u, v)$. The *metric representation* (or *code*) of a vertex w with respect to a vertex set $S = \{s_1, \ldots, s_n\} \subseteq V$ (denoted by $c(w|S)$) is an $n-$tuple $(d_G(w, s_1), \ldots, d_G(w, s_n))$. A vertex s *resolves* or *distinguishes* two distinct vertices, u and v, of V when $c(u|s) \neq c(v|s)$, i.e., $d_G(u, s) \neq d_G(v, s)$, considering $S = \{s\}$. In the same sense, S is said to be a *resolving set* (also known as a *locating set* or *metric generator*) for G if for every two distinct vertices u, v of V, we have $c(u|S) \neq c(v|S)$, i.e., for any such $u \neq v$, there exists a vertex $s \in S$ that resolves u, v. If no $s \neq u, v$ is found to satisfy the above criteria, then we include one among u or v in S (cf. for an n-vertex complete graph K_n, S contains $n - 1$ vertices in it). This means that for every pair of distinct vertices of G, their codes differ in at least one position. Every simple undirected connected graph G has a resolving set since the vertex set V itself forms a resolving set. The smallest possible resolving set is said to be a *metric basis*, and its cardinality is called the *metric dimension* (or *locating number*) of the graph G (in short, $\dim(G)$). For convenience, we use $\beta(G)$ to denote the metric dimension of a graph G for the whole exposition. Though there are graphs with unique metric bases, it is interesting to note that metric bases do not need to be unique for every graph [7].

The notion of the metric basis of connected graphs was introduced independently by Slater [21], and by Harary and Melter [12] in 1975 and 1976, respectively, for uniquely identifying every vertex in a graph. Calculating the exact metric dimension in general for graphs is challenging. To construct a resolving set one needs to provide an algorithm looking at the adjacency of every node, and therefore, for general graphs since every edge is not necessarily to be present in a shortest path with an element of a resolving set as an endpoint, the problem becomes complicated. Determining metric dimension is NP-hard [10] for many restricted classes of graphs, such as planar graphs, split graphs, bipartite and co-bipartite graphs, line graphs of bipartite graphs, subcubic graphs, interval and permutation graphs of diameter two, and for directed graphs. However, there are efficient algorithms to compute the metric dimension for trees [14, 21] in linear time. Sometimes the elements of a metric basis (also known as *landmarks*) are treated as sensors in a real-world network to preserve system security by transferring information or messages within a fixed group only. Finding such a minimal group of landmarks (satellites) is crucial in robot navigation problems [14] where the robot can uniquely determine its position by the presence of distinctly labeled landmarks. Similar applications of metric basis and its variants have been found in tracking the spread of disease between cities, to find patient zero and other items in a complex network during pandemics like COVID-19. In a deterministic epidemic model, the times of infection of the sensor nodes are converted to graph distances between sensors and the source of infection (patient zero), considering the time of infection of patient zero is known. The minimum number of sensors required to always detect the source of the disease is same as the metric dimension of that particular network [25]. Resolving sets are also used for distinguishing different chemical structures, determining a source of misinformation circulating in a social network in addition to recognizing intruders, finding "connected joins" in graphs, developing strategies for the Mastermind game, incorporating symbolic data in low-dimensional Euclidean spaces and many others. According to the available databases of many science repositories, interest in the topic of metric dimension and its variants has exploded over the last two decades due to its vast applications in several disciplines, including group theory and topology [4], information theory and extremal combinatorics [5, 23]. Two recent surveys [16, 22] discuss a complete compendium containing mostly the important findings on the metric dimension of graphs. Some other significant recent works on this literature are worth mentioning, for instance, [6, 9, 11, 17, 24].

The power graph has been extensively explored in the past due to its intriguing features and wide range of applications in parallel computing, signal processing, VLSI designing, and coding theory because of its increased connectivity in an existing network. The metric dimension of paths as well as the middle and total graphs of paths were studied by Ali et al. [3] in 2012; they also computed the constant metric dimension for square and cube of path graphs. Later on, Alholi et al. [2] determined some exact values and upper bound on the metric dimension of the power of paths by proving $\beta(P_n^k) \leq k, \beta(P_n^3) = 3, \beta(P_n^4) = 4$. Chartrand et al. [8] formulated the metric dimension of trees that are not paths as well as bounds for the dimension of unicyclic graphs. Their findings characterize all graphs of order n having metric dimension 1 (paths), $n-1$ (complete graphs), and $n-2$. In 2021, Nawaz et al. [18] proved that the metric dimension of the total graph of path power three and four is unbounded; they also proved some results on the edges of the power of path and total graph of power of path. Saha et al. [20] presented a lower bound on the metric dimension of P_n^r, and then built up a resolving set with cardinality that is the same as that of the lower bound. Javid et al. [13] initiated the study of metric dimension for square of cycles. The metric dimension of cycles with $n(\geq 3)$ vertices is 2. In due course of study, other advancements are made by determining the exact metric dimension for the power of cycles; values of $\beta(C_n^t)$, $2 \leq t \leq 5$ are available in [15]. In 2022, metric dimension of square of grid graphs was determined by providing an optimal resolving set with cardinality 3 [19]. Also, they carried out an investigation over the bounds on metric dimension for the square of trees recently. Due to the widespread applications of power graphs, and motivated by the above results, in this article, we study *cube of trees* $T^3 = (V, E)$ where any two vertices $u, v \in V$ are adjacent if and only if $d_T(u, v) \leq 3$.

The rest of the paper is organized as follows: Firstly, Section 2 represents a detailed explanation of all the terms and expressions that will be used later on to establish the corresponding results for the metric dimension of T^3. In Section 3, we have proved some essential lemmas on the properties of resolvability in T^3 that facilitate determining the resolving set of the cube of a tree. In Section 4, first, we provide the necessary and sufficient conditions of a vertex subset of V to become a resolving set for T^3. Next in Section 5 and in Section 6, we build the tight bounds (lower and upper) on $\beta(T^3)$ depending upon the number of *short legs*, *long legs*, *major stems*, and their positional appearances in the tree T. Constructing a resolving set to prove the upper bound on $\beta(T^3)$ is a worthy task. Furthermore, we prove that for every positive integer, there exists a family of cube of trees with the same metric dimension as those integer values. In Section 7, we analyze the metric dimension or the bounds on it for some well-known cube of trees, including caterpillars, lobsters, spiders, d-regular trees. Lastly, in Section 8, we restrict our findings to those cube of trees that have pendants as their legs and all of their stems lie on a central path and characterize such graph classes that satisfy $\beta(T^3) = \beta(T)$. In conclusion, we keep the challenge open to determine the bounds on metric dimension for any power of trees T^r (say) where $r \geq 4$.

2. Preliminaries

For a tree $T = (V, E)$, a vertex $v \in V$ of degree at least three is called a *core vertex* or *core*, a vertex of degree two and one is said to be a *path vertex* and *leaf*, respectively [1]. If we remove a vertex v from T, then $T \setminus \{v\}$ induces a $\deg(v)$ number of subtrees or components. A *branch* at a vertex v is the subgraph induced by v and one of the components of $T \setminus \{v\}$. A branch B of T at v, which is a path, is called *branch path* (also known as *leg*) [21]. The vertex v in a branch path satisfying $\deg(v) \geq 3$ is called *stem* of the branch path [21]. It is easy to observe that not every core vertex is a stem.

Definition 1. A vertex of a tree $T = (V, E)$ is said to be a *major stem* if it is a stem containing at least two legs. Other stems are called *minor stems*. A leg of length greater than or equal to three is said to be a *long leg*, other legs that have a length less than three are said to be *short legs*. We call a short leg of length two a *mid leg*, and a short leg of length one a *pendant*.

Observation 1. Let $T = (V, E)$ contain at least one stem. Then the following conditions are true:
i) Two legs adjacent to the same stem vertex $v \in V$ are disjoint except for the common stem v.
ii) Any two legs adjacent to two distinct stems must be disjoint.

Theorem 2. *[21] Let $T = (V, E)$ be a tree of order $|V| \geq 3$. Then $S \subseteq V$ forms a resolving set if and only if for each vertex x there are vertices from S on at least $deg(x) - 1$ of the $deg(x)$ components of $T \setminus \{x\}$.*

Theorem 3. *[14] Let $T = (V, E)$ be a tree that is not a path. If l_v is the number of legs attached to the vertex v. Then*

$$\beta(T) = \sum_{v \in V : l_v > 1} (l_v - 1) \tag{2.1}$$

Since the minor stems of a tree cannot have more than one leg as its branch, it is important to note the following from Theorem 3.

Corollary 1. *Let T be a tree that is not a path. Then $\beta(T) = \sum_{v \in V'} (l_v - 1)$ where V' denotes the set of all major stems of T and l_v is the number of legs attached to the major stem v.*

The problem of computing the metric dimension of trees is linear [14, 21], since from the above corollary we get to know that the metric dimension of a tree is the difference of the total number of leaves and the number of major stems of it, both of which can be computed in linear time.

Notation. Let $P = (u, u_1, \ldots, v_1, v)$ be the path on a tree T between the vertices u and v. Here u_1, v_1 are either the intermediate vertices of the above path P considering $d_T(u, v) \geq 2$ (u_1 can be equal to v_1 also when $d_T(u, v) = 2$) or end vertices when $d_T(u, v) = 1$ (i.e., $u_1 = v_1 = v$ or $u = u_1 = v_1$ or $u = u_1, v = v_1$). We denote $T_u(T_v)$ to be the component of T containing the vertex $u(v)$, obtained after deletion of the edge $uu_1(v_1v)$. A vertex x is said to be within the same component of u and v (say $T_{u,v}$) only when x occurs within the intermediate path of u, v, or it lies in some branch of T attached to some intermediate vertex of u, v.

Definition 2. Let $T = (V, E)$ be a tree. A graph $T^3 = (V, \hat{E})$ is said to be *cube of tree* of T if the vertex set V remains same as in T and the edge set $\hat{E} = E \cup \{uv \,|\, 2 \leq d_T(u, v) \leq 3\}$.

The distance between any two vertices u, v in T^3 is measured by $d_{T^3}(u, v) = \left\lceil \frac{d_T(u,v)}{3} \right\rceil$. We will use the notations $V(T^3)$ and $E(T^3)$ to denote the vertex set and edge set of T^3.

3. Properties regarding resolvability in T^3

In this section, we give some basic properties and results of the resolving set of T^3. We have established certain essential lemmas that are beneficial for determining the resolving set of T^3.

Lemma 1. *Let $T = (V, E)$ be a tree. Then every resolving set of T^3 is also a resolving set of T.*

Proof. Let S be a resolving set of T^3, and $u, v \in V$ be any two vertices. Since $V(T^3) = V(T)$ and S is a resolving set of T^3, there exists a vertex $s \in S$ such that $d_{T^3}(s, u) \neq d_{T^3}(s, v)$, which implies $\lceil \frac{d_T(s,u)}{3} \rceil \neq \lceil \frac{d_T(s,v)}{3} \rceil$. Hence we get $d_T(s, u) \neq d_T(s, v)$, i.e., s resolves the vertices u and v in T. Therefore, S forms a resolving set for T. $\square$

We can immediately draw some conclusions from the above lemma.

Corollary 2. *For any tree T, $\beta(T^3) \geq \beta(T)$.*

Proof. Let S be a metric basis for T^3. Then $|S| = \beta(T^3)$. Using Lemma 1, we get S to be a resolving set of T also. Therefore, $\beta(T) \leq |S| = \beta(T^3)$. □

Lemma 2. *Let $T = (V, E)$ be a tree and S be a resolving set of T^3. Then for every vertex $x \in V$, S contains a vertex from each component of $T \setminus \{x\}$ with one exception.*

Proof. On the contrary, let $T \setminus \{x\}$ have at least two components (say C_i, C_j) satisfying $S \cap V(C_i) = \emptyset$ and $S \cap V(C_j) = \emptyset$. Let $u \in S \cap V(C_i)$ and $v \in S \cap V(C_j)$ satisfy $d_T(x, u) = d_T(x, v)$. Now any vertex $w \in V \setminus (V(C_i) \cup V(C_j))$ must have to reach u or v via x. Therefore, $d_T(w, u) = d_T(w, x) + d_T(x, u) = d_T(w, x) + d_T(x, v) = d_T(w, v)$ and hence $d_{T^3}(w, u) = \lceil \frac{d_T(w,u)}{3} \rceil = \lceil \frac{d_T(w,v)}{3} \rceil = d_{T^3}(w, v)$. Therefore, a contradiction arises. Hence, the result follows. □

The following corollary is an essential tool for determining any resolving set of T^3.

Corollary 3. *Let v be a major stem of a tree T having m legs $L_1, L_2, \ldots, L_m$. Then, for every resolving set S of T^3, the following conditions hold.*

1. *$S \cap L_i \neq \emptyset$ for all $i \in \{1, 2, \ldots, m\}$ with one exception.*
2. *S contains at least $m - 1$ vertices from the legs adjacent to v.*

Lemma 3. *Let $T = (V, E)$ be a tree, and $v \in V$ be a core of degree m. If v is not a major stem, then there exist at least $m - 1$ components of T containing major stems.*

Proof. Since deg $(v) = m$, removing v from T will create m components. Now, as v is not a major stem, there can exist at most one branch attached to it, which is a path. Hence, there are $m - 1$ branches containing at least one vertex in each of the branches, which have at least two branches out from them. Each of these $m - 1$ branches is not the path. We consider one such branch B of v and a vertex u on B having deg $(u) \geq 3$ for which $d_T(v, u)$ is maximum. Therefore, one can verify that u must possess at least two branch paths, and hence u becomes a major stem of B, as well as of T from Definition 1. Similar logic holds true for all other branches of v that are not paths. Hence, the result follows. □

Corollary 4. *Let v be a core vertex of a tree T and $T \setminus \{v\}$ contain m components $C_1, \ldots, C_m$. If any component C_i contains l_i major stems where $1 \leq i \leq m$, then for every resolving set S of T^3, $|S \cap C_i| \geq \sum_{j=1}^{l_i} (n_j - 1)$, where n_j is the number of legs attached to a major stem in C_i.*

Proof. If C_i is not a branch path, then applying Lemma 2 for each major stem of C_i, it follows that S contains at least $\sum_{j=1}^{l_i}(n_j - 1)$ vertices from the legs adjacent to the major stems of C_i. $\square$

Lemma 4. *Let $T = (V, E)$ be a tree and uv be an edge in T^3. Then a vertex $x \neq u, v$ resolves u, v in T^3 if and only if the following conditions happen.*

- *If x belongs to at least one among T_u or T_v, then either*

$$d_T(u,v) = 3 \text{ or}$$
$$d_T(u,v) = 2 \text{ and } min\{d_T(x,u), d_T(x,v)\} \equiv 0 \text{ or } 2 \ (mod\ 3) \text{ or}$$
$$d_T(u,v) = 1 \text{ and } min\{d_T(x,u), d_T(x,v)\} \equiv 0 \ (mod\ 3)$$

- *If x belongs to $T_{u,v}$, then $d_T(u,v) = 3$ and $min\{d_T(x,u), d_T(x,v)\} \equiv 0 \ (mod\ 3)$.*

Proof. Since $uv \in E(T^3)$, $1 \leq d_T(u,v) \leq 3$ clearly.

Case I. Without loss of generality, first we consider the case when $x \in T_u$. Then we can write $d_T(x,u) = 3k + m$ and $d_T(x,v) = d_T(x,u) + d_T(u,v) = (3k+m) + d_T(u,v)$ for some integers k, m where $k \geq 0$, $0 \leq m < 3$. Hence $\min\{d_T(x,u), d_T(x,v)\} = d_T(x,u)$.

If $m = 0$, $d_{T^3}(x,u) = \lceil \frac{3k}{3} \rceil = k \neq k+1 = \lceil \frac{3k + d_T(u,v)}{3} \rceil = d_{T^3}(x,v)$ as $1 \leq d_T(u,v) \leq 3$. Therefore, when $\min\{d_T(x,u), d_T(x,v)\} = 3k \equiv 0 \pmod 3$, then x resolves u, v.

For $m = 1$ or 2, $d_{T^3}(x,u) = \lceil \frac{3k+m}{3} \rceil = k+1$ and $d_{T^3}(x,v) = \lceil \frac{(3k+m)+d_T(u,v)}{3} \rceil = k + \lceil \frac{m + d_T(u,v)}{3} \rceil$. Now x resolves u, v if and only if $d_{T^3}(x,v) = k+2$ (since $d_T(u,v) \leq 3$). This can only happen when $m + d_T(u,v) > 3$, i.e., when $m = 1$ and $d_T(u,v) = 3$ or when $m = 2$ and $2 \leq d_T(u,v) \leq 3$. Therefore, if $d_T(u,v) = 3$ and $\min\{d_T(x,u), d_T(x,v)\} \equiv 1$ or $2 \pmod 3$ or if $d_T(u,v) = 2$ and $\min\{d_T(x,u), d_T(x,v)\} \equiv 2 \pmod 3$, then x resolves u, v.

Case II. Next, we consider the case when x belongs to the same component of u and v, i.e., in $T_{u,v}$. Since $x \neq u, v$, $d_T(u,v) > 1$. Note that in this case, the only possibility of x resolving u, v is when $d_T(u,v) = 3$ and x occurs in some branch attached to u_1 or v_1, where $P = (u, u_1, v_1, v)$ is the path connecting u, v in T. Without loss of generality, we assume $\min\{d_T(x,u), d_T(x,v)\} = d_T(x,u)$. Then x must be attached to the branch of u_1. Let $d_T(x,u_1) = 3k + m$ for some nonnegative integers k, m satisfying $0 \leq m < 3$. Then $d_T(x,u) = d_T(x,u_1) + d_T(u_1,u) = (3k+m) + 1$ and $d_T(x,v) = d_T(x,u_1) + d_T(u_1,v) = (3k+m)+2$. Therefore, $d_{T^3}(x,u) = k + \lceil \frac{m+1}{3} \rceil$ and $d_{T^3}(x,v) = k + \lceil \frac{m+2}{3} \rceil$. One can easily verify now that x resolves $u, v \iff d_{T^3}(x,u) \neq d_{T^3}(x,v) \iff m = 2$. Therefore, $\min\{d_T(x,u), d_T(x,v)\} = d_T(x,u) = 3k + 3 \equiv 0 \pmod 3$. $\square$

If uv is an edge in T^3, then depending upon the different values of $d_T(u,v)$, we can impose restrictions on the vertices that can resolve u, v.

Corollary 5. *Let $T = (V, E)$ be a tree and uv be an edge in T^3. Then a vertex $x \neq u, v$ resolves u, v in T^3 if and only if the following conditions are true:*

1. *If $d_T(u,v) = 1$, then at least one among any three consecutive vertices chosen from $T_u \setminus \{u\}$ or $T_v \setminus \{v\}$ must coincide with x.*

2. *If $d_T(u,v) = 2$, then x must be one among any two consecutive vertices chosen from $T_u \setminus \{u\}$ or $T_v \setminus \{v\}$.*

3. *If $d_T(u,v) = 3$, then x is either in $T_u \setminus \{u\}$ or $T_v \setminus \{v\}$ or it is one among any three consecutive vertices from any branch attached to u_1 or v_1, where u_1 and v_1 are the intermediate vertices of the path (u, u_1, v_1, v) in T.*

Proof. It is easy to observe that the distance from a fixed vertex to any three (or two) consecutive vertices in T^3 must be different [1] computed in mod 3. The rest of the verification is immediate from Lemma 4. □

Lemma 5. *Let $T = (V,E)$ be a tree and u, v be two nonadjacent vertices in T^3. Then a vertex $x \neq u, v$ resolves u, v if and only if the following conditions are satisfied:*

- *If x belongs to $T_{u,v}$, then*

$$\begin{aligned} &\mathit{min}\ \{d_T(x,u), d_T(x,v)\} \equiv 0\ (\mathit{mod}\ 3) \text{ and } |d_T(x,v) - d_T(x,u)| \geq 1 \text{ or} \\ &\mathit{min}\ \{d_T(x,u), d_T(x,v)\} \equiv 1\ (\mathit{mod}\ 3) \text{ and } |d_T(x,v) - d_T(x,u)| \geq 3 \text{ or} \\ &\mathit{min}\ \{d_T(x,u), d_T(x,v)\} \equiv 2\ (\mathit{mod}\ 3) \text{ and } |d_T(x,v) - d_T(x,u)| \geq 2. \end{aligned}$$

- *Any x belonging to T_u or T_v can resolve u, v.*

Proof. In T^3, a vertex $x \neq u, v$ resolves u, v if and only if $d_{T^3}(x,u) \neq d_{T^3}(x,v)$. This implies $\lceil \frac{d_T(x,u)}{3} \rceil \neq \lceil \frac{d_T(x,v)}{3} \rceil$ and hence $d_T(x,u) \neq d_T(x,v)$, i.e., $|d_T(x,v) - d_T(x,u)| \geq 1$. Since u and v are nonadjacent in T^3, we have $d_T(u,v) > 3$. Consider the two cases below.

Case I. First we consider the case when x is in $T_{u,v}$. Let s be the intermediate vertex on the path $P = (u, u_1, \ldots, s, \ldots, v_1, v)$ connecting the unique path joining x to s in T. Now $d_T(x,u) \neq d_T(x,v) \iff d_T(s,u) \neq d_T(s,v)$. Without loss of generality, we assume min $\{d_T(s,u), d_T(s,v)\} = d_T(s,u)$. Then $d_T(x,u) = d_T(x,s) + d_T(s,u)$ and $d_T(x,v) = d_T(x,s) + d_T(s,v)$, and therefore min $\{d_T(x,u), d_T(x,v)\} = d_T(x,u)$. It is easy to note that $d_T(x,u) \geq 2$ always.

a) If $d_T(x,u) \equiv 0 \,(\mathrm{mod}\, 3)$, then $d_T(x,u) = 3k$ for some positive integer k, and $d_{T^3}(x,u) = k$. Since $d_T(x,v) > d_T(x,u)$, $d_T(x,v) \geq 3k+1$, which implies $d_T(x,v) - d_T(x,u) \geq 1$. Hence we get $d_{T^3}(x,v) \geq \lceil \frac{3k+1}{3} \rceil = k+1 > k = d_{T^3}(x,u)$.

b) If $d_T(x,u) \equiv 1 \,(\mathrm{mod}\, 3)$, then $d_T(x,u) = 3k+1$ for positive integer k and $d_{T^3}(x,u) = \lceil \frac{3k+1}{3} \rceil = k+1$. Since $d_T(x,v) > d_T(x,u)$, we have $d_T(x,v) \geq 3k+2$. Now $d_{T^3}(x,v) \neq$

[1] *it must be a 3-permutation (or 2-permutation) of the set* $\{0, 1, 2\}$

$d_{T^3}(x,u) \iff \lceil \frac{d_T(x,v)}{3} \rceil \neq k+1$. This implies that $d_T(x,v) \neq 3k+2, 3k+3$ and hence $d_T(x,v) \geq 3k+4$. Therefore, $d_T(x,v) - d_T(x,u) \geq 3$.

c) If $d_T(x,u) \equiv 2 \pmod 3$, then $d_T(x,u) = 3k+2$ for some integer $k \geq 0$ and $d_{T^3}(x,u) = \lceil \frac{3k+2}{3} \rceil = k+1$. Also, $d_T(x,v) > d_T(x,u)$ implies $d_T(x,v) \geq 3k+3$. Now $d_{T^3}(x,v) \neq d_{T^3}(x,u) \iff \lceil \frac{d_T(x,v)}{3} \rceil \neq k+1$. This implies that $d_T(x,v) \neq 3k+3$, and hence $d_T(x,v) \geq 3k+4$. Therefore, $d_T(x,v) - d_T(x,u) \geq 2$.

If x is an intermediate vertex of the $u-v$ path P, then considering $s = x$, a similar logic will follow.

Case II. Next, we consider the case when x is either in T_u or T_v. Without loss of generality, we assume that x is in T_u. Then $d_T(x,v) = d_T(x,u) + d_T(u,v) > d_T(x,u) + 3$ as u and v are nonadjacent in T^3. Therefore $d_{T^3}(x,v) > d_{T^3}(x,u) + 1$. Hence, any such x can resolve u, v. □

Corollary 6. *Let $T = (V,E)$ be a tree and u, v be two nonadjacent vertices in T^3 satisfying $4 \leq d_T(u,v) \leq 5$. Then a vertex $x \neq u, v$ resolves u, v in T^3 if and only if the following conditions are true:*

1. *If $d_T(u,v) = 4$, then x is either in $T_u \setminus \{u\}$ or $T_v \setminus \{v\}$, or it is one among any two consecutive vertices from any branch of T attached to the intermediate vertex u_1 or v_1 of the path (u, u_1, w, v_1, v) in T.*

2. *If $d_T(u,v) = 5$, then x is either in $T_u \setminus \{u\}$ or $T_v \setminus \{v\}$, or x coincides with u_1 or v_1 or any vertex on a branch attached to them, or it is one among any three consecutive vertices from any branch of T attached to the intermediate vertices w_1 or w_2 of the path $(u, u_1, w_1, w_2, v_1, v)$ in T.*

Proof. Let $d_T(u,v) = 4$. Without loss of generality, we assume x, y to be two consecutive vertices on a branch B attached to u_1. Then $\min\{d_T(x,u), d_T(x,v)\} = d_T(x,u)$ and $\min\{d_T(y,u), d_T(y,v)\} = d_T(y,u)$. Now $d_T(x,v) - d_T(x,u) = (d_T(x,u_1) + d_T(u_1,v)) - (d_T(x,u_1) + d_T(u_1,u)) = d_T(u_1,v) - d_T(u_1,u) = 2$ as $d_T(u,v) = 4$. Similarly, we get $d_T(y,v) - d_T(y,u) = 2$. Since the vertices x, y are consecutive along B, at least one among $d_T(x,u)$ or $d_T(y,u)$ takes a value from the set $\{0, 2\}$ computed in mod 3. Let $d_T(x,u) \equiv 0$ or $2 \pmod 3$. Then, by Lemma 5, x resolves u, v in T^3. Similar logic follows if $d_T(y,u) \equiv 0$ or $2 \pmod 3$.

The proof of resolvability for the case $d_T(u,v) = 5$ is analogous and can be verified using Lemma 5. □

4. The necessary and sufficient conditions for resolving sets of T^3

In the following, we present the necessary and sufficient conditions for a vertex subset to become a resolving set for cube of trees.

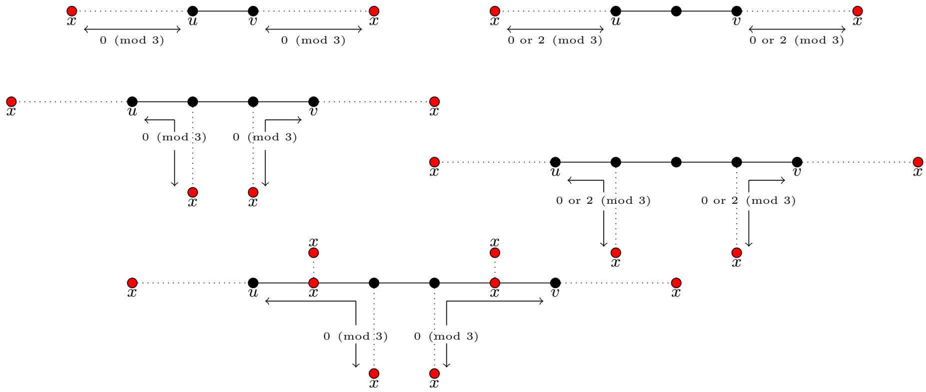

Figure 1. Resolvability conditions in T^3 depending on $d_T(u,v)$ (all possible positions of x that resolves u,v are depicted by red vertices)

Theorem 4. *Let $T=(V,E)$ be a tree. The necessary and sufficient conditions for a set $S \subset V$ to be a resolving set of T^3 are*

1. *For every edge $uv \in E(T)$, S contains at least one vertex x which is at distance 0 (mod 3) from u or v.*
2. *For every edge $uv \in E(T^2)$, S contains at least one vertex x in T_u or T_v satisfying min$\{d_T(x,u), d_T(x,v)\} \equiv 0$ or 2 (mod 3)*
3. *For every edge $uv \in E(T^3)$, S contains one vertex x either in T_u or T_v such that $|d_T(x,u) - d_T(x,v)| = 3$, otherwise min $\{d_T(x,u), d_T(x,v)\} \equiv 0$ (mod 3).*
4. *For every pair of four distance vertices u,v, S contains one vertex x either in T_u or T_v such that $|d_T(x,u) - d_T(x,v)| = 4$, otherwise min $\{d_T(x,u), d_T(x,v)\} \equiv 0$ or 2 (mod 3) when $d_T(x,u) \neq d_T(x,v)$.*
5. *For every pair of five distance vertices u,v, S contains one vertex x either in T_u or T_v such that $|d_T(x,u) - d_T(x,v)| = 5$, otherwise $|d_T(x,u) - d_T(x,v)| = 3$ or min $\{d_T(x,u), d_T(x,v)\} \equiv 0$ (mod 3).*

Proof. *Necessity:* Let S be a resolving set of T^3, and $x \in S$ resolves a pair of distinct vertices u,v. If $x \neq u,v$, then condition 1, condition 2, and condition 3 hold from Lemma 4. Also, condition 4 and condition 5 follow from Corollary 6. By triviality, all the conditions hold if $x = u$ or v.

Sufficiency: Let u,v be any two arbitrary vertices of T^3. We consider the following cases depending on their adjacency in T and prove the existence of a vertex $x \in S$ such that x resolves u,v in each case [2]. (see Figure 1 and Figure 2)

[2] *we omit the trivial case, i.e., when $x = u$ or v from rest of the part of this proof*

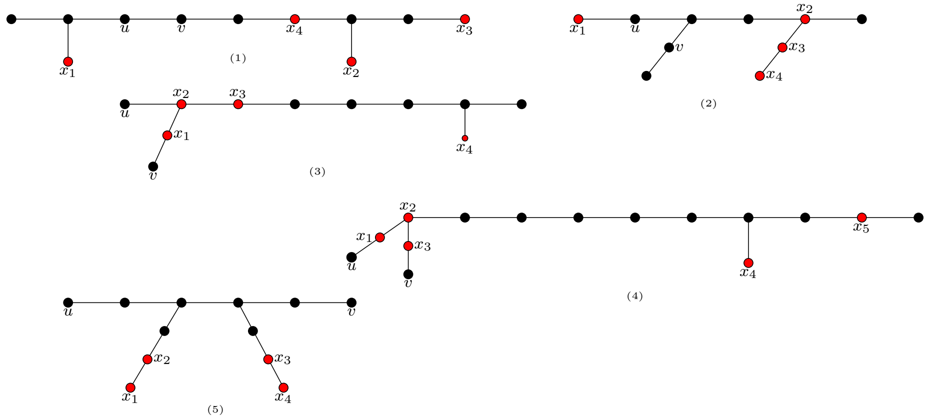

Figure 2. For the trees T_i, $1 \le i \le 5$ and S_i (set of all red vertices), all the conditions of Theorem 4 hold true except condition i, which fails for the pair u, v satisfying $d_T(u,v) = i$

Case 1. Let u and v be adjacent in T. From condition 1, for each edge uv, there exists a vertex (say x) from S such that $d_T(x,u) \equiv 0 \pmod 3$ or $d_T(x,v) \equiv 0 \pmod 3$. Without loss of generality, we assume $d_T(x,u) \equiv 0 \pmod 3$. Then we get min $\{d_T(x,u), d_T(x,v)\} = d_T(x,u) \equiv 0 \pmod 3$ and hence using Lemma 4, x resolves u and v.

Case 2. Let u and v be nonadjacent in T. Consider the following cases according to $d_T(u,v)$ is even or odd.

Subcase (2a). $d_T(u,v)$ is even.

Let $d_T(u,v) = 2$. Then, by condition 2, one can able to find some $x \in S$, which is at distance 0 or 2 (mod 3) from $uv \in E(T^2)$. Therefore, x resolves u, v follows from Lemma 4.

Let $d_T(u,v) = 4$. Then, from condition 4, we get the existence of some $x \in S$ such that $|d_T(x,u) - d_T(x,v)| = 4$ or min $\{d_T(x,u), d_T(x,v)\} \equiv 0$ or $2 \pmod 3$ when $d_T(x,u) \neq d_T(x,v)$. If $|d_T(x,u) - d_T(x,v)| = 4$, x is either in $T_u \setminus \{u\}$ or in $T_v \setminus \{v\}$. Hence, by Corollary 6, x resolves u, v in T^3. For the other case, x must be attached to some branch at s of the $u - v$ path satisfying min $\{d_T(x,u), d_T(x,v)\} \equiv 0$ or 2 (mod 3). Let $d_T(x,u) = \min\{d_T(x,u), d_T(x,v)\}$. Since $d_T(x,u) \neq d_T(x,v)$, we get $d_T(x,v) - d_T(x,u) = d_T(s,v) - d_T(s,u) = 2$. Therefore, x resolves u, v in T^3 by Lemma 5.

Next, we consider the case when $d_T(u,v) \ge 6$. Let $(u, \dots, u_1, u_0, w, v_0, v_1, \dots, v)$ be the path connecting u, v in T, where w is the middle vertex of the $u - v$ path satisfying $d_T(u,w) = d_T(w,v)$.

Consider the two vertices u_0, v_0 that occur on either side of w within the path $u - w, w - v$, respectively, satisfying $d_T(w,u_0) = d_T(w,v_0) = 1$. Applying condition 2 for the edge $u_0v_0 \in E(T^2)$ we get the existence of some $x \in S$. Without loss of generality, we assume $x \in T_{u_0}$.

If x occurs in the extended path of $u - u_0$, then by Lemma 5, x resolves u, v in T^3 as $d_T(x,v) - d_T(x,u) = d_T(u,v) \geq 6$.

Next, we consider x to be within the $u - u_0$ path or attached to some vertex s of the $u - u_0$ path. Then $d_T(x,u_0) = \min\{d_T(x,u_0), d_T(x,v_0)\} \equiv 0$ or $2 \pmod 3$. It is easy to note that $d_T(x,u) = \min\{d_T(x,u), d_T(x,v)\}$.

a) Let $d_T(x,u_0) \equiv 0 \pmod 3$.

Let $d_T(x,u) = d_T(x,s) + d_T(s,u) = d_T(x,u_0) - d_T(s,u_0) + d_T(s,u) = 3k - d_T(s,u_0) + d_T(s,u)$ for some integer $k \geq 0$. Then $d_T(x,v) = d_T(x,u_0) + d_T(u_0,w) + d_T(w,v) = 3k + 1 + (d_T(u,s) + d_T(s,u_0) + 1)$ as $d_T(w,v) = d_T(u,w)$. Therefore, $d_T(x,v) - d_T(x,u) = 2 + 2d_T(s,u_0) \geq 4$ when $d_T(s,u_0) \geq 1$. Hence, by Lemma 5, it follows that x resolves u, v. If $d_T(s,u_0) = 0$, then also x resolves u, v if $d_T(x,u) \equiv 0$ or $2 \pmod 3$ as $d_T(x,v) - d_T(x,u) = 2$.

Hence the case remains when $d_T(s,u_0) = 0$ (i.e., $s = u_0$) and $d_T(x,u) \equiv 1 \pmod 3$. Since $d_T(x,u_0) \equiv 0 \pmod 3$, we have $d_T(u,u_0) = d_T(v,v_0) \equiv 1 \pmod 3$. Therefore, $d_T(u,v) = d_T(u,u_0) + d_T(u_0,v_0) + d_T(v_0,v) \equiv (1+2+1) \pmod 3 \equiv 1 \pmod 3$. Since the distance between u and v is even, $d_T(u,v) \geq 10$.

Now consider the two vertices u_1, v_1 on either side of w satisfying $d_T(w,u_1) = d_T(w,v_1) = 2$. Then, applying condition 4 on the four distance vertices u_1, v_1, we get the existence of some $y \in S$. Two cases may arise here.

i) If y lies in any extended branch of $u_1 - u$, then by Lemma 5, it follows that y resolves u, v in T^3. Again, if y is attached to some vertex s of the $u_1 - u$ path or lies within the $u_1 - u$ path (i.e., $y = s$), then $d_T(y,v) - d_T(y,u) = (d_T(y,u_1) + d_T(u_1,v_1) + d_T(v_1,v)) - (d_T(y,u_1) + d_T(u,u_1) - 2d_T(s,u_1)) = 2d_T(s,u_1) + 4 \geq 4$ as $d_T(u,u_1) = d_T(v,v_1)$. Hence, applying Lemma 5, it is easy to conclude that y resolves u, v in T^3.

Similar logic follows if y is attached to some intermediate vertex of the $v_1 - v$ path or lies within or in the extended path of $v_1 - v$.

ii) If y is attached to u_0 satisfying $d_T(y,u_1) = \min\{d_T(y,u_1), d_T(y,v_1)\} \equiv 0$ or $2 \pmod 3$. Then $d_T(y,v) - d_T(y,u) = (d_T(y,u_0) + d_T(u_0,v_1) + d_T(v_1,v)) - (d_T(y,u_1) + d_T(u_1,u)) = (d_T(y,u_1) - 1) + 3 - d_T(y,u_1) = 2$ as $d_T(u,u_1) = d_T(v,v_1)$ and $\min\{d_T(y,u), d_T(y,v)\} = d_T(y,u) = d_T(y,u_1) + d_T(u_1,u) \equiv 0$ or $2 \pmod 3$ as $d_T(u,u_1) = d_T(u,u_0) - d_T(u_0,u_1) \equiv 0 \pmod 3$. Therefore, by Lemma 5, it follows that y resolves u, v in T^3.

Similarly, one can show that if y is attached to v_0 satisfying $d_T(y,v_1) = \min\{d_T(y,v_1), d_T(y,u_1)\} \equiv 0$ or $2 \pmod 3$, then y resolves u, v.

b) Let $d_T(x,u_0) \equiv 2 \pmod 3$. Then, analogous to the previous case, one can show that $d_T(x,v) - d_T(x,u) \geq 4$ if $d_T(s,u_0) \geq 1$. Therefore, x resolves u, v by Lemma 5.

Again, when $d_T(s,u_0)=0$, then $d_T(x,v)-d_T(x,u)=2$, therefore, if $d_T(x,u)\equiv 0$ or 2 (mod 3), then x resolves u,v.

Hence, the only case remains when $s=u_0$ and $d_T(x,u)\equiv 1 \pmod 3$. Then $d_T(u,u_0)=d_T(v,v_0)\equiv 2 \pmod 3$ as $d_T(x,u_0)\equiv 2 \pmod 3$. Hence $d_T(u,v)=d_T(u,u_0)+d_T(u_0,v_0)+d_T(v_0,v)\equiv 0 \pmod 3$.

Now consider two neighbours of u_0, one is u_1 on the path $u-u_0$, and another (say p_0) is on the path u_0-x. Then $d_T(u_1,p_0)=d_T(u_1,u_0)+d_T(u_0,p_0)=2$. Clearly, $d_T(u,u_1)\equiv 1 \pmod 3$. Applying condition 2 on u_1,p_0, we get the existence of some $y\in S$. Three cases may arise here.

i) If y occurs in an extended path from p_0, then min $\{d_T(y,p_0),d_T(y,u_1)\}=d_T(y,p_0)\equiv 0$ or 2 (mod 3). Then $d_T(y,v)=d_T(y,p_0)+d_T(p_0,u_0)+d_T(u_0,v_0)+d_T(v_0,v)$ and $d_T(y,u)=d_T(y,p_0)+d_T(p_0,u_0)+d_T(u_0,u)$. Therefore $d_T(y,v)-d_T(y,u)=2$ and min $\{d_T(y,v),d_T(y,u)\}=d_T(y,u)=d_T(y,p_0)+d_T(p_0,u_1)+d_T(u_1,u)\equiv 0$ or 2 (mod 3). Hence, y resolves u,v in T^3 by Lemma 5.

ii) If y occurs in the intermediate path of $u-u_0$, then $d_T(y,v)=d_T(y,u_0)+d_T(u_0,v_0)+d_T(v_0,v)$ and $d_T(y,u)=d_T(u,u_0)-d_T(y,u_0)$. Therefore, $d_T(y,v)-d_T(y,u)=2d_T(y,u_0)+2>3$ clearly. Hence, by Lemma 5, we can conclude that y resolves u,v in T^3.

iii) If y occurs in the extended path from $u-u_0$, then it also resolves u,v in T^3 by Lemma 5.

Subcase (2b). $d_T(u,v)$ is odd. Let $d_T(u,v)=2m+1$ for some positive integer $m\geq 1$.

When $m=1$, then $d_T(u,v)=3$. From condition 3, either there exists a $x\in S$ such that $|d_T(x,v)-d_T(x,u)|=3$ (i.e., $x\in T_u$ or T_v) or $\min\{d_T(x,u),d_T(x,v)\}\equiv 0$ (mod 3) and hence from Lemma 4 the result follows. Again, when $m=2$, i.e., $d_T(u,v)=5$. Let (u,u_0,w_1,w_2,v_0,v) be the path between u,v in T. Then, from condition 5, there exists a $x\in S$ either coming from T_u or T_v satisfying $|d_T(x,v)-d_T(x,u)|=5$, otherwise $|d_T(x,v)-d_T(x,u)|=3$ or min $\{d_T(x,u),d_T(x,v)\}=0 \pmod 3$. If $|d_T(x,v)-d_T(x,u)|=5$ or 3, then by Lemma 5, x resolves u,v in T^3. In the other case, when $\min\{d_T(x,u),d_T(x,v)\}=0 \pmod 3$ and x is in the same component $T_{u,v}$ of u,v, it must be attached to the vertex w_1 or w_2 satisfying $|d_T(x,v)-d_T(x,u)|=1$. Hence, by Lemma 5, x resolves u,v in T^3 .

Next, we consider the case when $m\geq 3$, i.e., $d_T(u,v)\geq 7$. We consider the $u-v$ path as $(u,\ldots,u_1,u_0,w_1,w_2,$
$v_0,v_1,\ldots,v)$ where $d_T(u,w_1)=d_T(v,w_2)=m, d_T(u_1,v_1)=5$.

a) Let $m\equiv 0 \pmod 3$.

Applying condition 1 for the edge w_1w_2, we get the existence of a vertex $x\in S$. Without loss of generality, we assume $x\in T_{w_1}$. Then $\min\{d_T(x,w_1),d_T(x,w_2)\}=$

$d_T(x, w_1) \equiv 0 \pmod 3$.

If x occurs in the extended path of $u - w_1$, then $d_T(x, v) - d_T(x, u) = d_T(u, v) \geq 7$. Again, if x occurs in a branch attached to some vertex s within the path $u - w_1$, then $d_T(x, v) = d_T(x, s) + d_T(s, w_1) + d_T(w_1, w_2) + d_T(w_2, v)$ and $d_T(x, u) = d_T(x, s) + d_T(s, u) = d_T(x, s) + d_T(u, w_1) - d_T(s, w_1)$. Therefore, $d_T(x, v) - d_T(x, u) = 2d_T(s, w_1) + 1 \geq 3$ when $d_T(s, w_1) \geq 1$. Again, if $d_T(s, w_1) = 0$, i.e., when $s = w_1$, we get $d_T(x, u) = d_T(x, w_1) + d_T(w_1, u) \equiv 0 \pmod 3$ and $d_T(x, v) - d_T(x, u) = 1$. Therefore, by Lemma 5, x resolves u, v in T^3 for the above cases.

b) Let $m \equiv 2 \pmod 3$.

It is easy to note that $d_T(u, u_1) \equiv 0 \pmod 3$ in this case. Applying condition 5 to the vertices u_1, v_1, we get to know the existence of a $x \in S$. Without loss of generality, we assume $x \in T_{w_1}$. Then $d_T(x, u_1) = \min\{d_T(x, u_1), d_T(x, v_1)\}$ and hence $d_T(x, u) = \min\{d_T(x, u), d_T(x, v)\}$.

When $|d_T(x, u_1) - d_T(x, v_1)| = 5$, then x is in the extended path of $u_1 - w_1$. Then $|d_T(x, v) - d_T(x, u)| = 5 \geq 3$. If $|d_T(x, u_1) - d_T(x, v_1)| = 3$, then $x = u_0$ or x is on a branch attached to u_0 as $x \in T_{w_1}$. Therefore $d_T(x, v) - d_T(x, u) = 3$. Again, if x is attached to w_1 satisfying $d_T(x, u_1) \equiv 0 \pmod 3$. Then $d_T(x, v) = d_T(x, w_1) + d_T(w_1, w_2) + d_T(w_2, v)$, $d_T(x, u) = d_T(x, w_1) + d_T(w_1, u)$, and therefore $d_T(x, v) - d_T(x, u) = 1$. Moreover, $d_T(x, u) = d_T(x, u_1) + d_T(u_1, u) \equiv 0 \pmod 3$ in this situation. Hence, by Lemma 5, x resolves u, v in T^3.

c) Let $m \equiv 1 \pmod 3$.

In this case, $d_T(u, u_0) \equiv 0 \pmod 3$. Since $d_T(u_0, v_0) = 3$, applying condition 3 on the edge u_0v_0 we get the existence of some $x \in S$. Without loss of generality, we assume $x \in T_{w_1}$. Therefore $d_T(x, u) = \min\{d_T(x, u), d_T(x, v)\}$. Now if x occurs in the extended path of $u - u_0$ or is attached to some intermediate vertex of the path $u - u_0$, then $|d_T(x, v) - d_T(x, u)| \geq 3$, therefore by Lemma 5, x resolves u, v in T^3.

Therefore, the case remains when x is attached to a branch at w_1 satisfying min $\{d_T(x, u_0), d_T(x, v_0)\} = d_T(x, u_0) \equiv 0 \pmod 3$. In this case, we have $d_T(x, v) - d_T(x, u) = (d_T(x, w_1) + d_T(w_1, w_2) + d_T(w_2, v)) - (d_T(x, w_1) + d_T(w_1, u)) = 1$ as $d_T(w_2, v) = d_T(w_1, u)$. Furthermore, we get $d_T(x, u) = \min\{d_T(x, u), d_T(x, v)\} = d_T(x, u_0) + d_T(u_0, u) \equiv 0 \pmod 3$. Hence, by Lemma 5, it follows that x resolves u, v in T^3.

Thus, we prove that S is a resolving set of T^3. □

5. Lower bound on the metric dimension of T^3

In this section, for a given tree T, we determine the lower bound on $\beta(T^3)$.

Lemma 6. *Let $T = (V, E)$ be a tree, and v_0 be a major stem of T containing n_0 legs. Then any metric basis of T^3 must contain $n_0 + m_0 - 2$ number of vertices from the legs of v_0, where $m_0 \geq 1$ is the number of mid legs attached to v_0.*

Proof. Let S be an arbitrary metric basis of T^3. Let $B(v_0)$ be the set of all leg vertices[3] corresponding to v_0 in T and $n_0 = p_0 + m_0 + l_0$, where p_0, m_0 and l_0 denote the number of pendants, mid legs, and long legs, respectively.

Consider any arbitrary pair of vertices $\{u, v\} \subset B(v_0)$ satisfying $d_T(v_0, u) = d_T(v_0, v)$. Now if both u, v are on short legs, then $d_T(u, v) = 2$ or 4. One can verify from Theorem 4 that no vertex $w \neq u, v$ can resolve them in T^3. Therefore, it is necessary to include at least $p_0 + 2m_0 - 2$ vertices in S when $m_0 \geq 1$. Since $|S|$ is minimum in comparison to any resolving set of T^3, there will always be a pair of vertices $\{a_0, b_0\} \subset B(v_0)$ satisfying $d_T(v_0, a_0) = 1, d_T(v_0, b_0) = 2$, left aside from vertex selection while constructing S coming from short legs when $m_0 \geq 1$.

Consider a long leg L attached to v_0, and let $\{x, y\} \subset B(v_0)$ be the pair of vertices on L satisfying $d_T(v_0, x) = 1$ and $d_T(v_0, y) = 2$, respectively. We consider the pair of vertices $\{a_0, x\}, \{b_0, y\}$. Clearly, $d_T(x, a_0) = 2$ and $d_T(y, b_0) = 4$. Now to resolve any of the above pairs and keep $|S|$ to be minimum, it is necessary to include one vertex z from L satisfying $d_T(x, z) \equiv 0$ or $2 \pmod 3$ by Theorem 4. It can be noted that any $z \neq x, y$ on L satisfying $d_T(v_0, z) \equiv 0$ or $1 \pmod 3$ will work. Since there are l_0 long legs attached to v_0, applying similar logic, it is necessary to include l_0 vertices in S from each of the long legs. Hence, the total number of vertex insertions necessary for constructing any metric basis S of T^3 is $p_0 + 2m_0 - 2 + l_0 = (p_0 + m_0 + l_0 - 1) + (m_0 - 1) = (n_0 - 1) + (m_0 - 1) = n_0 + m_0 - 2$. □

Theorem 5. *Let $T = (V, E)$ be a tree. Then*

$$\beta(T^3) \geq \beta(T) + \sum_{i=1, m_i \geq 1}^{k} m_i - k,$$

where k is the total number of major stems of T containing at least one mid leg and m_i denotes the number of mid legs attached to the major stem v_i, $1 \leq i \leq k$.

Proof. Let S be a resolving set of T^3. Then, by Lemma 1, it is also a resolving set of T. Let V' be the set of all major stems of T, and each $v_i \in V'$ contains n_i legs, $1 \leq i \leq |V'|$. From Lemma 6, we get to know that while constructing any metric basis of T^3, we necessarily need to insert $n_i + m_i - 2$ number of vertices from the legs of v_i where $m_i \geq 1$ and the number is $n_i - 1$ for the remaining major stems (where $m_i = 0$) from Corollary 3. Therefore, $|S| \geq \sum_{i=1}^{k}(n_i + m_i - 2) + \sum_{j=1}^{|V'|-k}(n_j - 1)$. This holds for every resolving set S of T^3, hence we get $\beta(T^3) \geq \sum_{i=1}^{|V'|}(n_i - 1) + \sum_{i=1}^{k}(m_i - 1) =$

[3] *vertices that are along the legs attached to some common major stem*

$\beta(T) + \sum\limits_{i=1, m_i \geq 1}^{k} m_i - k$ using Corollary 1. $\square$

Corollary 7. *Let $T = (V, E)$ be a tree. Then*

$$\beta(T^3) \geq \beta(T) + \sum_{i=1, m_i \geq 2}^{l} m_i - l$$

where l is the number of major stems of T containing at least two mid legs and m_i is the number of mid legs attached to the major stem $v_i, 1 \leq i \leq l$.

6. Upper bound on the metric dimension of T^3

In the following theorem, for a given tree T, we determine the upper bound on $\beta(T^3)$.

Theorem 6. *Let $T = (V, E)$ be a tree. Then*

$$\beta(T^3) \leq \beta(T) + \sum_{i=1, m_i \geq 2}^{l} (m_i - 1) + M + 1 - l$$

where M is the total number of major stems and l is the number of major stems containing at least two mid legs, and m_i denotes the number of mid legs attached to the major stem v_i where $1 \leq i \leq l$.

Proof. Let V' be the set of all major stems of T, and hence $|V'| = M$. Let p_v, m_v and l_v denote the number of pendants, mid legs, and long legs attached to an arbitrary major stem $v \in V'$ and $B(v)$ be the set of all leg vertices corresponding to v in T. We denote $B[v] = B(v) \cup \{v\}$. Now, depending on the number of different types of legs attached to each major stem, we build a resolving set S for T^3 in the following way:

Construction of S:

1) $\mathbf{m_v \geq 1}$

We choose all the vertices from every mid leg in S, leaving one mid leg aside as unpicked. Now if $l_v \geq 1$, we pick the vertex from each long leg, which is at a distance of three from v in $B(v)$. Again, if $p_v \geq 1$, then we include all the pendants of $B(v)$ in S.

2) $\mathbf{p_v \geq 1, m_v = 0}$

Except for one pendant, we choose all the pendants of $B(v)$ in S. Also, we include all distance three vertices of $B(v)$ that occur along long legs when $l_v \geq 1$.

3) $\mathbf{p_v = m_v = 0}$

It is easy to note that $l_v \geq 1$ as v is a major stem. In this case, except for one long leg, we include all vertices that are at a distance of 3 from v along long legs in S.

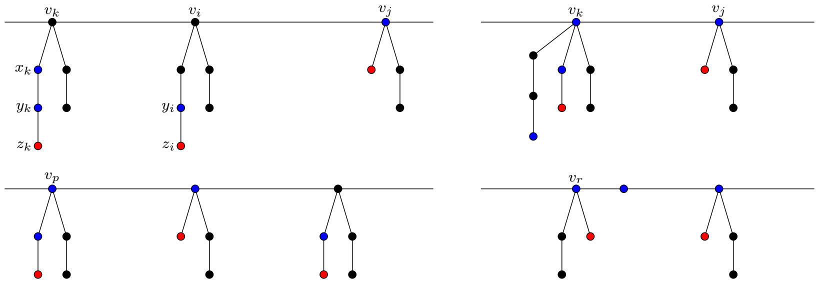

Figure 3. Tree T having red vertices as elements of a metric basis of it, blue vertices are extra inserted to form a metric basis S of T^3, above (left and right) figures correspond to the situation when T contains at least one major stem, and below figures indicate the situation when there is no major stem containing long legs in T

As per our above construction, $\beta(T)+\sum_{i=1,m_i\geq 2}^{l}(m_i-1)$ number of vertices has already been included in S. We now insert $M+1-l$ extra vertices in S. But this insertion of vertices depends on some circumstances listed below.

Method of insertion of $M+1-l$ extra vertices in S:

a) First, we consider the case when there is at least one major stem containing long legs in T. (see Figure 3)

i) If there is at least one major stem (say v_k) containing long legs satisfying $m_{v_k}\leq 1$, then we select a long leg (say L_k) attached to v_k from which the vertex z_k satisfying $d_T(v_k,z_k)=3$ has already been included in S. Next, we pick x_k, y_k from L_k satisfying $d_T(v_k,x_k)=1, d_T(v_k,y_k)=2$, and include them in S.

Now if $v_i\neq v_k$ be a major stem possessing long legs satisfying $m_{v_i}\leq 1$, then we select a long leg L_i of v_i from where z_i is already chosen for S satisfying $d_T(v_i,z_i)=3$. We pick y_i from L_i satisfying $d_T(v_i,y_i)=2$ and insert in S.

Also, we include all those major stems v_j in S for which $l_{v_j}=0$ and $m_{v_j}\leq 1$.

ii) If every major stem that contains at least one long leg also satisfies $m_v\geq 2$, then we insert one such major stem (say v_k) in S. We also insert all those major stems v_j in S that satisfy $l_{v_j}=0$ and $m_{v_j}\leq 1$.

b) Next, we consider the case when there is no major stem containing long legs in T.

If there exists at least one major stem v_p satisfying $m_{v_p}\geq 2$, then include v_p in S, otherwise, we include a neighbour of an arbitrary major stem v_r in S, which does not belong to $B(v_r)$. We also include all those major stems in S that contain at most one mid leg attached to them.

Therefore, the maximum number of extra vertex insertions in the aforementioned scenarios is $M+1-l$.

Proof showing that S is a resolving set of T^3:

We now show that S resolves every pair of vertices $u, v \in V \setminus S$. For this, it is sufficient to prove for the cases when $d_T(u,v) \leq 5$ as per Theorem 4. Recall that, in T, there always exists a unique path joining any two vertices. From the construction of S, one can observe that there always exists a major stem v_1 (say) ($v_1 = v_k$ or v_p or v_r in Figure 3) having three consecutive vertices of $B[v_1]$ (or two vertices from $B[v_1]$ and one is the neighbour of v_1 that does not belong to $B[v_1]$) and all other major stems having two consecutive vertices from their legs included in S that occur in the extended path of $u - v$ (i.e., in T_u or T_v) or within the same component of u, v (i.e., in $T_{u,v}$), then using Corollary 5 and Corollary 6, u, v can be resolved by one of these leg vertices that has been selected for S. $\square$

Theorem 7. *For any positive integer n, there always exists a tree T satisfying $\beta(T^3) = n$.*

Proof. First, we consider the situation when n is even. For this, we consider a tree T (see Figure 4) having $M = \frac{n}{2} - 1$ number of major stems. Here each of the two major stems v_0, v_1 contains exactly two mid legs satisfying $d_T(v_0, v_1) \equiv 0 \pmod 3$, other $k = M - 2 = \frac{n}{2} - 3$ major stems $w_i, 1 \leq i \leq k$, contain pendants as their only legs, where $d_T(v_0, w_i) \equiv 1 \pmod 3$. Furthermore, we consider $k - 1$ of these major stems to contain exactly two pendants and one among them to contain exactly three pendants. From Theorem 3, it is clear that the metric dimension of T, i.e., $\beta(T) = \frac{n}{2}$.

Below, we construct a resolving set S of T^3. Since T contains exactly two major stems having two mid legs, from Theorem 5 it follows that we need to insert at least two more vertices from these mid legs in S. Also, we need to include k $(= \frac{n}{2} - 3)$ more vertices in S to resolve the following pair of vertices $\{u_1, w_1\}$, $\{u_2, w_2\}, \ldots, \{u_k, w_k\}$ in T^3. We insert $w_1, w_2, \ldots, w_k$ in S. Again, no vertex of S inserted so far can resolve the vertices v_0, x_0 in T^3, therefore, we include one more vertex v_0 in S. One can verify that by applying Theorem 4, S becomes a resolving set of T^3. Furthermore, $|S| \geq \beta(T) + 2 + k + 1 = \frac{n}{2} + 3 + \frac{n}{2} - 3 = n$. From Theorem 6, it follows that $\beta(T^3) \leq \beta(T) + \sum\limits_{i=1, m_i \geq 2}^{l} (m_i - 1) + M + 1 - l = \frac{n}{2} + 2 + (\frac{n}{2} - 1) + 1 - 2 = n$. Therefore, $\beta(T^3) = n$ and hence S becomes a metric basis of T^3.

Next, we consider the case when n is odd. Then we consider a tree T (see Figure 4) having $M = \frac{n-1}{2}$ number of major stems, where each of the two major stems v_0, v_1 contains exactly two mid legs satisfying $d_T(v_0, v_1) \equiv 0 \pmod 3$ and the other $k = M - 2 = \frac{n-5}{2}$ number of major stems $w_i, 1 \leq i \leq k$, contain two pendants each satisfying $d_T(v_0, w_i) \equiv 1 \pmod 3$. Proceeding similarly as above, one can verify that $\beta(T) = \frac{n-1}{2}$ and a minimum resolving set S of T^3 contains exactly n vertices, hence $\beta(T^3) = n$. $\square$

The following corollary is immediate from the proof of the above theorem:

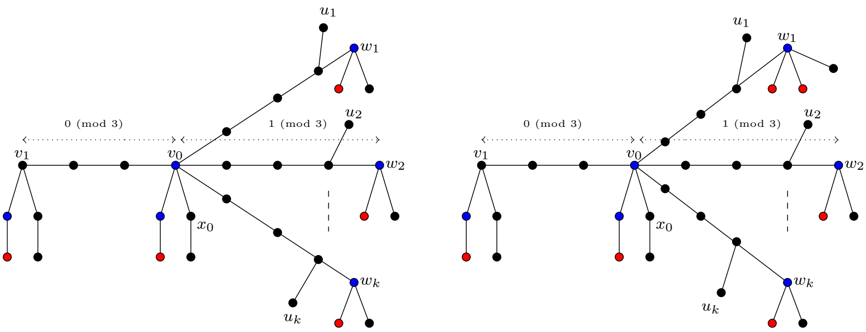

Figure 4. Tree T with $\beta(T^3) = n$ (left when n is odd, right when n is even) where the red vertices form the metric basis of T and the blue vertices are extra inserted to form a metric basis of T^3

Corollary 8. *For a tree T, there always exists a family of trees attaining every value between the lower and upper bounds on $\beta(T^3)$.*

7. Metric dimension of some well-known cube of trees

In this section, we present some well-known cube of trees (e.g., caterpillar, lobster tree, spider tree, and d-regular tree) that have attained the expected bounds on the metric dimension.

Let P be the central path [4] of caterpillar/lobster, and v_0, v_n be the starting and ending major stems on P. The total number of major stems of any of the trees above-mentioned containing at least two mid legs is denoted by l. On the other hand, m_i denotes the number of mid legs attached to the major stem v_i, where $1 \leq i \leq l$. Below, we construct the resolving sets S_0 and S of T and T^3, respectively. In each of the figures in this section, the red vertices form S_0. One can verify that such choices can be made by Corollary 1. Furthermore, S can be obtained by inserting the blue vertices in S_0. Following Theorem 4, it can be verified that S resolves any two arbitrary vertices of V. One can find the lower and upper bounds on $\beta(T^3)$ by applying Corollary 7 and Theorem 6, respectively.

Example 1. Let $T = (V, E)$ be a caterpillar. It is easy to observe that there can not be any mid leg (or long leg) attached to any stem except v_0 or v_n. Furthermore, if there is any mid leg or long leg attached to v_0 or v_n, then that should be one in number. Also, no long leg and mid leg can occur simultaneously at v_0 or v_n. Again, while constructing S, first we consider that v_0 contains a long leg (or mid leg) attached to it. A similar choice of vertices can be made for S if v_n contains a long leg (or mid leg) attached to v_n and v_0 contains only pendants. Another case remains when v_0, v_n contain only pendants attached to them. Hence, $\beta(T) \leq \beta(T^3) \leq \beta(T) + 3$. (see Figure 5)

[4] *longest path between any two pendant vertices of a tree*

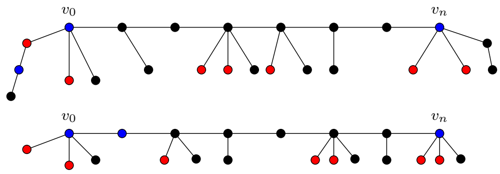

Figure 5. caterpillar

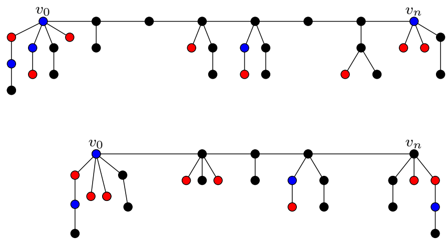

Figure 6. Lobster

Example 2. Let T be a lobster tree. Then either v_0 or v_n, or both of them, only contain a single long leg, and the other major stems contain only mid legs and pendants. In this case, $\beta(T) + \sum_{i=1, m_i \geq 2}^{l} m_i - l \leq \beta(T^3) \leq \beta(T) + \sum_{i=1, m_i \geq 2}^{l} m_i - l + 3$. (see Figure 6)

Example 3. Let T be a spider tree. If it is a star, then $\beta(T^3) = \beta(T) + 1$, otherwise, we have $\beta(T) + \sum_{i=1, m_i \geq 2}^{l} m_i - l \leq \beta(T^3) \leq \beta(T) + \sum_{i=1, m_i \geq 2}^{l} m_i - l + 2$. (see Figure 7)

In a d-regular tree T, only pendants can be attached to every major stem. Let the length of a central path P in T is $2t$, where t is the depth of T. Then the total number of pendants in T is $d(d-1)^{t-1}$.

Example 4. Let T be a d-regular tree ($d \geq 3$) with depth t. If $t \leq 2$, then $\beta(T) \leq \beta(T^3) \leq \beta(T) + d$ and for $t \geq 3$, $\beta(T) \leq \beta(T^3) \leq \beta(T) + d(d-1)^{t-3}(d-2)$. (see Figure 8)

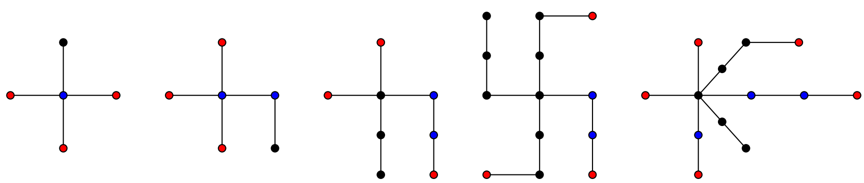

Figure 7. Spider

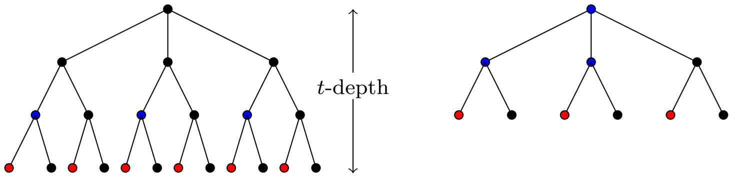

Figure 8. **A d-regular tree**

8. Characterization of some restricted T^3 satisfying $\beta(T^3) = \beta(T)$

Proposition 1. *Let $T = (V, E)$ be a tree having at least two major stems. If $\beta(T^3) = \beta(T)$, then there exists at least one pair of major stems v_i, v_j satisfying $d_T(v_i, v_j) \equiv 1 \; or \; 2 \; (mod\, 3)$.*

Proof. On the contrary, let every pair of major stems have their distances as 0 (mod 3). Using Theorem 3, we observe that since $\beta(T^3) = \beta(T)$, except for one, from all the legs of every major stem of T, we can pick at most one vertex for the metric basis of T^3. Hence, from Theorem 5, it can be easily verified that the number of mid legs attached to any major stem is at most one.

Claim 1. *To choose vertices for a metric basis S of T^3, if we select a vertex from a long leg (or a mid leg) attached to any major stem v (say), it is mandatory to choose the vertex that is at a distance 0 or 1 (mod 3) from v on the same leg.*

Proof. If we select a vertex (say y) in S from a long leg/mid leg L attached to the major stem v satisfying $d_T(v, y) \equiv 2$ (mod 3), then the vertices x and x' will possess the same code in T^3 measured from y, z where x, x' are two neighbours of v on the legs L, L', respectively, where L' is the leg that is left aside from vertex selection for S and z is a vertex from any branch of v apart from L and L'. Hence $d_T(v, y) \equiv 0$ or 1 (mod 3). □

Using Claim 1, we construct a vertex subset S of V by inserting a vertex from each leg (apart from one) of all the major stems that are at a distance of 0 or 1 (mod 3) from the major stems.

Claim 2. *There will always remain at least one pair of vertices in T^3 that can not be resolved by any vertex of S.*

Proof. Let v_1 and v_2 be two major stems satisfying $d_T(v_1, v_2) = 3m$ for some integer m. Now consider the vertices u_0, v_0 of an edge $e(= u_0v_0) \in E$ on the intermediate path joining the vertices v_1, v_2 in T so that $d_T(v_1, u_0) \equiv 1$ (mod 3) and $d_T(v_2, v_0) \equiv 1$

(mod 3). Using the result of Claim 1, one can verify that there is no vertex x coming from the legs of v_1, v_2, which can resolve u_0, v_0 as $\min\{d_T(x,u_0), d_T(x,v_0)\} \equiv 1$ or 2 (mod 3). Similarly, it can be verified that u_0, v_0 can not be resolved in T^3 by any x coming from the legs of some other major stems that occur in the extended path of v_1 or v_2 as $d_T(v_i, v_j) \equiv 0 \pmod 3$ for all $v_i \neq v_j$.

Now, we show that u_0 and v_0 can not be resolved by any vertex x that comes from the leg of a major stem v_3 connected to an intermediate vertex s of the path joining v_1 and v_2. For this, first, we consider the case when $s = u_0$ or v_0. Without loss of generality, if $v_0 = s$, then $d_T(v_3, s) \equiv 2 \pmod 3$ as $d_T(v_3, v_2) \equiv 0 \pmod 3$. Therefore, $d_T(v_1, v_3) = d_T(v_1, u_0) + d_T(u_0, v_0) + d_T(v_0, v_3) \equiv 1 + 1 + 2 \pmod 3 \equiv 1 \pmod 3$. This introduces a contradiction. Next, we consider the case when $s \neq u_0, v_0$. Without loss of generality, we assume $\min\{d_T(v_3, v_0), d_T(v_3, u_0)\} = d_T(v_3, v_0)$. Therefore, s must lie within the intermediate path of $v_0 - v_2$.

If $d_T(v_3, s) \equiv 1 \pmod 3$, then $d_T(s, v_2) \equiv 2 \pmod 3$ as $d_T(v_2, v_3) \equiv 0 \pmod 3$. Hence, $d_T(v_0, s) = d_T(v_0, v_2) - d_T(s, v_2) \equiv 2 \pmod 3$. Therefore, $d_T(v_1, v_3) = d_T(v_1, u_0) + d_T(u_0, v_0) + d_T(v_0, s) + d_T(s, v_3) \equiv 1 + 1 + 2 + 1 \pmod 3 \equiv 2 \pmod 3$, which is not true as per our assumption.

If $d_T(v_3, s) \equiv 2 \pmod 3$, then we get $d_T(v_1, v_3) \equiv 1 \pmod 3$, therefore, a similar contradiction arises.

If $d_T(v_3, s) \equiv 0 \pmod 3$, then $d_T(v_0, s) = d_T(v_0, v_2) - d_T(s, v_2) = 1 - 0 \pmod 3 \equiv 1 \pmod 3$. Hence, $d_T(v_0, v_3) = d_T(v_0, s) + d_T(s, v_3) = 1 + 0 \pmod 3 \equiv 1 \pmod 3$.

Therefore, following Lemma 4, one can verify that u_0, v_0 can not be resolved by any vertex $x \in S$ coming from the legs of v_3 as $d_T(x, v_3) \equiv 0$ or 1 (mod 3) from Claim 1 implies $d_T(x, v_0) \equiv 1$ or 2 (mod 3). □

Since $\beta(T^3) = \beta(T)$, any metric basis of T^3 can only be constructed in the above way, as we did for S. But from Claim 2 we will always get a pair of vertices in T^3 that can not be resolved by any vertex of S. Hence, we get a contradiction. Therefore, there always exists a pair of major stems (say $\{v_i, v_j\}$) satisfying $d_T(v_i, v_j) \equiv 1$ or 2 (mod 3). □

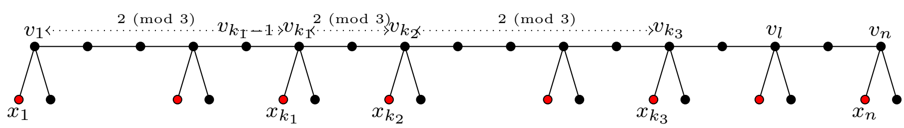

Figure 9. **Trees satisfying $\beta(T^3) = \beta(T)$ (set of all red vertices forms a metric basis of T^3)**

Below, we characterize those cube of trees that possess all their stems on one of their central paths [5], stems contain pendants only as their legs and have their metric

[5] *also known as diametral paths*

dimension similar to the metric dimension of their associated trees.

Theorem 8. *Let $T = (V, E)$ be a tree where every stem contains pendants only as their legs in T. If all the stems lie on a central path, $P = (x_1, v_1, v_2, \dots, v_n, x_n)$ [6] of T (see Figure 9), then $\beta(T^3) = \beta(T)$ if and only if the following conditions are satisfied:*

1. *There are at least three major stems, $v_{k_1}, v_{k_2}, v_{k_3}$ between v_1 and v_n along P such that $d_T(v_1, v_{k_1}) \equiv 2 \pmod 3$, $d_T(v_1, v_{k_2}) \equiv 1 \pmod 3$, $d_T(v_1, v_{k_3}) \equiv 0 \pmod 3$. Distances between any of the above pairs considered to be minimum satisfying the above criteria.*

2. *There does not exist any stem between v_1 and v_{k_1} that is at $1 \pmod 3$ distance from v_1 along P.*

3. *If $d_T(v_p, v_n) \equiv 1 \pmod 3$ for some stem $v_p < v_n$ [7] along P, then there must exist some major stem v_l satisfying $v_p < v_l < v_n$ such that $d_T(v_l, v_n) \equiv 2 \pmod 3$.*

Proof. *Necessity:* Let $\beta(T^3) = \beta(T)$. Then any metric basis of T^3 must contain one except all the pendants attached to every major stem of the tree T by Corollary 3. Also, if S is a metric basis of T^3, then all the conditions of Theorem 4 must hold.

Proof of condition 1 *and* 2. To resolve v_1 and x_1, there must exist a pendant $x_{k_1} \in S$ such that $d_T(v_1, x_{k_1}) \equiv 0 \pmod 3$ by condition 1 of Theorem 4. Hence, without loss of generality, we choose v_{k_1} to be the minimum distance major stem from v_1 satisfying $d_T(v_1, v_{k_1}) \equiv 2 \pmod 3$. Let v_{k_1-1} be the neighbour of the major stem v_{k_1} satisfying $v_1 < v_{k_1-1} < v_{k_1}$ along P. Now to resolve v_{k_1-1}, v_{k_1}, we need a major stem v_{k_2} and its pendant $x_{k_2} \in S$ such that $\min\{d_T(v_{k_1}, x_{k_2}), d_T(v_{k_1-1}, x_{k_2})\} \equiv 0 \pmod 3$ from condition 1 of Theorem 4. If $v_1 < v_{k_2} < v_{k_1}$ on P, then $d_T(v_1, v_{k_2}) = d_T(v_1, v_{k_1}) - (d_T(v_{k_1}, v_{k_1-1}) + d_T(v_{k_1-1}, v_{k_2})) \equiv 2 \pmod 3$, which is not possible by the choice of v_{k_1}. Therefore, $v_{k_2} > v_{k_1}$ on P. We consider v_{k_2} to be the minimum distance major stem from v_{k_1} satisfying $d_T(v_{k_1}, v_{k_2}) \equiv d_T(v_1, v_{k_1}) + d_T(v_{k_1}, v_{k_2}) \equiv 2 \pmod 3$.

Let there be a stem v_m between v_1 and v_{k_1} satisfying $d_T(v_1, v_m) \equiv 1 \pmod 3$, and let x_m be the pendant of v_m which is not in S. Then, to resolve x_m, v_{m-1}, there must exist a pendant $x_r \in S$ attached to some major stem v_r within v_1, v_m along P such that $d_T(x_r, v_{m-1}) \equiv 0$ or $2 \pmod 3$ by condition 2 of Theorem 4. Therefore, $d_T(v_r, v_{m-1}) \equiv 1$ or $2 \pmod 3$. If $d_T(v_r, v_{m-1}) \equiv 1 \pmod 3$, we get $d_T(v_1, v_r) = d_T(v_1, v_m) - (d_T(v_r, v_{m-1}) + d_T(v_{m-1}, v_m)) \equiv 2 \pmod 3$, which contradicts the choice of v_{k_1}. Again, if $d_T(v_r, v_{m-1}) \equiv 2 \pmod 3$, then we get $d_T(v_1, v_r) \equiv 1 \pmod 3$. Since $v_1 < v_r < v_{k_1}$ and $d_T(v_1, v_r) \equiv 1 \pmod 3$, proceeding similarly as above, a contradiction arises after finite steps when we get the minimum distance stem from v_1 at $1 \pmod 3$ distance.

[6] *$v_1, \dots v_n$ are path vertices and x_1, x_n are pendants attached to v_1, v_n, respectively. It is to be noted that v_1, v_n must be major stems since P is the central path and T contains legs as only pendants.*
[7] *$v_i \leq v_j$ indicates that v_i occurs left to v_j along P*

Further, to resolve v_{k_2-1}, v_{k_2}, there must exist some pendant $x_{k_3} \in S$ attached to some major stem v_{k_3} such that $\min\{d_T(x_{k_3}, v_{k_2}), d_T(x_{k_3}, v_{k_2-1})\} \equiv 0 \pmod 3$ by condition 1 of Theorem 4. If $v_{k_1} < v_{k_3} < v_{k_2}$ along P, then $d_T(v_{k_1}, v_{k_3}) = d_T(v_{k_1}, v_{k_2}) - d_T(v_{k_2}, v_{k_3}) \equiv 2 \pmod 3$, which contradicts the choice of v_{k_2}. Again, if $v_1 < v_{k_3} < v_{k_1}$, then $d_T(v_1, v_{k_3}) = d_T(v_1, v_{k_1}) - d_T(v_{k_1}, v_{k_3}) \equiv 1 \pmod 3$, which is not possible from the above paragraph. Therefore, $v_{k_2} < v_{k_3} < v_n$, which implies $d_T(v_1, v_{k_3}) = d_T(v_1, v_{k_2}) + d_T(v_{k_2}, v_{k_3}) \equiv 0 \pmod 3$. We consider v_{k_3} to be the minimum distance major stem from v_{k_2} satisfying $d_T(v_{k_2}, v_{k_3}) \equiv 2 \pmod 3$.

Proof of condition 3. Let $d_T(v_p, v_n) \equiv 1 \pmod 3$ for some stem $v_p < v_n$ along P. Let u be a pendant of v_p, which is not in S and $v = v_{p+1}$. Then to resolve u, v, there must exist some $x_l \in S$ attached to a major stem v_l satisfying $v_p < v_l < v_n$, such that $d_T(v_{p+1}, x_l) \equiv 0$ or $2 \pmod 3$ by condition 2 of Theorem 4. This implies $d_T(v_{p+1}, v_l) \equiv 2$ or $1 \pmod 3$. If $d_T(v_{p+1}, v_l) \equiv 1 \pmod 3$, then $d_T(v_l, v_n) = d_T(v_p, v_n) - d_T(v_p, v_{p+1}) - d_T(v_{p+1}, v_l) \equiv 2 \pmod 3$.

Again, if $d_T(v_{p+1}, v_l) \equiv 2 \pmod 3$, then $d_T(v_l, v_n) \equiv 1 \pmod 3$. Therefore, following a similar argument as did in the above paragraph, after finite steps, we can find some major stem v_l such that $d_T(v_l, v_n) \equiv 2 \pmod 3$.

Sufficiency: We consider T to be a tree that satisfies all the given conditions. From Corollary 2, it is already known that $\beta(T^3) \geq \beta(T)$. Hence, to prove $\beta(T^3) = \beta(T)$, it is sufficient to show that $\beta(T^3) \leq \beta(T)$. Let S be any metric basis of T, then all except one pendant from every major stem of T are the only members of S by Theorem 3. Now we show that S is a resolving set of T^3 also. For this, following Theorem 4, it is sufficient to prove that any two vertices $u, v \in V \setminus S$ satisfying $d_T(u, v) \leq 5$ can be resolved by at least one vertex of S. Let $x_1, x_{k_1}, x_{k_2}, x_{k_3}$ and x_n be the pendants attached to the major stems $v_1, v_{k_1}, v_{k_2}, v_{k_3}$ and v_n, respectively, which are included in S. (see Figure 9)

i) If u and v both appear on the central path P or one of them is attached to a stem on P.

Let v_l be the minimum distance major stem from v_n satisfying $d_T(v_l, v_n) \equiv 2 \pmod 3$ and $x_l \in S$ be a pendant attached to v_l. We will show that at least one among $x_1, x_{k_1}, x_{k_2}, x_{k_3}, x_l$ and x_n resolves u, v in T^3.

Condition 1 guarantees the presence of the above pendants when $d_T(u, v) \neq 2$. Then following Lemma 4 and Lemma 5, the result follows. Next, we consider the situation when $d_T(u, v) = 2$ and u is a pendant attached to some stem v_p and v is on the central path P.

Let $v = v_{p+1}$. If $d_T(v_p, v_n) \equiv 0$ or $2 \pmod 3$, then x_n resolves u, v in T^3 by Lemma 4. Next, we consider the situation when $d_T(v_p, v_n) \equiv 1 \pmod 3$. Then by using condition 3, we get an existence of a major stem v_l satisfying $v_p < v_l < v_n$ such that $d_T(v_l, v_n) \equiv 2 \pmod 3$. Without loss of generality, we can assume v_l to be the minimum distance vertex from v_n satisfying $d_T(v_l, v_n) \equiv 2 \pmod 3$. Hence, any pendant $x_l \in S$ attached to v_l resolves u, v by Lemma 4.

Let $v = v_{p-1}$. Then, by Lemma 4, at least one among x_{k_1}, x_1 resolves u, v when $v_{k_1} \leq v_p \leq v_n$ along P. Again, if $v_1 < v_p < v_{k_1}$, then $d_T(v_1, v_p) \not\equiv 2 \pmod 3$ from the definition of v_{k_1}. Further, $d_T(v_1, v_p) \not\equiv 1 \pmod 3$ by condition 2. Hence $d_T(v_1, v_p) \equiv 0 \pmod 3$ and therefore $d_T(x_1, v_{p-1}) = d_T(x_1, v_1) + d_T(v_1, v_{p-1}) \equiv 0 \pmod 3$. Hence, x_1 resolves u, v in T^3 by Lemma 4.

ii) If both u and v are pendants attached to two different stems, v_{m_1} and v_{m_2}, respectively. Let $v_{m_1} < v_{m_2}$ on P.

First, we consider the situation when $d_T(u, v) = 3$.

If $d_T(v_1, v_{m_1}) \equiv 1 \pmod 3$, then $d_T(x_1, u) \equiv 0 \pmod 3$. Then x_1 resolves u, v by Lemma 4, since $\min\{d_T(x_1, u), d_T(x_1, v)\} \equiv 0 \pmod 3$.

Next, if $d_T(v_1, v_{m_1}) \equiv 2 \pmod 3$, then $d_T(v_1, v_{m_2}) \equiv 0 \pmod 3$. Clearly, $v_{k_2} \neq v_{m_2}$. If $v_{k_2} > v_{m_2}$ along P, then $d_T(v_{m_2}, v_{k_2}) = d_T(v_1, v_{k_2}) - d_T(v_1, v_{m_1}) - d_T(v_{m_1}, v_{m_2}) \equiv 1 \pmod 3$. Therefore, $\min\ \{d_T(x_{k_2}, v), d_T(x_{k_2}, u)\} = d_T(x_{k_2}, v) = d_T(x_{k_2}, v_{k_2}) + d_T(v_{k_2}, v_{m_2}) + d_T(v_{m_2}, v) \equiv 0 \pmod 3$. Hence x_{k_2} resolves u, v in T^3 by Lemma 4. Again, if $v_{k_2} < v_{m_2}$, then $v_{k_2} < v_{m_1}$ clearly. Hence proceeding similarly one can verify that x_{k_2} resolves u, v in T^3.

Again, if $d_T(v_1, v_{m_1}) \equiv 0 \pmod 3$, then $v_{k_1} < v_{m_1}$, otherwise, $d_T(v_1, v_{m_2}) = 1 \pmod 3$ when $v_{m_2} < v_{k_1}$ contradicts condition 2. Therefore, $d_T(v_{k_1}, v_{m_1}) = d_T(v_1, v_{m_1}) - d_T(v_1, v_{k_1}) \equiv 1 \pmod 3$. Hence $\min\ \{d_T(x_{k_1}, u), d_T(x_{k_1}, v)\} = d_T(x_{k_1}, u) = d_T(x_{k_1}, v_{k_1}) + d_T(v_{k_1}, v_{m_1}) + d_T(v_{m_1}, u) \equiv 0 \pmod 3$. Hence x_{k_1} resolves u, v in T^3 by Lemma 4.

Next, we consider the situation when $d_T(u, v) = 4$. Then x_1 resolves u, v when $d_T(v_1, v_{m_1}) \equiv 0$ or $1 \pmod 3$. Again, if $d_T(v_1, v_{m_1}) \equiv 2 \pmod 3$, then if $v_{m_1} < v_{k_1}$, then $v_{k_1} > v_{m_2}$ clearly. Therefore, condition 2 gets contradicted since $d_T(v_1, v_{m_2}) \equiv 1 \pmod 3$. Hence $v_{k_1} \leq v_{m_1}$ and $d_T(v_{k_1}, v_{m_1}) \equiv 0 \pmod 3$, therefore x_{k_1} resolves u, v in T^3 following Lemma 5.

Lastly, if $d_T(u, v) = 5$, then $|d_T(x_1, u) - d_T(x_1, v)| = 5$ or $|d_T(x_1, u) - d_T(x_1, v)| = 3$ as per the situation $v_{m_1} \neq v_1$ or $v_{m_1} = v_1$. Hence, following Lemma 5, x_1 resolves u, v in T^3 in both circumstances.

Since all the conditions of Theorem 4 are satisfied, S becomes a resolving set of T^3. $\square$

9. Conclusion

In this article, we have determined the necessary and sufficient conditions for a resolving set to be a metric basis for the cube of trees. Also, we developed the upper and lower bounds on the metric dimension of the same graph class. Further, we discussed the characterization of some restricted class of cube of trees satisfying $\beta(T^3) = \beta(T)$. The following open problems are immediate from our study:

Problem 1. *Find the bounds on the metric dimension of* T^r *for any positive integer* $r \geq 4$.

Problem 2. *Characterize the class of trees that satisfy* $\beta(T^r) = \beta(T)$ for any positive integer r.

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