Communications in Combinatorics and Optimization
Vol. xx, No. x (xxxx), pp. 1-17
https://doi.org/10.22049/cco.2025.29874.2200

CCO
Commun. Comb. Optim.

***Research Article***

# Bilevel vector variational inequalities and multiobjective bilevel optimization problems

**Mohsine Jennane[1,*], El Mostafa Kalmoun[2]**

[1]FSDM, Department of mathematics, Sidi Mohamed Ben Abdellah University, Fez, Morocco
*mohsine.jennane@usmba.ac.ma

[2]School of Science and Engineering, Al Akhawayn University in Ifrane,
PO Box 104, Ifrane, 53000, Morocco
E.Kalmoun@aui.ma

*Received: 10 July 2024; Accepted: 7 April 2025*
*Published Online: 13 April 2025*

**Abstract:** In this paper, we introduce the concepts of bilevel vector variational inequalities (BVVI) of both Minty and Stampacchia types. Additionally, we establish connections between BVVI and multiobjective bilevel optimization problems (MBOP), focusing on the use of tangential subdifferentials. We investigate the relationship between the vector efficient points of MBOP and the solutions of BVVI, particularly under conditions of generalized convexity.

**Keywords:** multiobjective bilevel optimization, bilevel vector variational inequalities, tangential subdifferential, generalized convexity, monotonicity.

**AMS Subject classification:** 49J52, 90C29, 49J40

## 1. Introduction

In this paper, we focus on the Multiobjective Bilevel Optimization Problem (MBOP) introduced by Dempe [3]:

$$\min_{x,y} F(x,y) = (F_1, \ldots, F_{p_u})(x,y) \text{ subject to } G(x,y) \leq 0 \text{ and } y \in S(x), \tag{1.1}$$

where $S(x)$ represents the set of solutions to the lower-level problem, which is parameterized by $x$:

$$\min_{y} f(x,y) = (f_1, \ldots, f_{p_l})(x,y) \text{ subject to } g(x,y) \leq 0.$$

* *Corresponding Author*

© 2025 Azarbaijan Shahid Madani University

In this context, $x \in \mathbb{R}^{n_u}$ and $y \in \mathbb{R}^{n_l}$, with $F : \mathbb{R}^{n_u} \times \mathbb{R}^{n_l} \to \mathbb{R}^{p_u}$ representing the upper-level objective functions and $f : \mathbb{R}^{n_u} \times \mathbb{R}^{n_l} \to \mathbb{R}$ representing the lower-level objective functions. Additionally, the vector-valued mappings $G : \mathbb{R}^{n_u} \times \mathbb{R}^{n_l} \to \mathbb{R}^{m_u}$ represent the upper-level constraints, and $g : \mathbb{R}^{n_u} \times \mathbb{R}^{n_l} \to \mathbb{R}^{m_l}$ represent the lower-level constraints.
The feasible set of (1.1) is defined as:

$$\Pi = \{(x, y) \in \mathbb{R}^{n_u} \times \mathbb{R}^{n_l} : g(x, y) \leq 0, G(x, y) \leq 0, y \in S(x)\}.$$

A point $(\overline{x}, \overline{y}) \in \Pi$ is considered an efficient solution of (1.1) if, for any $(x, y) \in \Pi$,

$$F(x, y) - F(\overline{x}, \overline{y}) = (F_1(x, y) - F_1(\overline{x}, \overline{y}), \dots, F_{p_u}(x, y) - F_{p_u}(\overline{x}, \overline{y})) \notin -\mathbb{R}^{p_u}_+ \setminus \{0\}.$$

This condition ensures that the difference between the objective functions at the efficient solution and any other feasible point is not dominated by non-negative multiples of vectors in the set $-\mathbb{R}^{p_u}_+ \setminus \{0\}$.
Bilevel optimization problems offer a flexible framework for modeling complex decision-making scenarios involving multiple decision-makers, such as in supply chain management, transportation planning, and energy systems. They facilitate the coordination between different decision-makers by explicitly modeling their interactions and objectives.
To establish optimality conditions for MBOP, various approaches have been explored. The optimal value function, introduced by Outrata [17], stands out in optimistic bilevel programs and features prominently in works like [9, 21]. Similarly, the KKT (Karush-Kuhn-Tucker) approach remains central to deriving optimality conditions for MBOP, as seen in [4, 23].
In parallel, vector variational inequalities have gained traction as robust tools for addressing optimization problems with multiple objectives and constraints. Introduced by Giannessi [5] as extensions of Stampacchia variational inequalities [19], these inequalities have driven recent progress in optimality conditions for vector optimization, evident in studies such as [6–8, 16].
Despite the relevance of both MBOPs and VVIs, their connection remains scarcely investigated. The only known link is due to Kohli [10], who reformulated MBOP as a single-level problem via the value function and then applied variational inequality theory. In contrast, our work establishes a direct connection between MBOPs and VVIs without relying on intermediate reformulations. Moreover, although Minty- and Stampacchia-type variational inequalities offer distinct and complementary viewpoints in classical variational theory, their bilevel counterparts have not been addressed in the literature. This work introduces and analyzes these bilevel forms, which provides a unified variational framework for multiobjective bilevel optimization.
The main contributions of this paper are as follows:

1. We introduce Minty and Stampacchia-type bilevel variational inequalities (BVVIs) to formulate and analyze multiobjective bilevel problems under generalized convexity. This framework provides a new approach for establishing optimality conditions and developing computational strategies for MBOPs.

2. We derive our results using the tangential subdifferential, which offers greater precision compared to other generalized subdifferentials (e.g., Fréchet, Clarke, Michel-Penot). To our knowledge, this is the first application of the tangential subdifferential in the context of vector variational inequalities.

The paper is structured as follows: Section 2 provides an introduction to the essential concepts from variational analysis that will be integral to our subsequent discussions. In Section 3, we delve into the core of our study by introducing the BVVI of Minty and Stampacchia types. Within this section, we establish sufficient and necessary optimality conditions under a generalized convexity condition. Moving on to Section 4, we present the results related to the existence of solutions to BVVIs. Finally, we encapsulate our findings and insights in the concluding section.

## 2. Preliminaries

In this section, we revisit essential definitions and results that will serve as the foundation for our subsequent discussions.
Let $x := (x_1, \ldots, x_n)$ and $y := (y_1, \ldots, y_n)$ denote two vectors in $\mathbb{R}^n$. We use the following notation:

$$x = y \Leftrightarrow x_i = y_i \text{ for all } i = 1, 2, \ldots, n.$$
$$x \geqq y \Leftrightarrow x_i \geq y_i \text{ for all } i = 1, 2, \ldots, n \Leftrightarrow x - y \in \mathbb{R}^n_+.$$
$$x \geq y \Leftrightarrow x_i \geq y_i \text{ for all } i = 1, 2, \ldots, n \text{ and } x \neq y \Leftrightarrow x - y \in \mathbb{R}^n_+ \setminus \{0\}.$$
$$x > y \Leftrightarrow x_i > y_i \text{ for all } i = 1, 2, \ldots, n \Leftrightarrow x - y \in \text{int}(\mathbb{R}^n_+).$$

Furthermore, for all $x \in \mathbb{R}^n$ and $x^* = (x_1^*, \ldots, x_p^*) \in \mathbb{R}^{np}$ where each $x_i^* \in \mathbb{R}^n$ for $i = 1, \cdots, p$, we adopt the notation:

$$\langle x^*, x\rangle_p = \left(\langle x_1^*, x\rangle, \ldots, \langle x_p^*, x\rangle\right).$$

We recall that a function $\phi : \mathbb{R}^n \to \mathbb{R}$ is said to be tangentially convex at $z \in \mathbb{R}^n$ [18] if its directional derivative (also known as the Dini derivative) at $z$,

$$\phi'(z, \xi) := \lim_{t \downarrow 0} \frac{\phi(z + t\xi) - \phi(z)}{t},$$

is finite for any direction $\xi \in \mathbb{R}^n$ and convex in this argument. Note that the directional derivative of any tangentially convex function is sublinear due to its positive homogeneity.
The tangential subdifferential of $\phi$ at $z \in \mathbb{R}^n$ is given by [12, 18]:

$$\partial^{\mathcal{T}} \phi(z) := \{y^* \in \mathbb{R}^n : \langle y^*, \xi\rangle \leq \phi'(z, \xi) \text{ for all } \xi \in \mathbb{R}^n\}. \tag{2.1}$$

We point out that for a tangentially convex function, this subdifferential is nonempty, compact, and convex (see [13]).
Tangentially convex functions form a broad class that includes convex functions on open domains where the tangential subdifferential reduces to the classical Fréchet subdifferential, Gâteaux differentiable functions on open domains with a tangential subdifferential reduced to the gradient. This class also encompasses locally Lipschitz functions that are either Clarke regular [2] or Michel-Penot regular [15], with their tangential subdifferential equal to that of Clarke in the first case and Michel-Penot in the second.
Notably, if $\phi$ is tangentially convex at $z$, it follows from the sublinearity of $\phi'(z,\cdot)$ that (2.1) is equivalent to:

$$\phi'(z,\xi) = \max_{z^*\in\partial^{\mathcal{T}}\phi(z)} \langle z^*,\xi\rangle, \text{ for all } \xi\in\mathbb{R}^n.$$

This equivalence implies that:

$$\inf_{z^*\in\partial^{\mathcal{T}}\phi(z)} \langle z^*,\xi\rangle \le \phi_d^+(z,\xi) = \phi'(z,\xi) = \phi_d^-(z,\xi) \le \sup_{z^*\in\partial^{\mathcal{T}}\phi(z)} \langle z^*,\xi\rangle, \text{ for all } \xi\in\mathbb{R}^n.$$

Here, $\phi_d^+(z,\xi)$ and $\phi_d^-(z,\xi)$ represent, respectively, the upper and lower Dini directional derivatives of $\phi$ at $z$ in the direction $\xi$. Consequently, $\partial^{\mathcal{T}}\phi(z)$ serves as a convexificator of $\phi$ at $z$.
It's worth noting that in the case of tangentially convex functions, the definition of the tangential subdifferential coincides with that of the upper regular convexificator, as $\phi'(z,\xi) = \phi_d^+(z,\xi) = \sup_{z^*\in\partial^{\mathcal{T}}\phi(z)} \langle z^*,\xi\rangle$, for all $\xi\in\mathbb{R}^n$.
A vector function $\Phi = (\phi_1,\ldots,\phi_p) : \mathbb{R}^n \to \mathbb{R}^p$ is called tangentially convex if so is each of its coordinates $\phi_i$, and its tangential subdifferential at $z\in\mathbb{R}^n$ is given by:

$$\partial^{\mathcal{T}}\Phi(z) := \partial^{\mathcal{T}}\phi_1 \times \partial^{\mathcal{T}}\phi_2 \times \ldots \times \partial^{\mathcal{T}}\phi_p.$$

In the following, we present the Mean Value Theorem for tangentially convex functions, a recent extension that has been established in [14].

**Theorem 1.** *[14, Theorem 6] Assume that $\phi : \mathbb{R}^n \to \mathbb{R}\cup\{\infty\}$ is continuous on a convex set $C \subseteq \phi^{-1}(\mathbb{R})$ and tangentially convex on $C \setminus \text{ext}C$. If $\overline{z}$ and $z$ are two distinct points in $C$, then there exists $t_0 \in (0,1)$ such that*

$$\phi(z) - \phi(\overline{z}) \in \langle \partial^{\mathcal{T}}\phi\left(\overline{z} + t_0\left(z-\overline{z}\right)\right), z-\overline{z}\rangle.$$

In the context of vector-valued functions, we define specific forms of tangential convexity that incorporate the tangential subdifferential.

**Definition 1.** A tangentially convex vector-valued function $\Phi : \mathcal{S} \subseteq \mathbb{R}^n \to \mathbb{R}^p$ is

- $\partial^{\mathcal{T}}$-convex over $\mathcal{S}$ if, for all $z, \overline{z} \in \mathcal{S}$, one has

$$\Phi(z) \geqq \Phi(\overline{z}) + \langle \xi, z - \overline{z} \rangle_p, \quad \text{for all } \xi \in \partial^{\mathcal{T}} \Phi(\overline{z}).$$

- $\partial^{\mathcal{T}}$-quasiconvex over $\mathcal{S}$ if, for all $z, \overline{z} \in \mathcal{S}$, one has

$$\langle \xi, z - \overline{z} \rangle_p \geq 0 \Rightarrow \Phi(z) \geq \Phi(\overline{z}), \quad \text{for all } \xi \in \partial^{\mathcal{T}} \Phi(\overline{z}).$$

- $\partial^{\mathcal{T}}$-pseudoconvex over $\mathcal{S}$ if, for all $z, \overline{z} \in \mathcal{S}$, one has

$$\Phi(z) \leq \Phi(\overline{z}) \Rightarrow \langle \xi, z - \overline{z} \rangle_p \leq 0, \quad \text{for all } \xi \in \partial^{\mathcal{T}} \Phi(\overline{z}).$$

**Remark 1.** (i) Every $\partial^{\mathcal{T}}$-convex function is $\partial^{\mathcal{T}}$-quasiconvex and $\partial^{\mathcal{T}}$-pseudoconvex; however, the converse is not always true.
(ii) There is no inherent relationship between $\partial^{\mathcal{T}}$-quasiconvexity and $\partial^{\mathcal{T}}$-pseudoconvexity.
(iii) For $\Phi$ being $\partial^{\mathcal{T}}$-convex over $\mathcal{S}$, it satisfies the convexity property:

$$\Phi\left(\alpha z + (1-\alpha)\overline{z}\right) \leqq \alpha \Phi(z) + (1-\alpha)\Phi(\overline{z}),$$

for all $z, \overline{z} \in \mathcal{S}$ and $\alpha \in [0, 1]$.
For illustrative examples that clarify relationships between $\partial^{\mathcal{T}}$-convexity, $\partial^{\mathcal{T}}$-quasiconvexity and $\partial^{\mathcal{T}}$-pseudoconvexity, we refer the reader to [20].

We also introduce the property of monotonicity with respect to the tangential subdifferential.

**Definition 2.** The tangential subdifferential $\partial^{\mathcal{T}}\Phi$ of $\Phi : \mathcal{S} \subseteq \mathbb{R}^n \to \mathbb{R}^p$ is said to be monotone on $\mathcal{S}$ if, for all $z, \overline{z} \in \mathcal{S}$, $z^* \in \partial^{\mathcal{T}}\Phi(z)$ and $\overline{z}^* \in \partial^{\mathcal{T}}\Phi(\overline{z})$, the following inequality holds:

$$\langle z^* - \overline{z}^*, z - \overline{z} \rangle_p \geqq 0.$$

In the following theorem, we establish a relationship between the $\partial^{\mathcal{T}}$-convexity of a vector-valued function and the property of being monotone with respect to its tangential subdifferential on a convex set.

**Theorem 2.** *Assume that $\mathcal{S}$ is a nonempty convex subset of $\mathbb{R}^n$. If $\Phi$ is $\partial^{\mathcal{T}}$-convex on $\mathcal{S}$, then $\partial^{\mathcal{T}}\Phi$ is monotone on $\mathcal{S}$.*

*Proof.* Suppose that $\Phi$ is $\partial^{\mathcal{T}}$-convex on $\mathcal{S}$. Then, for all $z, \overline{z} \in \mathcal{S}$, $z^* \in \partial^{\mathcal{T}}\Phi(z)$ and $\overline{z}^* \in \partial^{\mathcal{T}}\Phi(\overline{z})$, we have

$$\Phi(z) - \Phi(\overline{z}) \geqq \langle \overline{z}^*, z - \overline{z} \rangle_p,$$

and

$$\Phi(\overline{z}) - \Phi(z) \geqq \langle z^*, \overline{z} - z \rangle_p.$$

Adding the above inequalities, for all $z, \overline{z} \in \mathcal{S}$, $z^* \in \partial^{\mathcal{T}}\Phi(z)$, and $\overline{z}^* \in \partial^{\mathcal{T}}\Phi(\overline{z})$, we obtain

$$\langle z^* - \overline{z}^*, z - \overline{z} \rangle_p \geqq 0,$$

which implies that $\partial^{\mathcal{T}}\Phi$ is monotone on $\mathcal{S}$. $\square$

## 3. Relationship between BVVI and MBOP

We begin this section by defining the set of constraints as follows

$$K = \{(x,y) \in \mathbb{R}^{n_u} \times \mathbb{R}^{n_l} : g(x,y) \leq 0,\ G(x,y) \leq 0\}.$$

We define also the following two notations

$$T_x = \{y \in \mathbb{R}^{n_l} : g(x,y) \leq 0\}, \quad \forall x \in \mathbb{R}^{n_l}$$

and

$$\phi_x(y) = f(x,y), \quad \forall x \in \mathbb{R}^{n_l}.$$

Recall from [22] that a pair $(\overline{x},\overline{y}) \in \Pi$ is an efficient solution to (1.1) if it is an optimal solution to the following problem:

$$\min_{(x,y)\in\Pi} F(x,y).$$

Note that $\Pi = \{(x,y) \in K : y \in \arg\min \phi_x\}$.
For the remainder of this paper, we assume that $F$ is tangentially convex at all $(x,y) \in K$, so $F$ admits a compact tangential subdifferential at any $(x,y) \in K$, denoted as $\partial^{\mathcal{T}} F(x,y)$. Additionally, we also assume that $\phi_x$ is tangentially convex at all $y \in T_x$, so $\phi_x$ has a compact tangential subdifferential at any $y \in T_x$, denoted as $\partial^{\mathcal{T}} \phi_x(y)$.
We now introduce two types of bilevel variational inequalities, specifically, the Bilevel Variational Inequalities of Minty-Type (MBVVI) and the Bilevel Variational Inequalities of Stampacchia-Type (SBVVI) as follows:
(MBVVI) Find $(\overline{x},\overline{y}) \in \Pi_M$ such that for all $(x,y) \in \Pi_M$ and $(x^*,y^*) \in \partial^{\mathcal{T}} F(x,y)$, we have

$$\langle (x^*,y^*),(x,y)-(\overline{x},\overline{y})\rangle_{p_u} \nleq 0,$$

where $\Pi_M = \{(x,y) \in K : y \in S_M(x)\}$, and $S_M(x)$ denotes the set of all solutions of the following lower-level variational inequality problem:
Find $\overline{y} \in T_x$ such that for all $y \in T_x$ and $y^* \in \partial^{\mathcal{T}} \phi_x(y)$, we have

$$\langle y^*, y-\overline{y}\rangle \geq 0.$$

(SBVVI) Find $(\overline{x},\overline{y}) \in \Pi_S$ such that for all $(x,y) \in \Pi_S$, there is $(\overline{x}^*,\overline{y}^*) \in \partial^{\mathcal{T}} F(\overline{x},\overline{y})$ such that

$$\langle (\overline{x}^*,\overline{y}^*),(x,y)-(\overline{x},\overline{y})\rangle_{p_u} \nleq 0,$$

where $\Pi_S = \{(x,y) \in K : y \in S_S(x)\}$, and $S_S(x)$ denotes the set of all solutions of the following lower-level variational inequality problem:
Find $\overline{y} \in T_x$ such that for all $y \in T_x$, there is $\overline{y}^* \in \partial^{\mathcal{T}} \phi_x(\overline{y})$ such that

$$\langle \overline{y}^*, y-\overline{y}\rangle \geq 0.$$

### 3.1. Relations Between Feasible Sets of MBOP, MBVVI, and SBVVI

We examine how the feasible sets $\Pi$, $\Pi_M$, and $\Pi_S$ associated with MBOP, MBVVI, and SBVVI are related. We begin by establishing the inclusion $\Pi_M \subseteq \Pi$ under suitable convexity assumptions on the lower-level objective and constraint functions.

**Proposition 1.** *Suppose $\phi_x$ is $\partial^{\mathcal{T}}$-pseudoconvex over $T_x$ and that $g$ is quasiconvex. Then $\Pi_M \subseteq \Pi$.*

*Proof.* Let $(x, y) \in \Pi_M$, so $y \in S_M(x)$, the set of solutions to (MBVVI) for the fixed $x$. Thus, for all $\widetilde{y} \in T_x$ and all $\widetilde{y}^* \in \partial^{\mathcal{T}}\phi_x(\widetilde{y})$, we have $\langle \widetilde{y}^*, \widetilde{y} - y\rangle \geq 0$.
The quasiconvexity of $g$ ensures $T_x$ is convex. For any sequence $\{\lambda_n\} \downarrow 0$ with $\lambda_n \in (0, 1]$, it follows that $y_n = y + \lambda_n(\widetilde{y} - y) \in T_x$. Then there exists $y_n^* \in \partial^{\mathcal{T}}\phi_x(y_n)$ satisfying $\langle y_n^*, y_n - y\rangle \geq 0$, which implies $\lambda_n\langle y_n^*, \widetilde{y} - y\rangle \geq 0$.
The compactness of $\partial^{\mathcal{T}}\phi_x(y_n)$ guarantees boundedness. Up to a subsequence, we can assume $y_n^* \to y^*$. Because $\partial^{\mathcal{T}}\phi_x$ is closed and $y_n \to y$ as $n \to \infty$, we conclude $y^* \in \partial^{\mathcal{T}}\phi_x(y)$, and thus $\langle y^*, \widetilde{y} - y\rangle \geq 0$.
Given that $\phi_x$ is $\partial^{\mathcal{T}}$-pseudoconvex over $T_x$, it follows that $\phi_x(\widetilde{y}) \geq \phi_x(y)$. This implies $y \in S(x)$, so $(x, y) \in \Pi$. Hence, $\Pi_M \subseteq \Pi$. □

We establish conditions ensuring that the solution set of the lower-level problem in (SBVVI) is contained within the solution set of the lower-level problem in (1.1).

**Proposition 2.** *Suppose $\phi_x$ is $\partial^{\mathcal{T}}$-pseudoconvex over $T_x$. Then $\Pi_S \subseteq \Pi$.*

*Proof.* Assume $(x, y) \in \Pi_S$. To prove $(x, y) \in \Pi$, we proceed by contrapositive. Suppose $y \notin S(x)$. Then there exists $\widehat{y} \in T_x$ such that $\phi_x(\widehat{y}) < \phi_x(y)$.
Given the $\partial^{\mathcal{T}}$-pseudoconvexity of $\phi_x$, it follows that for all $y^* \in \partial^{\mathcal{T}}\phi_x(y)$, the inequality $\langle y^*, \widehat{y} - y\rangle < 0$ holds. This implies $y \notin S_S(x)$. Consequently, $S_S(x) \subseteq S(x)$, and thus $\Pi_S \subseteq \Pi$. □

In the following, we establish the opposite inclusions to those in the previous two propositions. The next result shows when the set of solutions to the lower-level problem of (1.1) is included in the set of solutions to the lower-level problem of (MBVVI).

**Proposition 3.** *Assume that $\phi_x$ is $\partial^{\mathcal{T}}$-quasiconvex over $T_x$. Then $\Pi \subseteq \Pi_M$.*

*Proof.* Suppose $(x, y) \notin \Pi_M$, that is $y \notin S_M(x)$. Then there exist $\widehat{y} \in T_x$ and $\widehat{y}^* \in \partial^{\mathcal{T}}\phi_x(\widehat{y})$ such that $\langle \widehat{y}^*, \widehat{y} - y\rangle < 0$.
Given the $\partial^{\mathcal{T}}$-quasiconvexity of $\phi_x$ over $T_x$, this implies $\phi_x(\widehat{y}) < \phi_x(y)$. Hence, $y \notin S(x)$. Consequently, $S(x) \subseteq S_M(x)$, and thus $\Pi \subseteq \Pi_M$. □

The next proposition shows that the set of solutions to the lower-level problem of (1.1) is included in the set of solutions to the lower-level problem of (SBVVI).

**Proposition 4.** *Assume that $\phi_x$ is $\partial^{\mathcal{T}}$-quasiconvex over $T_x$ and that $g$ is quasiconvex. Then $\Pi \subseteq \Pi_S$.*

*Proof.* Let $(x, y) \in \Pi$, so $y \in S(x)$. Consider any $\widehat{y} \in T_x$. The quasiconvexity of $g$ ensures $T_x$ is convex. Thus, for any sequence $\{\lambda_n\} \downarrow 0$ with $\lambda_n \in (0, 1]$, $y_n = y + \lambda_n(\widehat{y} - y) \in T_x$.
Because $y \in S(x)$, it follows that $\phi_x(y_n) \geq \phi_x(y)$ for all $y_n \in T_x$. Given that $\phi_x$ is $\partial^{\mathcal{T}}$-quasiconvex at $y_n \in T_x$, there exists $y_n^* \in \partial^{\mathcal{T}}\phi_x(y_n)$ such that $\langle y_n^*, y_n - y\rangle \geq 0$. Substituting $y_n - y = \lambda_n(\widehat{y} - y)$ and noting $\lambda_n > 0$, we obtain $\langle y_n^*, \widehat{y} - y\rangle \geq 0$.
The boundedness of $\partial^{\mathcal{T}}\phi_x(y_n)$ allows us to assume, up to a subsequence, that $y_n^* \to y^*$. Since $\partial^{\mathcal{T}}\phi_x$ is closed and $y_n \to y$ as $n \to \infty$, we conclude $y^* \in \partial^{\mathcal{T}}\phi_x(y)$. Hence, $\langle y^*, \widehat{y} - y\rangle \geq 0$ holds for some $y^* \in \partial^{\mathcal{T}}\phi_x(y)$. This implies $y \in S_S(x)$, so $(x, y) \in \Pi_S$. Therefore, $\Pi \subseteq \Pi_S$. □

We conclude this subsection with this corollary which can be easily proven using Remark 1.

**Corollary 1.** *Assume that $\phi_x$ is $\partial^{\mathcal{T}}$-convex over $T_x$ and that $g$ is quasiconvex. Then*

$$\Pi = \Pi_M = \Pi_S.$$

### 3.2. Main results

The relationship between (MBVVI) and (SBVVI) can be elucidated in the following theorem.

**Theorem 3.** *Suppose that $F$ and $\phi_x$, for any $x \in \mathbb{R}^{n_u}$, are $\partial^{\mathcal{T}}$-convex over $\Pi$ and $T_x$, respectively and $g$ is quasiconvex. Any solution to (SBVVI) is also a solution to (MBVVI).*

*Proof.* Applying the contrapositive, let us assume that $(\overline{x}, \overline{y})$ does not solve (MBVVI). This means that there exist $(x, y) \in \Pi_M$ and $(x^*, y^*) \in \partial^{\mathcal{T}}F(x, y)$ such that

$$\langle (x^*, y^*), (x, y) - (\overline{x}, \overline{y})\rangle_{p_u} \leq 0.$$

Since $\phi_x$ is $\partial^{\mathcal{T}}$-convex over $T_x$ and that $g$ is quasiconvex, then by Corollary 1 $\Pi = \Pi_M = \Pi_S$.
According to Theorem 2, the $\partial^{\mathcal{T}}$-convexity of $F$ over $\Pi$ yields the monotonicity of $\partial^{\mathcal{T}}F$ over $\Pi$, respectively. Hence, for all $(x^*, y^*) \in \partial^{\mathcal{T}}F(x, y)$ and all $(\overline{x}^*, \overline{y}^*) \in \partial^{\mathcal{T}}F(\overline{x}, \overline{y})$, we get

$$\langle (x^*, y^*) - (\overline{x}^*, \overline{y}^*), (x, y) - (\overline{x}, \overline{y})\rangle_{p_u} \geqq 0.$$

Then, we deduce that there exist $(x,y)\in\Pi_S$ such that for all $(\overline{x}^*,\overline{y}^*)\in\partial^{\mathcal{T}}F(\overline{x},\overline{y})$ we have

$$\langle(\overline{x}^*,\overline{y}^*),(x,y)-(\overline{x},\overline{y})\rangle_{p_u}\leq 0.$$

Consequently, $(\overline{x},\overline{y})$ is not a solution of (SBVVI). □

We establish the conditions under which every efficient solution of (1.1) also qualifies as a solution of (MBVVI).

**Theorem 4.** *If $F$ is $\partial^{\mathcal{T}}$-quasiconvex over $\Pi$, $\phi_x$ (for any $x\in\mathbb{R}^{n_u}$) is $\partial^{\mathcal{T}}$-pseudoconvex over $T_x$, and $g$ is quasiconvex, then any efficient solution of (1.1) also serves as a solution to (MBVVI).*

*Proof.* Suppose that $(\overline{x},\overline{y})$ is not a solution to (MBVVI). Then, there exist $(x,y)\in\Pi_M$ and $(x^*,y^*)\in\partial^{\mathcal{T}}F(x,y)$ such that

$$\langle(x^*,y^*),(\overline{x},\overline{y})-(x,y)\rangle_{p_u}\geq 0.$$

Since $\phi_x$ is $\partial^{\mathcal{T}}$-convex over $T_x$ and that $g$ is quasiconvex, then by Proposition 1 $(x,y)\in\Pi$.
Due to the $\partial^{\mathcal{T}}$-quasiconvexity of $F$ over $\Pi$, we conclude

$$F(\overline{x},\overline{y})\geq F(x,y).$$

Consequently, $(\overline{x},\overline{y})$ is not an efficient solution of (1.1). □

We now show that, under the assumption of $\partial^{\mathcal{T}}$-convexity of $F$ and $\phi_x$, the converse of the above theorem holds.

**Theorem 5.** *Assume that $\Pi$ is convex. Additionally, suppose that $F$ and $\phi_x$, $x\in\mathbb{R}^{n_u}$, are $\partial^{\mathcal{T}}$-convex over $\Pi$ and $T_x$, respectively, $g$ is quasiconvex and $F_i$ is continuous for all $i\in\{1,...,p_u\}$. Any solution to (MBVVI) is also an efficient solution to (1.1).*

*Proof.* Suppose that $(\overline{x},\overline{y})$ is not an efficient solution of (1.1). Then, there exists $(x,y)\in\Pi$ such that:

$$F(x,y)-F(\overline{x},\overline{y})\leq 0. \tag{3.1}$$

By the $\partial^{\mathcal{T}}$-convexity of $F$ over $\Pi$, we have for all $\alpha\in[0,1]$

$$F(x_\alpha,y_\alpha)-F(\overline{x},\overline{y}))\leqq\alpha\big(F(x,y)-F(\overline{x},\overline{y})\big),$$

where $(x_\alpha, y_\alpha) = (\overline{x}, \overline{y}) + \alpha\big((x, y) - (\overline{x}, \overline{y})\big)$ with $(x_\alpha, y_\alpha) \in \Pi$ since $\Pi$ is convex. Then, for each $i \in \{1, ..., p_u\}$ and $\alpha \in (0, 1)$,

$$F_i(x_\alpha, y_\alpha) - F_i(\overline{x}, \overline{y}) \leq \alpha\big(F_i(x, y) - F_i(\overline{x}, \overline{y})\big). \tag{3.2}$$

Let $i \in \{1, ..., p_u\}$. By Theorem 1, there exist $\widehat{\alpha}_i \in (0, 1)$ and $(\widetilde{x}^*_{\widehat{\alpha}_i}, \widetilde{y}^*_{\widehat{\alpha}_i}) \in \partial^{\mathcal{T}} F_i(\widetilde{x}_{\widehat{\alpha}_i}, \widetilde{y}_{\widehat{\alpha}_i})$ with $(\widetilde{x}_{\widehat{\alpha}_i}, \widetilde{y}_{\widehat{\alpha}_i}) = (\overline{x}, \overline{y}) + \widehat{\alpha}_i\big((x_\alpha, y_\alpha) - (\overline{x}, \overline{y})\big) \in \Pi$ such that

$$F_i(x_\alpha, y_\alpha) - F_i(\overline{x}, \overline{y}) = \langle(\widetilde{x}^*_{\widehat{\alpha}_i}, \widetilde{y}^*_{\widehat{\alpha}_i}), (x_\alpha, y_\alpha) - (\overline{x}, \overline{y})\rangle,$$

which means

$$F_i(x_\alpha, y_\alpha) - F_i(\overline{x}, \overline{y}) = \langle(\widetilde{x}^*_{\widehat{\alpha}_i}, \widetilde{y}^*_{\widehat{\alpha}_i}), \alpha\big((x, y) - (\overline{x}, \overline{y})\big)\rangle. \tag{3.3}$$

In combining (3.2) and (3.3), we deduce

$$\langle(\widetilde{x}^*_{\widehat{\alpha}_i}, \widetilde{y}^*_{\widehat{\alpha}_i}), (x, y) - (\overline{x}, \overline{y})\rangle \leq F_i(x, y) - F_i(\overline{x}, \overline{y}). \tag{3.4}$$

**Case 1:** Suppose $\widehat{\alpha}_1 = \widehat{\alpha}_2 = ... = \widehat{\alpha}_{p_u} = \widehat{\alpha}$. Multiplying both the sides of (3.4) by $\widehat{\alpha}$, we get

$$\langle(\widetilde{x}^*_{\widehat{\alpha}_i}, \widetilde{y}^*_{\widehat{\alpha}_i}), \widehat{\alpha}\big((x, y) - (\overline{x}, \overline{y})\big)\rangle \leq \widehat{\alpha}\big(F_i(x, y) - F_i(\overline{x}, \overline{y})\big).$$

Consequently,

$$\langle(\widetilde{x}^*_{\widehat{\alpha}_i}, \widetilde{y}^*_{\widehat{\alpha}_i}), (x_{\widehat{\alpha}}, y_{\widehat{\alpha}}) - (\overline{x}, \overline{y})\rangle \leq \widehat{\alpha}\big(F_i(x, y) - F_i(\overline{x}, \overline{y})\big). \tag{3.5}$$

**Case 2:** Suppose $\widehat{\alpha}_1$, $\widehat{\alpha}_2$, ..., $\widehat{\alpha}_{p_u}$ are not all equal. Without loss of generality, we may assume that $\widehat{\alpha}_1 \neq \widehat{\alpha}_2$. Then, from (3.4), for some $(\widetilde{x}^*_{\widehat{\alpha}_1}, \widetilde{y}^*_{\widehat{\alpha}_1}) \in \partial^{\mathcal{T}} F_1(\widetilde{x}_{\widehat{\alpha}_1}, \widetilde{y}_{\widehat{\alpha}_1})$ and $(\widetilde{x}^*_{\widehat{\alpha}_2}, \widetilde{y}^*_{\widehat{\alpha}_2}) \in \partial^{\mathcal{T}} F_2(\widetilde{x}_{\widehat{\alpha}_2}, \widetilde{y}_{\widehat{\alpha}_2})$, one has

$$\begin{aligned}
&\langle(\widetilde{x}^*_{\widehat{\alpha}_1}, \widetilde{y}^*_{\widehat{\alpha}_1}), (x, y) - (\overline{x}, \overline{y})\rangle \leq F_1(x, y) - F_1(\overline{x}, \overline{y}),\\
&\langle(\widetilde{x}^*_{\widehat{\alpha}_2}, \widetilde{y}^*_{\widehat{\alpha}_2}), (x, y) - (\overline{x}, \overline{y})\rangle \leq F_2(x, y) - F_2(\overline{x}, \overline{y}).
\end{aligned}$$

Since $F_1$ and $F_2$ are $\partial^{\mathcal{T}}$-convex over $K$, both $\partial^{\mathcal{T}} F_1$ and $\partial^{\mathcal{T}} F_2$ are monotone. This implies that for all $(\widetilde{x}^*_{12}, \widetilde{y}^*_{12}) \in \partial^{\mathcal{T}} F_1(\widetilde{x}_{\widehat{\alpha}_2}, \widetilde{y}_{\widehat{\alpha}_2})$ and $(\widetilde{x}^*_{21}, \widetilde{y}^*_{21}) \in \partial^{\mathcal{T}} F_2(\widetilde{x}_{\widehat{\alpha}_1}, \widetilde{y}_{\widehat{\alpha}_1})$, the following inequalities hold:

$$\begin{aligned}
&\langle(\widetilde{x}^*_{\widehat{\alpha}_1}, \widetilde{y}^*_{\widehat{\alpha}_1}) - (\widetilde{x}^*_{12}, \widetilde{y}^*_{12}), (\widetilde{x}_{\widehat{\alpha}_1}, \widetilde{y}_{\widehat{\alpha}_1}) - (\widetilde{x}_{\widehat{\alpha}_2}, \widetilde{y}_{\widehat{\alpha}_2})\rangle \geq 0,\\
&\langle(\widetilde{x}^*_{\widehat{\alpha}_2}, \widetilde{y}^*_{\widehat{\alpha}_2}) - (\widetilde{x}^*_{21}, \widetilde{y}^*_{21}), (\widetilde{x}_{\widehat{\alpha}_2}, \widetilde{y}_{\widehat{\alpha}_2}) - (\widetilde{x}_{\widehat{\alpha}_1}, \widetilde{y}_{\widehat{\alpha}_1})\rangle \geq 0.
\end{aligned}$$

Then

$$\langle(\widetilde{x}^*_{12},\widetilde{y}^*_{12}),(\widetilde{x}_{\widehat{\alpha}_1},\widetilde{y}_{\widehat{\alpha}_1})-(\widetilde{x}_{\widehat{\alpha}_2},\widetilde{y}_{\widehat{\alpha}_2})\rangle\leq\langle(\widetilde{x}^*_{\widehat{\alpha}_1},\widetilde{y}^*_{\widehat{\alpha}_1}),(\widetilde{x}_{\widehat{\alpha}_1},\widetilde{y}_{\widehat{\alpha}_1})-(\widetilde{x}_{\widehat{\alpha}_2},\widetilde{y}_{\widehat{\alpha}_2})\rangle,$$
$$\langle(\widetilde{x}^*_{21},\widetilde{y}^*_{21}),(\widetilde{x}_{\widehat{\alpha}_2},\widetilde{y}_{\widehat{\alpha}_2})-(\widetilde{x}_{\widehat{\alpha}_1},\widetilde{y}_{\widehat{\alpha}_1})\rangle\leq\langle(\widetilde{x}^*_{\widehat{\alpha}_2},\widetilde{y}^*_{\widehat{\alpha}_2}),(\widetilde{x}_{\widehat{\alpha}_2},\widetilde{y}_{\widehat{\alpha}_2})-(\widetilde{x}_{\widehat{\alpha}_1},\widetilde{y}_{\widehat{\alpha}_1})\rangle.$$

Hence

$$\langle(\widetilde{x}^*_{12},\widetilde{y}^*_{12}),(\widehat{\alpha}_1-\widehat{\alpha}_2)\big((x,y)-(\overline{x},\overline{y})\big)\rangle\leq\langle(\widetilde{x}^*_{\widehat{\alpha}_1},\widetilde{y}^*_{\widehat{\alpha}_1}),(\widehat{\alpha}_1-\widehat{\alpha}_2)\big((x,y)-(\overline{x},\overline{y})\big)\rangle,$$
$$\langle(\widetilde{x}^*_{21},\widetilde{y}^*_{21}),(\widehat{\alpha}_2-\widehat{\alpha}_1)\big((x,y)-(\overline{x},\overline{y})\big)\rangle\leq\langle(\widetilde{x}^*_{\widehat{\alpha}_2},\widetilde{y}^*_{\widehat{\alpha}_2}),(\widehat{\alpha}_2-\widehat{\alpha}_1)\big((x,y)-(\overline{x},\overline{y})\big)\rangle.$$

If $\widehat{\alpha}_1>\widehat{\alpha}_2$, it follows that

$$\langle(\widetilde{x}^*_{12},\widetilde{y}^*_{12}),(x,y)-(\overline{x},\overline{y})\rangle\leq F_1(x,y)-F_1(\overline{x},\overline{y});$$

otherwise,

$$\langle(\widetilde{x}^*_{21},\widetilde{y}^*_{21}),(x,y)-(\overline{x},\overline{y})\rangle\leq F_2(x,y)-F_2(\overline{x},\overline{y}).$$

So taking for example the case where $\widehat{\alpha}_1>\widehat{\alpha}_2$, we get the existence of $(\widetilde{x}^*_{12},\widetilde{y}^*_{12})\in\partial^{\mathcal{T}}F_1(\widetilde{x}_{\widehat{\alpha}_2},\widetilde{y}_{\widehat{\alpha}_2})$ and $(\widetilde{x}^*_{\widehat{\alpha}_2},\widetilde{y}^*_{\widehat{\alpha}_2})\in\partial^{\mathcal{T}}F_2(\widetilde{x}_{\widehat{\alpha}_2},\widetilde{y}_{\widehat{\alpha}_2})$, such that

$$\langle(\widetilde{x}^*_{12},\widetilde{y}^*_{12}),(x,y)-(\overline{x},\overline{y})\rangle\leq F_1(x,y)-F_1(\overline{x},\overline{y}),$$
$$\langle(\widetilde{x}^*_{\widehat{\alpha}_2},\widetilde{y}^*_{\widehat{\alpha}_2}),(x,y)-(\overline{x},\overline{y})\rangle\leq F_2(x,y)-F_2(\overline{x},\overline{y}).$$

Similar finding is obtained for the other case of $\widehat{\alpha}_2>\widehat{\alpha}_1$. Consequently, setting $\widehat{\alpha}':=\min\{\widehat{\alpha}_1,\widehat{\alpha}_2\}$, we find $(\widetilde{x}^*_{\widehat{\alpha}'_1},\widetilde{y}^*_{\widehat{\alpha}'_1})\in\partial^{\mathcal{T}}F_1(\widetilde{x}_{\widehat{\alpha}'},\widetilde{y}_{\widehat{\alpha}'})$ and $(\widetilde{x}^*_{\widehat{\alpha}'_2},\widetilde{y}^*_{\widehat{\alpha}'_2})\in\partial^{\mathcal{T}}F_2(\widetilde{x}_{\widehat{\alpha}'},\widetilde{y}_{\widehat{\alpha}'})$, such that

$$\langle(\widetilde{x}^*_{\widehat{\alpha}'_1},\widetilde{y}^*_{\widehat{\alpha}'_1}),(x,y)-(\overline{x},\overline{y})\rangle\leq F_1(x,y)-F_1(\overline{x},\overline{y}),$$
$$\langle(\widetilde{x}^*_{\widehat{\alpha}'_2},\widetilde{y}^*_{\widehat{\alpha}'_2}),(x,y)-(\overline{x},\overline{y})\rangle\leq F_2(x,y)-F_2(\overline{x},\overline{y}).$$

By continuing this process, we can find $\widehat{\alpha}\in(0,\alpha)$ such that $\widehat{\alpha}:=\min\{\widehat{\alpha}_1,\widehat{\alpha}_2,\ldots,\widehat{\alpha}_{p_u}\}$, and for all $i\in\{1,\ldots,p_u\}$, there is $(\widetilde{x}^*_{\widehat{\alpha}_i},\widetilde{y}^*_{\widehat{\alpha}_i})\in\partial^{\mathcal{T}}F_i(\widetilde{x}_{\widehat{\alpha}},\widetilde{y}_{\widehat{\alpha}})$ such that

$$\langle(\widetilde{x}^*_{\widehat{\alpha}_i},\widetilde{y}^*_{\widehat{\alpha}_i}),(x,y)-(\overline{x},\overline{y})\rangle\leq F_i((x,y)-F_i(\overline{x},\overline{y}).$$

Multiplying both the sides of the above inequalities by $\widehat{\alpha}$, we obtain (3.5) as in Case 1. Henceforth, by (3.1), we deduce

$$\langle(\widehat{x}^*,\widehat{y}^*),(\widetilde{x}_{\widehat{\alpha}},\widetilde{y}_{\widehat{\alpha}})-(\overline{x},\overline{y})\rangle_{p_u}\leq 0,$$

for some $(\widehat{x}^*,\widehat{y}^*)=\big((\widetilde{x}^*_{\widehat{\alpha}_1},\widetilde{y}^*_{\widehat{\alpha}_1}),...,(\widetilde{x}^*_{\widehat{\alpha}_{p_u}},\widetilde{y}^*_{\widehat{\alpha}_{p_u}})\big)\in\partial^{\mathcal{T}}F(\widetilde{x}_{\widehat{\alpha}},\widetilde{y}_{\widehat{\alpha}})$.

Now, since $\phi_x$ is $\partial^{\mathcal{T}}$-convex over $T_x$ and that $g$ is quasiconvex, then by Corollary 1 $(\widetilde{x}_{\widehat{\alpha}},\widetilde{y}_{\widehat{\alpha}})\in\Pi_M$. This means $(\overline{x},\overline{y})$ is not a solution of (MBVVI). $\square$

Building upon the observation in Remark 1 that $\partial^{\mathcal{T}}$-convexity implies $\partial^{\mathcal{T}}$-quasiconvexity, we can readily derive the following corollary, which provides both necessary and sufficient conditions for a solution to (MBVVI) to also qualify as an efficient solution to (1.1).

**Corollary 2.** *Assume that $\Pi$ is convex. Additionally, suppose that $F$ and $\phi_x$, $x \in \mathbb{R}^{n_u}$, are $\partial^{\mathcal{T}}$-convex over $\Pi$ and $T_x$, respectively, $g$ is quasiconvex and $F_i$ is continuous for all $i \in \{1, ..., p_u\}$. Then, $(\overline{x}, \overline{y})$ is an efficient solution of (1.1) if and only if it is a solution of (MBVVI).*

We will now establish that, given the assumption of $\partial^{\mathcal{T}}$-pseudoconvexity of $F$ and the $\partial^{\mathcal{T}}$-quasiconvexity of $\phi_x$, satisfying (SBVVI) is sufficient for a solution to be efficient to (1.1).

**Theorem 6.** *Suppose that $F$ is $\partial^{\mathcal{T}}$-pseudoconvex over $\Pi$, $\phi_x$ (for all $x \in \mathbb{R}^{n_u}$) is $\partial^{\mathcal{T}}$-quasiconvex over $T_u$ and $g$ is quasiconvex. Any solution of (SBVVI) is also an efficient solution of (1.1).*

*Proof.* Suppose that $(\overline{x}, \overline{y})$ is not an efficient solution of (1.1). Then there is $(x, y) \in \Pi$ such that

$$F(x, y) - F(\overline{x}, \overline{y}) \leq 0.$$

By the $\partial^{\mathcal{T}}$-pseudoconvexity of $F$ over $\Pi$, we deduce that for all $(\overline{x}^*, \overline{y}^*) \in \partial^{\mathcal{T}} F(\overline{x}, \overline{y})$ verifying

$$\langle (\overline{x}^*, \overline{y}^*), (x, y) - (\overline{x}, \overline{y}) \rangle_{p_u} \leq 0.$$

Since $\phi_x$ is $\partial^{\mathcal{T}}$-quasiconvex over $T_x$ and that $g$ is quasiconvex, then by Proposition 4 $(x, y) \in \Pi_S$. Consequently, $(\overline{x}, \overline{y})$ is not a solution of (SBVVI). $\square$

We can easily derive the following corollary based on the observation in Remark 1 that $\partial^{\mathcal{T}}$-convexity implies $\partial^{\mathcal{T}}$-pseudoconvexity.

**Corollary 3.** *Suppose that $F$ and $\phi_x$, $x \in \mathbb{R}^{n_u}$, are $\partial^{\mathcal{T}}$-convex over $\Pi$ and $T_x$, respectively and $g$ is quasiconvex. Any solution of (SBVVI) is also an efficient solution of (1.1).*

**Remark 2.** The theorems presented in this paper extend several related results in the literature, including those in [11] and [1].

Now, we illustrate some of the obtained results through the following example.

**Example 1.** Consider $n_u = n_l = 1$, $p_u = p_l = 2$, $m_u = m_l = 2$ and let the leader's objectives and constraints be given by

$$F_1(x,y) = |x| + y^2, \qquad F_2(x,y) = x^2 + |y|,$$
$$G_1(x,y) = x, \qquad G_2(x,y) = -x - 1.$$

The follower's objective and constraints are as follows

$$f(x,y) = \phi_x(y) = yx^2 + x,$$
$$g_1(x,y) = x - y, \qquad g_2(x,y) = y.$$

First, after straightforward calculations, we find that the feasible set of (1.1) and the set of constraints are respectively $\Pi = [-1,0] \times \{0\}$ and $K = [-1,0] \times [-1,0]$. For all $(x,y) \in K$, we have the following tangential subdifferentials of the leaders' objectives:

$$\partial^{\mathcal{T}} F_1(x,y) = \begin{cases} \{(1,2y)\}, & \text{if } x > 0 \\ \{(-1,2y),(1,2y)\}, & \text{if } x = 0 \\ \{(-1,2y)\}, & \text{if } x < 0 \end{cases}$$

$$\partial^{\mathcal{T}} F_2(x,y) = \begin{cases} \{(2x,1)\}, & \text{if } y > 0 \\ \{(2x,-1),(2x,1)\}, & \text{if } y = 0 \\ \{(2x,-1)\}, & \text{if } y < 0. \end{cases}$$

Additionally, we have

$$\partial^{\mathcal{T}} \phi_x(y) = \{x^2\}.$$

To verify $\partial^*$-convexity of $F = (F_1, F_2)$ over $\Pi$, consider a point $(x,y), (\overline{x}, \overline{y}) \in \Pi$. We have

$$\begin{aligned} F_1(x,y) - F_1(\overline{x},\overline{y}) &= |x| + y^2 - |\overline{x}| - \overline{y}^2 \\ &= \overline{x} - x + (y + \overline{y})(y - \overline{y}). \end{aligned}$$

On the other hand, for all $\xi_1 \in \partial^* F_1(\overline{x},\overline{y})$ one has

$$\langle \xi_1, (x,y) - (\overline{x},\overline{y}) \rangle = \begin{cases} \langle (\pm 1, 2\overline{y}), (x, y - \overline{y}) \rangle = \pm x + 2\overline{y}(y - \overline{y}), & \text{if } \overline{x} = 0 \\ \langle (-1, 2\overline{y}), (x - \overline{x}, y - \overline{y}) \rangle = \overline{x} - x + 2\overline{y}(y - \overline{y}), & \text{if } \overline{x} < 0 \end{cases}$$

Then

$$\begin{aligned} F_1(x,y) - F_1(\overline{x},\overline{y}) - \langle \xi_1, (x,y) - (\overline{x},\overline{y}) \rangle &= \begin{cases} (y - \overline{y})^2 \text{ or } -2x + (y - \overline{y})^2, & \text{if } \overline{x} = 0 \\ (y - \overline{y})^2, & \text{if } \overline{x} < 0 \end{cases} \\ &\geq 0. \end{aligned}$$

In the same way we show that

$$F_2(x,y) - F_2(\overline{x},\overline{y}) - \langle \xi_2, (x,y) - (\overline{x},\overline{y}) \rangle \geq 0.$$

Consequently, for all $(x,y),(\overline{x},\overline{y}) \in \Pi$ and $\xi \in \partial^{\mathcal{T}} F(\overline{x},\overline{y}) = \partial^{\mathcal{T}} F_1(\overline{x},\overline{y}) \times \partial^{\mathcal{T}} F_1(\overline{x},\overline{y})$ one has

$$F(x,y) \geqq F(\overline{x},\overline{y}) + \langle \xi, (x,y) - (\overline{x},\overline{y}) \rangle_2.$$

Hence $F$ is $\partial^{\mathcal{T}}$-convex over $\Pi$.
Let us now check the $\partial^{\mathcal{T}}$-convexity of $\phi_x$ over $T_x = [-1,0]$. Let $y, \overline{y} \in T_x$ and $\zeta \in \partial^* \phi_x(\overline{y}) = \{x^2\}$. We have

$$\phi_x(y) - \phi_x(\overline{y}) - \langle \zeta, y - \overline{y} \rangle = x^2(y - \overline{y}) - x^2(y - \overline{y}) = 0.$$

Hence $\phi_x$ is $\partial^{\mathcal{T}}$-convex over $T_x$.
Letting $(\overline{x},\overline{y}) := (0,0)$, we get

$$\partial^{\mathcal{T}} F(\overline{x},\overline{y}) = \{(-1,0),(1,0)\} \times \{(0,-1),(0,1)\} \text{ and } \partial^{\mathcal{T}} \phi_x(\overline{y}) = \{(0,1)\}.$$

Since for all $y \in T_x$, there is $\overline{y}^* = 0 \in \partial^{\mathcal{T}} \phi_x(\overline{y})$ such that

$$\langle \overline{y}^*, y - \overline{y} \rangle \geq 0.$$

Then $(\overline{x},\overline{y}) \in \Pi_S$.
For all $(x,y) \in \Pi_S$, there exists $(\overline{x}^*, \overline{y}^*) = \big((-1,0),(0,1)\big) \in \partial^{\mathcal{T}} F(\overline{x},\overline{y})$ verifying

$$\langle (\overline{x}^*, \overline{y}^*), (x,y) - (\overline{x},\overline{y}) \rangle_2 = (-x, y) \in [0,1] \times \{0\}.$$

Therefore, we can see that the point $(\overline{x},\overline{y})$ solves (SBVVI) since

$$\langle (\overline{x}^*, \overline{y}^*), (x,y) - (\overline{x},\overline{y}) \rangle_2 \notin -\mathbb{R}^2_+ \setminus \{(0,0)\}.$$

We can conclude that all conditions of Theorem 3 are fulfilled and therefore $(\overline{x},\overline{y})$ is a solution of (MBVVI).
Also all the conditions stated in Corollary 3 have been satisfied, which means $(\overline{x},\overline{y})$ qualifies as an efficient solution for (1.1).

## 4. Conclusions and future directions

This paper introduced bilevel variational inequalities (BVVIs) of Minty and Stampacchia types and established their connection to multiobjective bilevel optimization problems under generalized convexity. We derived sufficient and necessary optimality conditions based on the tangential subdifferential. This subdifferential provides more accurate optimality characterizations than commonly used alternatives such as the Fréchet, Clarke, or Michel-Penot subdifferentials. The results offer a direct variational formulation of MBOPs that does not rely on intermediate transformations.
The main advantages of our approach are:

- It provides a direct link between MBOPs and vector variational inequalities without the need for value-function or KKT-based reformulations.
- It extends the existing variational inequality theory to the bilevel setting under generalized convexity.
- It applies the tangential subdifferential, which yields sharper optimality conditions than those derived from other subdifferential concepts.

This work also points to several research directions:

- Extending the results to bilevel problems that involve interval-valued or set-valued objective functions.
- Investigating other solution concepts such as quasi-efficient solutions and approximate solutions.
- Developing the theory of BVVIs on more general geometric structures such as Riemannian and Hadamard manifolds.
- Exploring the use of other subdifferential types to broaden the applicability of the results.

One limitation of this study is its reliance on generalized convexity assumptions. These assumptions may not hold in certain real-world applications. Addressing bilevel problems outside the generalized convexity setting remains an open and challenging question. Nevertheless, this variational inequality framework provides a solid foundation for future theoretical developments and algorithm design in bilevel multiobjective optimization.

**Conflict of Interest:** The authors declare that they have no conflict of interest.

**Data Availability:** Data sharing is not applicable to this article as no data sets were generated or analyzed during the current study.

## References

[1] S. Al-Homidan and Q.H. Ansari, *Generalized minty vector variational-like inequalities and vector optimization problems*, J. Optim. Theory Appl. **144** (2010), no. 1, 1–11.
https://doi.org/10.1007/s10957-009-9591-7.

[2] F.H. Clarke, *Optimization and Nonsmooth Analysis*, Society for Industrial and Applied Mathematics (SIAM, 3600 Market Street, Floor 6, Philadelphia, PA 19104), 1990.

[3] S. Dempe, *First-order necessary optimality conditions for general bilevel programming problems*, J. Optim. Theory Appl. **95** (1997), no. 3, 735–739.
https://doi.org/10.1023/A:1022646611097.

[4] A. Fischer, A.B. Zemkoho, and S. Zhou, *Semismooth newton-type method for bilevel optimization: global convergence and extensive numerical experiments*, Optim. Methods Softw. **37** (2022), no. 5, 1770–1804.
https://doi.org/10.1080/10556788.2021.1977810.

[5] F. Giannessi, *Theorems of the alternative: quadratic programs and complementarity problems*, pp. 151–186, John Wiley & Sons Ltd, New York, 1980.

[6] P. Gupta and S.K. Mishra, *On minty variational principle for nonsmooth vector optimization problems with generalized approximate convexity*, Optimization **67** (2018), no. 8, 1157–1167.
https://doi.org/10.1080/02331934.2018.1466884.

[7] H. Huang, *Strict vector variational inequalities and strict Pareto efficiency in nonconvex vector optimization*, Positivity **19** (2015), no. 1, 95–109.
https://doi.org/10.1007/s11117-014-0285-5.

[8] M. Jennane, L. El Fadil, and E.M. Kalmoun, *On local quasi efficient solutions for nonsmooth vector optimization problems*, Croat. Oper. Res. Rev. **11** (2020), no. 1, 1–10.
https://doi.org/10.17535/crorr.2020.0001.

[9] M. Jennane, E.M. Kalmoun, and L. El Fadil, *Optimality conditions for nonsmooth multiobjective bilevel optimization using tangential subdifferentials*, RAIRO Oper. Res. **55** (2021), no. 5, 3041–3048.
https://doi.org/10.1051/ro/2021139.

[10] B. Kohli, *Variational inequalities and optimistic bilevel programming problem via convexifactors*, pp. 243–255, Springer New York, New York, NY, 2011.

[11] V. Laha and S.K. Mishra, *On vector optimization problems and vector variational inequalities using convexificators*, Optimization **66** (2017), no. 11, 1837–1850.
https://doi.org/10.1080/02331934.2016.1250268.

[12] C. Lemaréchal, *An introduction to the theory of nonsmooth optimization*, Optimization **17** (1986), no. 6, 827–858.
https://doi.org/10.1080/02331938608843204.

[13] J.E. Martínez-Legaz, *Optimality conditions for pseudoconvex minimization over convex sets defined by tangentially convex constraints*, Optim. Lett. **9** (2015), no. 5, 1017–1023.
https://doi.org/10.1007/s11590-014-0822-y.

[14] ———, *A mean value theorem for tangentially convex functions*, Set-Valued Var. Anal. **31** (2023), no. 2, Article numbe 13.
https://doi.org/10.1007/s11228-023-00674-3.

[15] P. Michel and J.P. Penot, *A generalized derivative for calm and stable functions*, Diff. Integ. Eq. **5** (1992), no. 2, 433–454.

[16] S.K. Mishra and V. Laha, *On minty variational principle for nonsmooth vector optimization problems with approximate convexity*, Optim. Lett. **10** (2016), no. 3, 577–589.
https://doi.org/10.1007/s11590-015-0883-6.

[17] J.V. Outrata, *On the numerical solution of a class of Stackelberg problems*, Zeitschrift für Operations Research **34** (1990), no. 4, 255–277.

https://doi.org/10.1007/BF01416737.

[18] B.N. Pshenichnyi, *Necessary Conditions for an Extremum*, CRC Press, 2020.

[19] G. Stampacchia, *Formes bilineaires coercitives sur les ensembles convexes*, Comptes Rendus Hebdomadaires Des Seances De L Academie Des Sciences **258** (1964), no. 18, 4413.

[20] S.K. Suneja and B. Kohli, *Optimality and duality results for bilevel programming problem using convexifactors*, J. Optim. Theory Appl. **150** (2011), no. 1, 1–19. https://doi.org/10.1007/s10957-011-9819-1.

[21] J.J. Ye and D.L. Zhu, *Optimality conditions for bilevel programming problems*, Optimization **33** (1995), no. 1, 9–27. https://doi.org/10.1080/02331939508844060.

[22] A. Yezza, *First-order necessary optimality conditions for general bilevel programming problems*, J. Optim. Theory Appl. **89** (1996), no. 1, 189–219. https://doi.org/10.1007/BF02192648.

[23] A.B. Zemkoho and S. Zhou, *Theoretical and numerical comparison of the Karush–Kuhn–Tucker and value function reformulations in bilevel optimization*, Comput. Optim. Appl. **78** (2021), no. 2, 625–674. https://doi.org/10.1007/s10589-020-00250-7.