Communications in Combinatorics and Optimization
Vol. xx, No. x (xxxx), pp. 1-13
https://doi.org/10.22049/cco.2025.29949.2235

CCO
Commun. Comb. Optim.

Research Article

Exact double domination in subdivision, Mycielskian and middle graphs

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Received: 11 August 2024; Accepted: 27 April 2025
Published Online: 5 May 2025

Abstract: An exact doubly dominating set (also called an efficient doubly dominating set in [F. Harary and T.W. Haynes, Double domination in graphs, Ars Combin. 55 (2000), 201–213]) for a graph $G = (V, E)$ is a subset D of vertices such that each vertex of G is dominated by exactly two vertices of D. In this paper we show that subdivision graphs admit exact doubly dominating sets under specific conditions, while Mycielskian and middle graphs do not. We provide some characterizations and we investigate the existence of exact doubly dominating sets for their complements.

Keywords: exact double domination, subdivision, middle, Mycielskian.

AMS Subject classification: 05C69, 05C76

1. Introduction

Let $G = (V, E)$ be a finite and simple graph with vertex set $V = V(G)$ and edge set $E = E(G)$. The set $N_G(u)$ denotes the (open) neighborhood of $u \in V(G)$, which means the set of all adjacent vertices to u in G, the closed neighborhood of u is $N_G[u] = N_G(u) \cup \{u\}$ and the degree of u is $\deg_G(u) = |N_G(u)|$. An isolated vertex is a vertex of degree zero. A clique C in G is a subset of vertices of G such that every two distinct vertices in C are adjacent and hence, the induced subgraph of G on it is a complete graph. A matching in G is a set of pairwise non-incident edges of E and, a perfect matching is a matching in which every vertex of the graph is incident to exactly one edge of the matching. For a subset of vertices $X \subseteq V$, $G[X]$

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denotes the subgraph induced by X. Note that X is an independent set in G If and only if $G[X]$ contains no edge. The complement graph of G is denoted by $\overline{G}$. In recent years much attention drawn to the domination theory which has different applications in diverse areas and is an interesting branch in graph theory. Each vertex of a graph is said to dominate every vertex in its closed neighborhood. A subset S of $V(G)$ is a dominating set for G if each vertex in $V(G) \setminus S$ is adjacent to at least one vertex in S. The domination number of G, denoted by $\gamma(G)$, is the minimum size of a dominating set of G. In [8] Fink and Jacobson generalized the concept of dominating sets. Let k be a positive integer. A subset D of vertices in G is a k-dominating set if each vertex in $V(G) \setminus D$ is adjacent to at least k vertices in D. The k-domination number $\gamma_k(G)$ is the minimum cardinality of a k-dominating set of G. Hence, for $k = 1$, 1-dominating sets are the classical dominating sets. A vertex subset D is a perfect k-dominating set if each vertex v of G, not in D, is adjacent to exactly k vertices of D. The perfect k-domination problem is NP-complete for general graphs, see [4]. A possible application for perfect k-domination is provided by a specialist giving radiation (or a powerful drug) to a patient. In order to be more effective, there must be precisely k units administered to the neighboring cells (any more may be very dangerous). The cells where the drug is given directly are weakened and harmed. Thus, we wish to minimize the number of spots or cells where it is given. Hence, we would want a minimum perfect k-dominating set, see [4]. Note that every nontrivial graph has a perfect k-dominating set, since the entire vertex set is such a set and there are graphs whose only perfect k-dominating set is their entire vertex set (consider the stars $K_{1,t}$ for $1 < k < t$). A paired-dominating set is a dominating set of vertices whose induced subgraph has a perfect matching. A set $S \subseteq V$ is a double dominating set for G if each vertex in V is dominated by at least two vertices in S, note that this concept is different from 2-domination. Cockayne et al. in [6] called a vertex subset D of V to be a perfect dominating set of G if every vertex in $V(G) \setminus D$ is adjacent to exactly one vertex of D, see also [7]. Note that sets that are both perfect dominating and independent are called perfect codes by Biggs in [3] or efficient dominating sets by Bange, Barkauskas and Slater in [2]. Analogously to perfect or efficient domination, Harary and Haynes in [12] defined an efficient doubly dominating set as a subset D of vertices such that each vertex of G is dominated by exactly two vertices of D, i.e $|N_G[v] \cap D| = 2$ for each $v \in V$, see also [11]. This concept generalizes efficient domination by ensuring each vertex is covered exactly twice, unlike classical domination which allows flexibility in coverage. Chellali, Khelladi and Maffray in [5] prefer to use the phrase exact doubly dominating set for this concept and they show that the complexity of the problem of deciding whether a graph admits an exact doubly dominating set is NP-complete. This concept can be rich in applications. For instant, consider prisoners and guards where the concept of domination indicates that each prisoner can be seen by some guard. In exact double domination, securing of the prisoners as well as safety for the guards is considered by providing a designated backup for each guard. This increases the security by requiring that each prisoner is guarded by two guards. A similar argument can be investigated for networks. Also, this concept may finds applications in diverse fields like coding theory, parallel computing, wireless ad hoc networks and fault-tolerant networks, see [5] and [13]. Note that not all graphs admit an exact doubly dominating set (for example consider the 4-cycle C_4 or the stars). In [5] a constructive characterization of those trees that admit an exact doubly dominating set is provided, and they establish a necessary and sufficient condition for the existence of an exact doubly dominating set in a connected cubic graph. Note that the following result show that an exact doubly dominating set is a paired-dominating set but obviouslu the converse is not true in general. Also, an exact doubly dominating set is a perfect 2-dominating set but the converse is not true in general.

Theorem 1. *[5] The vertex set of every exact doubly dominating set induces a matching. Moreover, if G has an exact doubly dominating set, then all such sets have the same size.*

Theorem 2. *[5]*
a) A path P_n has an exact doubly dominating set if and only if $n \equiv 2 \pmod 3$. If this holds, then the size of any such set is $\frac{2(n+1)}{3}$.
b) A cycle C_n has an exact doubly dominating set if and only if $n \equiv 0 \pmod 3$. If this holds, then the size of any such set is $\frac{2n}{3}$.

2. Subdivision graphs

The subdivision operation is an operation that replaces an edge with a path of length at least two by inserting new vertices. If each edge is replaced by a path of order three (i.e., 1-subdividing each edge of G), then the subdivision graph is denoted by $S(G)$, see $S(P_5)$ in Figure 1. Domination number and identifying code number of the subdivision of some famous families of graphs are investigated and determined in [1]. Some upper and lower bounds for the mixed metric dimension of $S(G)$ is provided in [9]. The minimum number of edges that must be subdivided in order to increase the total k-rainbow domination number of a graph is considered in [15]. Also, $2-$rainbow domination number of the subdivision graph of some famous families of graphs is determined in [16]. Recall that an edge contraction is an operation that removes an edge while simultaneously merging the two (end) vertices that it previously joined. In the following, we first investigate the existence of exact doubly dominating sets in the subdivision graph $S(G)$ (see Theorem 3).

Lemma 1. *Let G be an n-vertex graph and assume that there exists an exact doubly dominating set D for its subdivision $S(G)$. Then, $\big|D \cap V(G)\big| \le n-1$ and the equality holds if and only if $G \in \{P_3, C_3\}$.*

Proof. Assume that $V(G) = \{v_1, v_2, ..., v_n\}$ and $V(S(G)) = V(G) \cup Z$ in which

$$Z = \big\{z_{ij} : \ v_iv_j \in E(G)\big\}, \ N_{S(G)}(z_{ij}) = \{v_i, v_j\}.$$

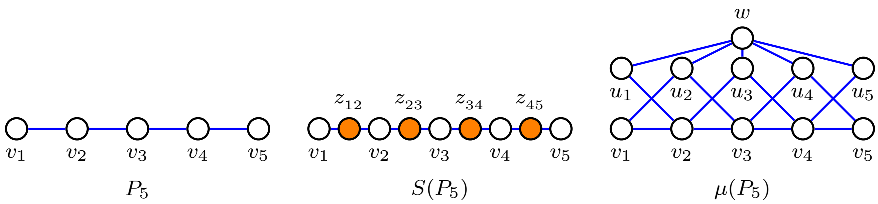

Figure 1. **A path, it's subdivision and Mycielskian**

Note that $V(G)$ induces an independent set in $S(G)$ and for each $v_i \in V(G)$ we have

$$|N_{S(G)}[v_i] \cap V(G)| = |\{v_i\}| = 1 \neq 2.$$

Hence, $D \neq V(G)$. If $V(G) \subsetneqq D$, then for each $z_{ij} \in Z \cap D$ we obviously have

$$|N_{S(G)}[z_{ij}] \cap D| = |\{v_i, v_j, z_{ij}\}| = 3,$$

which is a contradiction. Thus, $\big|D \cap V(G)\big| \leq n-1$.
It is easy to check that the bound is attained for $G \in \{P_3, C_3\}$. This follows from the fact that in P_3 and C_3, the placement of vertices ensures exact double domination without surplus or deficiency. Now let G be a graph for which $\big|D \cap V(G)\big| = n-1$. Assume that $V(G) \setminus D = \{v_i\}$. Since D is an exact doubly dominating set and $v_i \notin D$, there exist exactly two indices j and j' such that $\{z_{ij}, z_{ij'}\} \subseteq D \cap Z$. Thus, $\{v_iv_j, v_iv_{j'}\} \subseteq E(G)$. If there exists an index j'' such that $v_iv_{j''} \in E(G)$, then $z_{ij''} \notin D$ and this implies that $N_{S(G)}[z_{ij''}] \cap D = \{v_{j''}\}$, which is a contradiction. Thus, v_i is a vertex of degree two in G. Since $V(G) \setminus \{v_i\} \subseteq D$, for each $z_{rs} \in Z \setminus \{z_{ij}, z_{ij'}\}$ we have $\{v_r, v_s\} \subseteq N_{S(G)}[z_{rs}] \cap D$. Hence, $z_{rs} \notin D$. This means that for each vertex $v_r \notin \{v_i, v_j, v_{j'}\}$ (if any one exists!) we have

$$N_{S(G)}[v_r] \cap D = \{v_r\},$$

which is a contradiction. Thus, $V(G) = \{v_i, v_j, v_{j'}\}$ and hence, $G \in \{P_3, C_3\}$. $\square$

Definition 1. Let Γ be the family of all finite graphs containing a perfect matching. Suppose that $H \in \Gamma$ and $M \subseteq E(H)$ is a perfect matching for H. Define the new graph $S_M(H)$ to be the graph obtained from H by 1-subdividing each edge of M. Note that $|V(H)| = 2|M|$ and

$$|V(S_M(H))| = |V(H)| + |M| = 3\,|M| = \frac{3\,|V(H)|}{2},$$

which specially implies that $|V(S_M(H))|$ is a multiple of 3. Also, let

$$\mathcal{G} = \big\{S_M(H) :\ H \in \Gamma \text{ and } M \text{ is a perfect matching for } H\big\}.$$

Theorem 3. *Let G be a graph. Then, there exists an exact doubly dominating set D for $S(G)$ if and only if $G \in \mathcal{G}$. In this case we have $|D| = \frac{4|V(G)|}{3}$.*

Proof. At first, assume that $G \in \mathcal{G}$. Hence, $G = S_M(H)$ for some graph H and some perfect matching M of H. Let Ω be the set of (new) vertices of G which are obtained by 1-subdividing each edge of matching M in H. Hence, $V(G) = V(H) \cup \Omega$. Assume that $V(G) = \{v_1, v_2, ..., v_n\}$ and $V(S(G)) = V(G) \cup Z$ in which

$$Z = \big\{z_{ij} : \ v_iv_j \in E(G)\big\}, \qquad N_{S(G)}(z_{ij}) = \{v_i, v_j\}.$$

Note that for each $v_i \in \Omega$ we have $\deg_G(v_i) = 2$. Since M is a perfect matching for H, for each pair of distinct vertices $v_i, v_{i'}$ in Ω we have $N_G(v_i) \cap N_G(v_{i'}) = \emptyset$ and

$$\bigcup_{v_i \in \Omega} N_G(v_i) = V(G) \setminus \Omega.$$

Specially, Ω is a dominating set for G and $|V(G)| = 3|\Omega|$ (or equivalently, $|\Omega| = \frac{n}{3}$). Now consider the subdivision graph $S(G)$ and let

$$D = \big(V(G) \setminus \Omega\big) \bigcup \Bigg(\bigcup_{v_i \in \Omega} N_{S(G)}(v_i) \Bigg).$$

For each $v_i \in \Omega$ we have

$$\big|D \cap N_{S(G)}[v_i]\big| = |N_{S(G)}(v_i)| = \deg_G(v_i) = 2.$$

For each $v_j \in V(G) \setminus \Omega$, there exists unique $v_i \in \Omega$ such that $v_jv_i \in E(G)$ and hence,

$$\big|D \cap N_{S(G)}[v_j]\big| = \big|\{v_j, z_{ij}\}\big| = 2.$$

For each $z_{rs} \in Z$, if $\{v_r, v_s\} \cap \Omega = \emptyset$, then we have

$$\big|D \cap N_{S(G)}[z_{rs}]\big| = \big|\{v_r, v_s\}\big| = 2,$$

and otherwise, we have $\big|\{v_r, v_s\} \cap \Omega\big| = 1$ and $z_{rs} \in D$, which implies that

$$\big|D \cap N_{S(G)}[z_{rs}]\big| = 2.$$

Therefore, D is an exact doubly dominating set for $S(G)$ and, we obviously have

$$|D| = (n - \frac{n}{3}) + \frac{n}{3} \times 2 = \frac{4n}{3},$$

as desired.
Now assume that G is an n-vertex graph such that there exists an exact doubly dominating set D for $S(G)$. We want to show that $G \in \mathcal{G}$. As before, assume that $V(G) = \{v_1, v_2, ..., v_n\}$ and $V(S(G)) = V(G) \cup Z$ in which

$$Z = \{z_{ij} : \ v_i v_j \in E(G)\}, \qquad N_{S(G)}(z_{ij}) = \{v_i, v_j\}.$$

Let $\Omega = V(G) \setminus D$. By Lemma 1, we have $|D \cap V(G)| \leq n-1$. Thus, $|V(G) \setminus D| \geq 1$ and hence, $\Omega \neq \emptyset$. Since D is an exact doubly dominating set for $S(G)$, for each $z_{ij} \in V(S(G))$ we have

$$2 = \big|D \cap N_{S(G)}[z_{ij}]\big| = \big|D \cap \{v_i, z_{ij}, v_j\}\big|.$$

Thus, if $v_i \in \Omega$, then $\{v_j, z_{ij}\} \subseteq D$ for each $v_j \in N_G(v_i)$. Specially, $\big|D \cap N_{S(G)}[v_i]\big| = 2$ implies that $\deg_G(v_i) = 2$. By Theorem 1, D induces a matching in $S(G)$. Thus, if $v_{j'} \in D$, then there exists unique $v_{i'} \in N_G(v_{j'})$ such that $z_{i'j'} \in D \cap N_{S(G)}(v_{j'})$. Hence, $v_{i'} \notin D$ which means that $v_{i'} \in \Omega$. Now the previous statement implies that $\deg_G(v_{i'}) = 2$. Therefore, for each pair of different vertices $v_i, v_{i'}$ in Ω we have $N_G(v_i) \cap N_G(v_{i'}) = \emptyset$ and

$$\bigcup_{v_i \in \Omega} N_G(v_i) = V(G) \setminus \Omega.$$

Specially, we can see that

$$|V(G)| = |\Omega| + |V(G) \setminus \Omega| = |\Omega| + 2|\Omega| = 3|\Omega|,$$

and

$$|D| = |D \cap V(G)| + |D \cap Z| = \big(|V(G)| - |\Omega|\big) + 2|\Omega| = 4|\Omega| = 4\frac{|V(G)|}{3}\ .$$

Now consider the graph G and for each $v_i \in \Omega$, contract exactly one of two incident edges of v_i in G and, let H be the resulting graph. Note that $|V(H)| = 2|\Omega|$ and $|E(H)| = |E(G)| - |\Omega|$. Also, let $M \subseteq E(H)$ be the set of remaining incident edges of v_i's (the elements of Ω). Note that by the previous facts and this construction, M is a perfect matching for H and $S_M(H) = G$. This means that $G \in \mathcal{G}$, as desired. $\square$

Example 1. Let G be a graph with the vertex set $\{a, b, c, d, e, f\}$ and the edge set $\{ab, ae, af, ef, bc, bd, cd\}$ as depicted in Figure 2 (i). Then, there exists an exact doubly dominating set for the subdivision graph $S(G)$ (consider the filled green vertices in Figure 2 (ii)). Note that $G = S_M(H) \in \mathcal{G}$ in which H is depicted in Figure 2 (ii) whose perfect mathching is M consists of two edges m_1 and m_2.

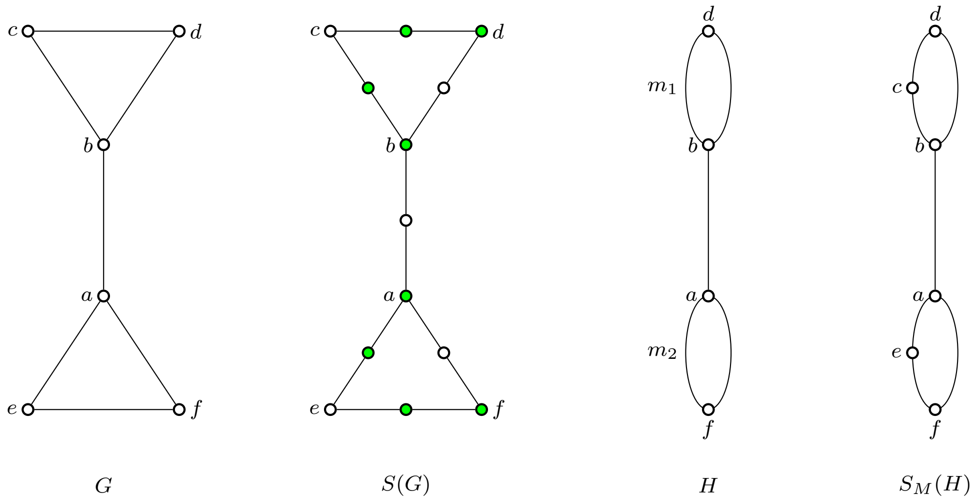

Figure 2. **Graphs corresponding to Example 1.**

Theorem 4. *Let G be a graph. Then, there exists an exact doubly dominating set D for $\overline{S(G)}$ if and only if G contains at least two isolated vertices and D consists of two isolated vertices of G.*

Proof. Obviously, each isolated vertex of G is adjacent to all other vertices of $\overline{S(G)}$ in $\overline{S(G)}$. Hence, if G contains at least two isolated vertices, then each set D consisting of two isolated vertices of G is an exact doubly dominating set for $\overline{S(G)}$. Note that if we use the previous notations, then $V(G)$ (and similarly, Z) induces an independent set in $S(G)$ and hence, induces a clique in its complement $\overline{S(G)}$.
Now assume that G is a graph such that there exists an exact doubly dominating set D for $\overline{S(G)}$. Since for each $v_i \in V(G)$ we have $\left|D \cap N_{\overline{S(G)}}[v_i]\right| = 2$ and $V(G)$ induces a clique in $\overline{S(G)}$, we must have $|D \cap V(G)| \leq 2$. Similarly, we have $|D \cap Z| \leq 2$.
If $|D \cap V(G)| = 0$, then by using Theorem 1, we obtain $|D \cap Z| = 2$ and hence, $D = \{z_{ij}, z_{rs}\}$ for two distinct vertices $z_{ij}, z_{rs} \in Z$. Note that $z_{ij} \in Z$ implies that $v_iv_j \in E(G)$ and hence, v_i is not adjacent to z_{ij} in $\overline{S(G)}$. Thus, $\left|D \cap N_{\overline{S(G)}}[v_i]\right| \leq 1$, which is a contradiction.
If $|D \cap V(G)| = 1$, then $D \cap V(G) = \{v_i\}$ for some $v_i \in V(G)$ and Theorem 1 implies that there exists unique vertex $z_{rs} \in D \cap Z$ which is adjacent to v_i in $\overline{S(G)}$ and hence, $v_i \notin \{v_r, v_s\}$. Since Z induces a clique, again Theorem 1 implies that $D = \{v_i, z_{rs}\}$. Thus, $\left|D \cap N_{\overline{S(G)}}[v_r]\right| = |\{v_i\}| = 1$, a contradiction.
Therefore, we must have $|D \cap V(G)| = 2$. Assume that $D \cap V(G) = \{v_i, v_j\}$. Note that v_i and v_j are adjacent in $\overline{S(G)}$. By Theorem 1, for each $z_{rs} \in D$ we must have $\{v_r, v_s\} \cap \{v_i, v_j\} \neq \emptyset$ because z_{rs} must be non-adjacent to v_i and v_j in $\overline{S(G)}$. If there

exists $z_{is} \in D$ for some $s \neq j$, then

$$\left|D \cap N_{\overline{S(G)}}[v_j]\right| = |\{v_i, v_j, z_{is}\}| = 3,$$

a contradiction. Similarly, $z_{js} \notin D$ for each $s \neq i$. Thus, $D \subseteq \{v_i, v_j, z_{ij}\}$ and now Theorem 1 implies that $D = \{v_i, v_j\}$. Specially, each member of Z must be adjacent to both v_i and v_j in $\overline{S(G)}$, which means that each member of Z is non-adjacent to both v_i and v_j in $S(G)$. Hence, v_i and v_j are two isolated vertices in G and the proof is complete. □

3. Mycielski graphs

Let G be graph with vertex set $V(G) = \{v_1, v_2, ..., v_n\}$. The Mycielski graph $\mu(G)$ of G is a graph of order $2n+1$ with the vertex set $V(G) \cup \{w, u_1, u_2, ..., u_n\}$ and with the edge set $E(G) \cup \{v_i u_j : \ v_i v_j \in E(G)\} \cup \{w u_i : \ 1 \leq i \leq n\}$, see $\mu(P_5)$ in Figure 1.

Theorem 5. *Let G be an arbitrary graph. Then, the Mycielskian graph $\mu(G)$ does not admit any exact doubly dominating set.*

Proof. Suppose on the contrary that $D \subseteq V(\mu(G))$ is an exact doubly dominating set for $\mu(G)$. Note that $N_{\mu(G)}(w) = \{u_1, u_2, ..., u_n\}$. Since w dominates each u_i, selecting w in D forces a unique second vertex in D, contracting the requirement that all vertices be covered exactly twice. More precisely, if $w \in D$, then Theorem 1 implies that there exists unique vertex u_i, $1 \leq i \leq n$, such that $u_i \in D$. Hence, $D \cap \{u_1, u_2, ..., u_n\} = \{u_i\}$. Since $\{w, u_i\} \subseteq D \cap N_{\mu(G)}[u_i]$ and $\left|D \cap N_{\mu(G)}[u_i]\right| = 2$, for each $v_j \in N_G(v_i) \subseteq N_{\mu(G)}(u_i)$ we have $v_j \notin D$. Thus, $N_{\mu(G)}[v_i] \cap D \subseteq \{v_i\}$ and hence, $|N_{\mu(G)}[v_i] \cap D| \neq 2$, which is a contradiction. This contradiction implies that $w \notin D$. Since $|N_{\mu(G)}[w] \cap D| = 2$, we must have $D \cap \{u_1, u_2, ..., u_n\} = \{u_i, u_j\}$ for some $i \neq j$. By Theorem 1, u_i is adjacent to a (unique) vertex of D. Hence, there exists unique vertex $v_r \in N_G(v_i) = \big(N_{\mu(G)}(u_i) \cap V(G)\big)$ such that $v_r \in D$. If $v_i \in D$, then $\{v_i, v_r, u_i\} \subseteq \big(N_{\mu(G)}[v_r] \cap D\big)$, which is a contradiction. Thus, $v_i \notin D$ and this implies that $N_{\mu(G)}[v_i] \cap D = \{v_r\}$. This also leads to another contradiction and hence, the proof is complete. □

Theorem 6. *The complement of the Mycielskian graph $\mu(G)$ (i.e., $\overline{\mu(G)}$) has an exact doubly dominating set D if and only if D consists of exactly two isolated vertices of G.*

Proof. Let $D \subseteq V(\overline{\mu(G)}) = V(\mu(G))$ be an exact doubly dominating set for $\overline{\mu(G)}$. Note that the set $\{u_1, u_2, ..., u_n\}$ induces a clique in $\overline{\mu(G)}$ and we have $N_{\overline{\mu(G)}}(w) = \{v_1, v_2, ..., v_n\}$.
If $w \in D$, then Theorem 1 implies that there exists unique vertex v_i, $1 \leq i \leq n$, such that $D \cap \{v_1, v_2, ..., v_n\} = \{v_i\}$. Since $v_i u_i \in E(\overline{\mu(G)})$ and $w u_i \notin E(\overline{\mu(G)})$,

the condition $|N_{\overline{\mu(G)}}[u_i] \cap D| = 2$ implies that $|D \cap \{u_1, u_2, ..., u_n\}| = 1$. This means that $|D| = 3$ which contradicts Theorem 1. Thus, $w \notin D$. Since $|N_{\overline{\mu(G)}}[w] \cap D| = 2$, there exist $i \neq j$ such that $D \cap \{v_1, v_2, ..., v_n\} = \{v_i, v_j\}$. Since the set $\{u_1, u_2, ..., u_n\}$ induces a clique in $\overline{\mu(G)}$, $v_i u_i \in E(\overline{\mu(G)})$ and $|N_{\overline{\mu(G)}}[u_i] \cap D| = 2$, we must have $|D \cap \{u_1, u_2, ..., u_n\}| \leq 1$. Thus,

$$|D| = \big|D \cap \{v_1, v_2, ..., v_n\}\big| + \big|D \cap \{u_1, u_2, ..., u_n\}\big| \leq 2 + 1.$$

Now Theorem 1 implies that $|D| = 2$. Hence, $D = \{v_i, v_j\}$ and again Theorem 1 implies that $v_i v_j \in E(\overline{\mu(G)})$. This means that $v_i v_j \notin E(\mu(G))$. Also, for each $v_r \in V(G) \setminus \{v_i, v_j\}$ the condition $|N_{\overline{\mu(G)}}[v_r] \cap D| = 2$ implies that $\{v_r v_i, v_r v_j\} \subseteq E(\overline{\mu(G)})$, which equivalently means that $v_r v_i, v_r v_j \notin E(G)$. Therefore, D consists of two isolated vertices v_i and v_j of G. The converse is obvious, and the proof is complete. □

4. Middle graphs

Recall that the line graph $L(G)$ of G is the graph with vertex set $E(G)$ in which e and e' are adjacent in $L(G)$ if and only if the corresponding edges share a common vertex in G. The concept of middle graph $M(G)$ of G was introduced by Hamada and Yoshimura in [10] as an intersection graph on the vertex set of G, whose vertex set is $V(G) \cup E(G)$ and two vertices a, b in its vertex set are adjacent whenever $a, b \in E(G)$ and a, b are adjacent in $L(G)$, or $a \in V(G), b \in E(G)$ and a, b are incident in G. When $V(G) = \{v_1, v_2, ..., v_n\}$, then for convenient we can set $V\big(M(G)\big) = V(G) \cup Z$, where $Z = \{z_{ij} :\ v_i v_j \in E(G)\}$ and

$$E\big(M(G)\big) = \{v_i z_{ij}, v_j z_{ij} :\ v_i v_j \in E(G)\} \cup E\big(L(G)\big).$$

Thus, $M(G)$ is a graph of order $|V(G)| + |E(G)|$ and size $2|E(G)| + |E(L(G))|$ and it contains the line graph $L(G)$ as an induced subgraph, see $M(C_4)$ in Figure 3. The domination number of the middle of some famous families of graphs such as star graphs, double stars, paths, cycles, wheels, complete graphs, complete bipartite graphs and friendship graphs is considered and determined in [14].

Theorem 7. *Let G be an arbitrary graph. Then, the middle graph $M(G)$ does not admit any exact doubly dominating set.*

Proof. Note that if $E(G) = \emptyset$, then $E(M(G)) = \emptyset$ and hence, $M(G)$ does not admit any exact doubly dominating set. Thus, let $E(G) \neq \emptyset$ and assume (on the contrary) that there exists an exact doubly dominating set $D \subseteq V(M(G))$ for $M(G)$. Choose an arbitrary vertex $v_i \in V(G)$ with $\deg_G(v_i) \geq 1$. Since $|N_{M(G)}[v_i] \cap D| = 2$ and $N_{M(G)}[v_i] \cap V(G) = \{v_i\}$, there exists at least one vertex $z_{ij} \in N_{M(G)}[v_i] \cap D$, see

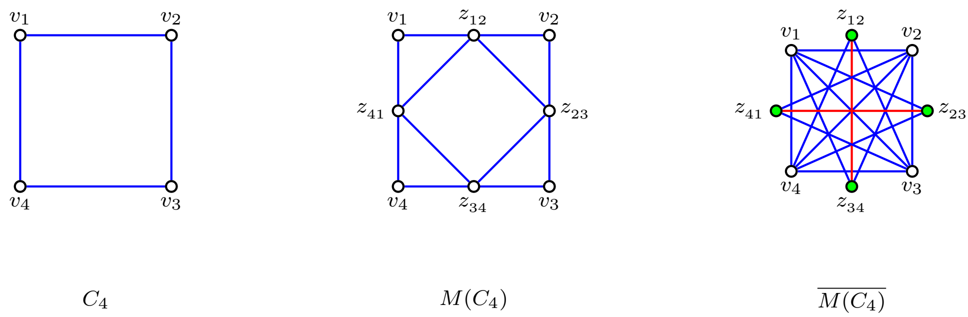

Figure 3. **The cycle C_4, it's middle graph and the complement of it's middle graph.**

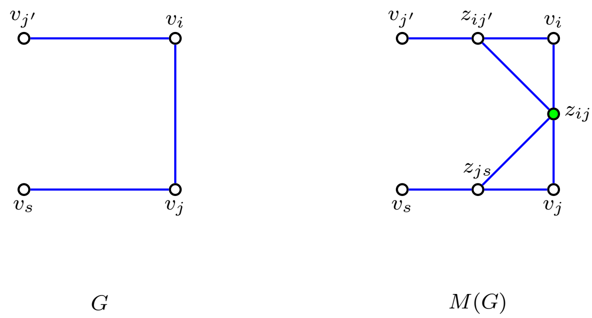

Figure 4. **A part of the middle graph of G.**

Figure 4. Hence, we have $v_j \in N_G(v_i)$.
Since $N_{M(G)}[v_i] \subseteq N_{M(G)}[z_{ij}]$ and

$$\left|N_{M(G)}[z_{ij}] \cap D\right| = 2 = \left|N_{M(G)}[v_i] \cap D\right|,$$

we must have $v_j \notin D$ and $z_{js} \notin D$ for each $v_s \in N_G(v_j) \setminus \{v_i\}$ (otherwise, $v_i \notin D$ and $z_{ij'} \notin D$ for each $v_{j'} \in N_G(v_i)$, which contradicts the fact $|N_{M(G)}[v_i] \cap D| = 2$). These facts imply that $N_{M(G)}[v_j] \cap D = \{z_{ij}\}$, which is a contradiction. Therefore, $M(G)$ has not any exact doubly dominating set. □

Theorem 8. *The complement of the middle graph $M(G)$ (i.e., $\overline{M(G)}$) has an exact doubly dominating set D if and only if G contains at least two isolated vertices and D consists of two isolated vertices of G or, G is (isomorphic to) the cycle C_4 and $D = E(C_4)$.*

Proof. For convenience, let $\mathscr{M} = M(G)$ and $\bar{\mathscr{M}} = \overline{M(G)}$. If G contains at least two isolated vertices, say x and y, then it is easy to see that $D = \{x, y\}$ is an exact doubly dominating set for $\bar{\mathscr{M}}$. Also, if $G = C_4$, then it is easy to check that $D = E(C_4)$ is an exact doubly dominating set for $\overline{M(C_4)}$, see Figure 3 in which the green vertices are the elements of D.

Now assume that G is a graph such that $\bar{\mathscr{M}}$ has an exact doubly dominating set D. Note that $V(G) \subseteq V(\mathscr{M})$ induces an independent set in $\mathscr{M}$ and hence, it induces a clique in $\bar{\mathscr{M}}$. Since $|N_{\bar{\mathscr{M}}}[v_i] \cap D| = 2$ for each $v_i \in V(G)$, we must have $|D \cap V(G)| \leq 2$. Hence, we consider the following three cases.

Case 1. $|D \cap V(G)| = 2$.
Suppose that $D \cap V(G) = \{v_i, v_j\}$. For each $v_{i'} \in V(G)$ we have $\{v_i, v_j\} \subseteq N_{\bar{\mathscr{M}}}[v_{i'}]$ and hence, $N_{\bar{\mathscr{M}}}[v_{i'}] \cap D = \{v_i, v_j\}$ because D is an exact doubly dominating set. If there exists $z_{rs} \in D \cap Z$, then the fact

$$N_{\bar{\mathscr{M}}}[v_i] \cap D = \{v_i, v_j\} = N_{\bar{\mathscr{M}}}[v_j] \cap D$$

implies that z_{rs} is non-adjacent to v_i and v_j in $\bar{\mathscr{M}}$ and hence, in G two vertices v_i and v_j are incident to the edge $v_r v_s$. This means that $v_r v_s = v_i v_j$. Therefore, $D = \{v_i, v_j, z_{ij}\}$ which contradicts Theorem 1. This contradiction shows that $D \cap Z = \emptyset$ and hence, $D = \{v_i, v_j\}$. If there exists an edge $v_i v_r$ in G which is incident to the vertex v_i in G, then we obtain $N_{\bar{\mathscr{M}}}[z_{ir}] \cap D \subseteq \{v_j\}$, a contradiction. Thus, v_i (and similarly v_j) is an isolated vertex in G. This means that D consists of two isolated vertices of G and hence, the proof is complete in this case.

Case 2. $|D \cap V(G)| = 1$.
Assume that $D \cap V(G) = \{v_i\}$. Since $|N_{\bar{\mathscr{M}}}[v_i] \cap D| = 2$, there exists a (unique) vertex $z_{rs} \in D \cap Z$ which is adjacent to v_i in $\bar{\mathscr{M}}$. This implies that $v_r v_s \in E(G)$ and $v_i \notin \{v_r, v_s\}$. Since $|N_{\bar{\mathscr{M}}}[v_r] \cap D| = 2$ and $v_i \in N_{\bar{\mathscr{M}}}[v_r]$, there exists a vertex $z_{r's'} \in D \cap Z$ which is adjacent to v_r in $\bar{\mathscr{M}}$. Thus, $v_{r'} v_{s'} \in E(G)$ and $v_r \notin \{v_{r'}, v_{s'}\}$. Note that by Theorem 1, v_i and z_{rs} are not adjacent to $z_{r's'}$ in $\bar{\mathscr{M}}$. Hence, we must have $v_i \in \{v_{r'}, v_{s'}\}$ and $v_s \in \{v_{r'}, v_{s'}\}$. These facts imply that $z_{r's'} = z_{is}$. Specially, we have $v_i v_s \in E(G)$. By Theorem 1, D induces a matching in $\bar{\mathscr{M}}$ and hence, there exists $z_{r''s''} \in D \cap Z$ which is adjacent to z_{is}. Similarly, Since v_i and z_{rs} are not adjacent to $z_{r''s''}$ in $\bar{\mathscr{M}}$, we must have $v_i \in \{v_{r''}, v_{s''}\}$ and $v_r \in \{v_{r''}, v_{s''}\}$. Thus, $z_{r''s''} = z_{ir}$ and hence, $v_i v_r \in E(G)$. Since two edges $v_i v_r$ and $v_i v_s$ in G share the common endpoint v_i, two vertices z_{ir} and z_{is} are not adjacent in $\bar{\mathscr{M}}$, which means z_{is} is not adjacent to $z_{r''s''}$ in $\bar{\mathscr{M}}$. This is a contradiction. Therefore, this case leads to a contradiction and hence, is impossible.

Case 3. $|D \cap V(G)| = 0$.
In this case, me must have $D \cap Z \neq \emptyset$. Assume that $z_{ij} \in D \cap Z$. Since $z_{ij} \in Z$, we have $v_i v_j \in E(G)$. By Theorem 1, there exists $z_{i'j'} \in D$ which is adjacent to z_{ij} in $\bar{\mathscr{M}}$. Therefore, $v_{i'} v_{j'} \in E(G)$ and $\{v_i, v_j\} \cap \{v_{i'}, v_{j'}\} = \emptyset$. Note that $z_{ij} \notin N_{\bar{\mathscr{M}}}[v_i]$ and hence, $\{z_{ij}, z_{i'j'}\}$ is a proper subset of D. For each $z_{i''j''} \in D \setminus \{z_{ij}, z_{i'j'}\}$, Theorem 1 implies that $z_{i''j''}$ is not adjacent to z_{ij} and $z_{i'j'}$ in $\bar{\mathscr{M}}$, and hence, we must have

$$\{i'', j''\} \cap \{i, j\} \neq \emptyset \neq \{i'', j''\} \cap \{i', j'\}.$$

Thus, we have $D = \{z_{ij}, z_{i'j'}, z_{ii'}, z_{jj'}\}$ or $D = \{z_{ij}, z_{i'j'}, z_{ij'}, z_{ji'}\}$. In each of these two cases, the four elements of D provides a cycle C_4 in G and D corresponds to the four edges of this 4-cycle. Since $|D| = 4$ and each vertex in $\overline{M(C_4)}$ is dominated exactly twice, G can not have extra vertices nor extra edges. This means that $G = C_4$ as desired. Now the proof is complete. $\square$

5. Conclusion

In this paper we study investigate the existence of exact doubly dominating sets for some famous graph operations and we show that subdivision graphs admit exact doubly dominating sets under specific conditions, while Mycielskian and middle graphs do not. We provide some characterizations and we investigate the existence of exact doubly dominating sets for their complements. Future work could explore exact double domination in Cartesian and lexicographic graph products or analyze its computational complexity in specific graph families.

Acknowledgements: The authors warmly thank the anonymous referees for reading this manuscript very carefully and providing numerous valuable corrections and suggestions which improve the quality of this paper.

Conflict of Interest: The authors declare that they have no conflict of interest.

Data Availability: Data sharing is not applicable to this article as no data sets were generated or analyzed during the current study.

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