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Research Article

Hierarchy of subfamilies of Ptolemaic graphs: Axiomatic characterizations and interval functions

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Abstract: Ptolemaic graphs are precisely the graphs that are both chordal and distance-hereditary. Markenzon et al. [L. Markenzon and C.F.E.M. Waga, New results on ptolemaic graphs, Discrete Appl. Math. 196 (2015), 135–140] established a hierarchy of Ptolemaic graphs comprising six subfamilies: laminar chordal graphs, block duplicate graphs, block graphs, AC graphs, trees, and paths. In this paper, we present a new proof of the characterization of AC graphs using forbidden induced subgraphs and identify an additional graph class that lies between AC graphs and paths within this hierarchy. The interval function is a well-studied tool in metric graph theory, and the characterization of the interval function of graph families is an interesting problem in metric graph theory having connections to first-order logic. In this paper, we propose a set of independent betweenness axioms for an arbitrary function known as a transit function and provide a characterization of the interval functions corresponding to graphs in the extended hierarchy of subgraphs of Ptolemaic graphs, specifically laminar chordal graphs, block duplicate graphs, and AC graphs.

Keywords: Ptolemaic graph, laminar chordal, block duplicate, AC graph, transit function, interval function.

AMS Subject classification: 05C12, 05C75

1. Introduction

The interval function I_G of a connected graph G is an important notion in metric graph theory and an essential concept in the study of the metric properties of graphs. It is

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defined as a function that maps for every pair of vertices u, v in G, the set $I_G(u,v) = \{w \mid w \text{ lies on a shortest } u,v\text{-path}\}$. When no confusion arises for the graph G, we write I instead of I_G. The first extensive study of the interval function is due to Mulder in [20], where the term interval function was coined. The axiomatic studies in metric graph theory captured attention due to the various characterizations of the interval function I by Nebeský [22, 24–26], Nebeský and Mulder [22]. The axiomatic characterization of the interval function of special class of graphs also became an interesting problem, For e.g. trees [11, 28], median graphs [20, 23, 29], geodetic graphs [26], block graphs [2], weakly modular graphs, partial cubes, their principal subclasses, and superclasses [7].

The interval function is generalized to the notion of a transit function R by Mulder in [21] to various setups in discrete structures such as graphs, hypergraphs, posets, etc. Formally, a transit function is defined on a non-empty set V. We consider only finite non-empty set V in this paper.

A *transit function* on V is a function $R : V \times V \to 2^V$ satisfying the following three axioms:

$(t1)$ $u \in R(u,v)$, for all $u, v \in V$,

$(t2)$ $R(u,v) = R(v,u)$, for all $u, v \in V$,

$(t3)$ $R(u,u) = \{u\}$, for all $u \in V$.

If V is the vertex set of a graph G, then we say that R is a transit function on G. The *underlying graph* G_R of a transit function R is the graph with vertex set V, where two distinct vertices u and v are joined by an edge if and only if $R(u,v) = \{u,v\}$. Note that if R is a transit function on G, then G_R does not need to be isomorphic with G, see [21].

Another important example is the induced path function J of G, given by $J(u,v) = \{w \mid w \text{ lies on an induced } u,v\text{-path}\}$. Axioms, as the above three, on a function R on the set V are called *transit axioms*.

For the induced path transit function, Nebeský [27] obtained a very interesting result: using first-order logic, he proved that characterization of J by transit axioms is impossible. This was an extra motivation to characterize the induced path function on special classes of graphs; see, for example, [8, 9, 19, 20].

The following betweenness axioms were considered by Mulder in [19].

$(b1)$: $x \in R(u,v)$, $x \neq v \Rightarrow v \notin R(u,x)$,

$(b2)$: $x \in R(u,v) \Rightarrow R(u,x) \subseteq R(u,v)$.

In [9] it is proved that if R satisfies the axioms $(b1)$ and $(b2)$, then the underlying graph G_R of R is connected and both the axioms $(b1)$ and $(b2)$ are necessary for the connectivity of G_R. Mulder also introduced the following axioms in [20].

$(c5)$: if $x \in R(u,v)$ and $y \in R(u,x)$, then $x \in R(y,v)$ for all u, v, x, y.

$(c4)$: if $x \in R(u,v)$, then $R(u,x) \cap R(x,v) = \{x\}$ for all u, v, x.

The axioms $(t1)$, $(t2)$, $(b2)$, $(c5)$ and $(c4)$ are known as five classical axioms of the interval function I_G of a connected graph, since I_G satisfies these axioms. It is clear that the axioms $(t1)$ and $(c4)$ imply the axiom $(t3)$. Hence, any function R from $V \times V \to 2^V$ satisfying the five classical axioms is a transit function. Furthermore, it is easy to see that we have the following implications. The axioms $(t1)$, $(t2)$ and $(c4)$ imply $(b1)$, see [2]. Axioms $(t1)$, $(t2)$, $(t3)$ and $(c5)$ imply axioms $(c4)$, see [22]. Hence if R is a transit function that satisfies the axiom $(c5)$ then R satisfies the axioms $(c4)$ and $(b1)$.

Ptolemaic graphs form an interesting subclass of chordal graphs (A graph G is *chordal* if it contains no induced cycles of length more than 3). It is precisely the graphs that are both chordal and distance-hereditary (A graph G is *distance-hereditary* if every induced path is also a shortest path in G). Ptolemaic graphs possess several characterizations based on the reduced clique graph, analyzing the behaviour of the one-vertex extensions, etc. [3, 13]. A hierarchy of Ptolemaic graphs is developed based on the characteristics of the minimal vertex separators in each subclass by Markenzon et al.[18] and analyzed the Ptolemaic graphs for their properties as chordal graphs and classified six subfamilies of Ptolemaic graphs, namely laminar chordal graphs, block duplicate graphs, block graphs, AC graphs, trees, and paths.

The main purpose of this paper is to show that the interval function of all these subfamilies of Ptolemaic graphs possesses an axiomatic characterization in terms of simple first-order axioms framed on an arbitrary transit function. In other words, these subfamilies of Ptolemaic graphs can be characterized in terms of their interval function. All graphs that we consider in this paper, possess a forbidden subgraphs characterizations. That is, these graphs can be characterized using a list of forbidden-induced subgraphs. Our approach is that, when we consider a graph that has a forbidden induced subgraph characterization, we identify axioms that correspond to one or more of these forbidden subgraphs. In a way, the proposed axiomatization is a reinterpretation of forbidden subgraphs for a particular graph that possesses a forbidden induced subgraphs characterization. The paper is organized as follows.

In the rest of the introductory section, we fix the graph theoretical terminologies required for the subsequent sections. In Section 2, we describe the properties and characterizations of the subclasses of Ptolemaic graphs that we discuss in this paper. In Section 3, the axiomatic characterizations of the interval function of the subclasses of Ptolemaic graphs, namely laminar chordal, block duplicate graphs, AC-graphs and Ptolemaic C-I graphs, are presented.

Now, we define certain graph-theoretic concepts that are required in this paper. Let $G = (V, E)$ be a connected graph, vertex set and edge set of G denoted as $V(G)$ and $E(G)$, respectively. The complement of G is denoted as $\overline{G}$. A graph H is said to be a *subgraph* of G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. H is an *induced subgraph* of G if for $u, v \in V(H)$ and $uv \in E(G)$ implies $uv \in E(H)$. A graph G is said to be *H-free*, if G has no induced subgraph isomorphic to H. A *complete graph* is a graph whose vertices are pairwise adjacent, denoted as K_n. A set $S \subseteq V(G)$ is a *clique* if the subgraph of G induced by S is a complete graph, and a *maximal clique* is a clique which is not contained by any other clique. A vertex v is called *simplicial vertex* if

its neighborhood induces a complete subgraph. The *bull, claw, dart, double-diamond, gem* and *n-cycle* (C_n) graphs are depicted in Fig. 1. A *chordal graph* is a graph that does not contain an induced cycle of length greater than 3.
A subset $S \subseteq V$ is a *separator* of G if at least two vertices in the same connected component of G are in two distinct connected components of $G[V \setminus S]$. The set S is a *minimal separator* of G if S is a separator and no proper subset of S separates the graph. A subset $S \subset V$ is a *vertex separator* for non adjacent vertices u and v (a uv-separator), if the removal of S from the graph separates u and v into distinct connected components. If no proper subset of S is a uv-separator then S is a *minimal uv-separator*. If S is minimal uv-separator for some pair of vertices, it is called a *minimal vertex separator*. An efficient algorithm to determine the set of minimal vertex separators can be found in [17].
The vertices u and v are *true twins* in G if they have the same closed neighborhoods and they are *false twins* if they have the same open neighborhoods. Let $G = (V, E)$ be a graph and $v \in V$, consider the set $V' = V \cup \{v'\}$ where $\{v'\} \cap V = \emptyset$. The graph with vertex set V' and edge set consisting of the edge set of G together with edges between v' and v and between v' and all neighbors of v in G is called the graph obtained by adding a true twin to G.

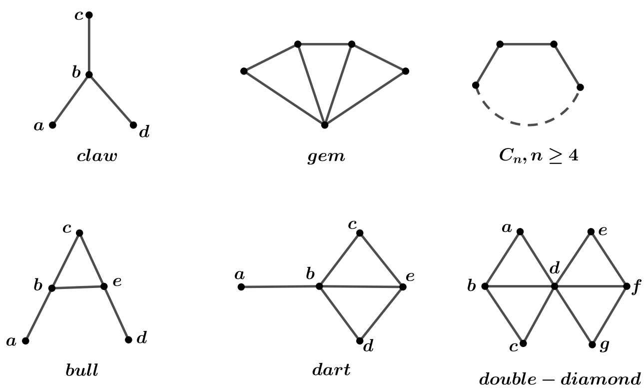

Figure 1. **Claw, gem, n-Cycle(C_n) for $n \geq 4$, bull, dart and double-diamond graph**

2. Subclasses of Ptolemaic graphs

In this section, we make a short survey on the following subfamilies of Ptolemaic graphs, namely the block duplicate graph, AC graph, laminar chordal graph, and Ptolemaic C-I graph. Ptolemaic graphs are explored in many ways.
A connected graph G is *Ptolemaic* if for any four vertices u, v, w, x of V,

$$d(u, v)d(w, x) \leq d(u, w)d(v, x) + d(u, x)d(v, w).$$

The following theorem by Howorka is a most celebrated characterization of Ptolemaic graphs.

Theorem 1 ([14]). *The following conditions are equivalent:*

1. *G is Ptolemaic.*
2. *G is gem-free and chordal.*
3. *G is distance-hereditary and chordal.*

The two known subclasses of Ptolemaic graphs are reviewed in terms of minimal vertex separators: block duplicate graphs and AC graphs. A graph G is a *block graph* if every block of G is a clique. A *block duplicate graph* is a graph obtained by adding zero or more true twins to each vertex of a block graph G (or equivalently to each cut-vertex, since adding a true twin to a non-cut vertex preserves the property of being a block graph). Block duplicate graph was introduced by Golumbic and Peled [12]. The class was also defined as strictly chordal graphs in [15] based on hypergraph properties. The class was proved to be gem-free and dart-free [12, 16] (see Figure 1, for the graphs gem and dart). Block duplicate graphs also have properties in terms of the structure of their minimal vertex separators, as proved in the following theorem. Based on the theorem, the recognition algorithm for block duplicate graphs becomes trivial.

Theorem 2 ([18]). *Let $G = (V, E)$ be a chordal graph and $\mathbb{S}$ be the set of minimal vertex separators of G. The following statements are equivalent:*

1. *G is a block duplicate graph.*
2. *For any distinct $S, S' \in \mathbb{S}$, $S \cap S' = \emptyset$.*
3. *G is gem-free and dart-free.*

Another subclass of Ptolemaic graphs, the AC graphs, were presented in [4]. An *AC graph* is a graph whose clique intersection graph is acyclic. Their characterization is simple and they are easy to recognize.

Theorem 3 ([5]). *A graph $G = (V, E)$ is an AC graph if and only if it is chordal and every vertex in G belongs to at most two maximal cliques.*

In the following theorem, we have provided new proof of the characterization of AC graphs using the forbidden induced subgraphs.

Theorem 4. *A graph G is an AC graph if and only if it is a $(C_n, gem, claw)$-free graph for $n \geq 4$.*

Proof. Let $G = (V, E)$ be a connected graph. Suppose that G is an AC graph; then by Theorem 3, G is chordal, and every vertex in G belongs to at most two maximal cliques. Hence, it is clear that G is a graph free of $(C_n, gem, claw)$ for $n \geq 4$.
Conversely, if G is a graph free of $(C_n, gem, claw)$ for $n \geq 4$, then we need to prove that G is an AC graph.
Suppose that G is not an AC graph; then by Theorem 3, G is either not chordal, or there exists a vertex in G belonging to at least three maximal cliques.
If G is not chordal, then G contains an induced cycle C_n for some $n \geq 4$, which is a contradiction to the assumption.
If there is a vertex u in G that belongs to at least three maximal cliques. Without loss of generality, u belongs to three maximal cliques, say M_1, M_2 and M_3. Let $M = M_1 \cap M_2 \cap M_3$, $S_i = V(M_i) \setminus V(M)$ for $i = 1, 2, 3$ and also clear that $u \in V(M)$ and $S_1 \cap S_2 \cap S_3 = \emptyset$. That is, M_1, M_2 and M_3 are maximal cliques containing u and S_i is the set of vertices in M_i by removing the common vertices in M_1, M_2 and M_3 and since M_i's are maximal cliques $S_i \neq \emptyset$ for $i = 1, 2, 3$. Then, the possible relationship between S_1, S_2 and S_3 are;

(i) $S_1 \cap S_2 = \emptyset$, $S_2 \cap S_3 = \emptyset$ and $S_1 \cap S_3 = \emptyset$.
In this case, the vertices $w \in S_1$, $x \in S_2$, and $y \in S_3$ together with the vertex u form an induced claw in G, a contradiction to the assumption.

(ii) $S_1 \cap S_2 \neq \emptyset$, $S_2 \cap S_3 = \emptyset$, and $S_1 \cap S_3 = \emptyset$.
In this case, there exist $w \in S_1$ and $x \in S_2$ such that $w \notin S_2$ and $x \notin S_1$ (such x and y exist since M_1 and M_2 are maximal cliques). Then, the vertices w, x and $y \in S_3$ together with the vertex u form an induced claw in G, a contradiction to the assumption.

(iii) $S_1 \cap S_2 \neq \emptyset$, $S_2 \cap S_3 \neq \emptyset$ and $S_1 \cap S_3 = \emptyset$.
In this case, if there exist $w \in S_1$, $x \in S_2$, and $y \in S_3$ such that $w \notin S_2 \cup S_3$, $x \notin S_1 \cup S_3$ and $y \notin S_1 \cup S_2$, then the vertices w, x, y together with the vertex u form an induced claw in G, a contradiction to the assumption.
Otherwise, that is, every element in S_2 is in S_1 or S_3. Then there exist $w' \in S_1$, $x' \in S_1 \cap S_2$, $y' \in S_2 \cap S_3$ and $z' \in S_3$ such that $w' \notin S_2$ and $z' \notin S_2$ (such that w', x', y' and z' exits because M_1, M_2, M_3 are maximal cliques and $S_1 \cap S_3 = \emptyset$ and $S_1 \cap S_2 \cap S_3 = \emptyset$). It is clear that the vertices w', x', y', z' together with the vertex u form an induced gem in G, a contradiction to the assumption.

(iv) $S_1 \cap S_2 \neq \emptyset$, $S_2 \cap S_3 \neq \emptyset$, and $S_1 \cap S_3 \neq \emptyset$.
In this case, if there exist $w \in S_1$, $x \in S_2$, and $y \in S_3$ such that $w \notin S_2 \cup S_3$, $x \notin S_1 \cup S_3$ and $y \notin S_1 \cup S_2$, then the vertices w, x, y together with the vertex u form an induced claw in G, a contradiction to the assumption.

If such w, x, and y do not exist, then the elements of any of the sets S_1, S_2, and S_3 are contained in the other two sets, say S_1. The elements in S_1 contain in S_2 and S_3. That is, $S_1 = (S_1 \cap S_2) \cup (S_1 \cap S_3)$. Then we see that $S_1 \cup (S_2 \cap S_3)$ is a clique and $S_1 \subset S_1 \cup (S_2 \cap S_3)$, not possible. So, in this case, there exist $w \in S_1$, $x \in S_2$, and $y \in S_3$ such that $w \notin S_2 \cup S_3$, $x \notin S_1 \cup S_3$ and $y \notin S_1 \cup S_2$.

Thus, we have proved that in all cases, G contains either an induced claw or an induced gem or an induced C_n, for $n \geq 4$, a contradiction to the assumption, and hence the theorem follows. □

Let $\mathcal{F} = \{F_1, \ldots, F_k\}$, $F_i \subseteq V$, $1 \leq i \leq k$, be a family of sets. $\mathcal{F}$ is a *laminar family* if $F_i \cap F_j \neq \emptyset$ implies that $F_i \subseteq F_j$ or $F_j \subseteq F_i$, $1 \leq i \leq j \leq k$. In other words, a family of sets is called *laminar* if two sets are disjoint or if one of them is a subset of the other. An empty family is a laminar family. A chordal graph G is called a *laminar chordal graph* if the set $\mathbb{S}$ of all minimal vertex separators is laminar. The family of laminar chordal graphs is characterized in [18] as follows.

Theorem 5 ([18]). *Let $G = (V, E)$ be a chordal graph. Then G is a laminar chordal graph if and only if G is gem-free and double-diamond-free.*

A *partially ordered set or poset* $P = (V, \leq)$ consists of a nonempty set V and a reflexive, anti-symmetric, transitive relation $\leq$ on V. If $u \leq v$ or $v \leq u$ in a poset, we say u and v are *comparable*, otherwise *incomparable*. If $u \leq v$ but $u \neq v$, then we write $u < v$. If u and v are in V, then v *covers* u in P if $u < v$ and there is no w in V with $u < w < v$, denoted by $u \lhd v$.
The *cover-incomparability*, shortly, *C-I graph* of a poset $P = (V, \leq)$ is defined as the graph G with the vertex set V and two vertices $u, v \in V$ are adjacent in G if u and v are incomparable or $u \lhd v$ or $v \lhd u$ in P. The C-I graphs were introduced in [6] as the underlying graphs of the so-called standard transit functions of a poset [6]. For an example, see Figure 2. A graph that is both Ptolemaic and C-I graph is called *Ptolemaic C-I graph*. The following theorem is proved in [1].

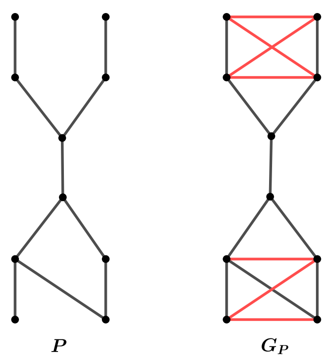

Figure 2. **P is a poset, and G_P is its corresponding C-I graph.**

Theorem 6 ([1]). *Let G be a C-I graph. Then, the following conditions are equivalent.*

(i) G is a Ptolemaic graph.

(ii) G is a laminar chordal graph.

(iii) G is a block duplicate graph.

(iv) G is an AC graph.

Based on the above theorem and the hierarchy of the Ptolemaic graphs established by Markenzon et al. [18], we conclude that the Ptolemaic C-I graph is also a subclass of the AC graph and a superclass of the path graphs. Thus, the family of Ptolemaic C-I graphs is another family of graphs which can be included in the hierarchy of subclass of Ptolemaic graphs, as shown in Figure 3.

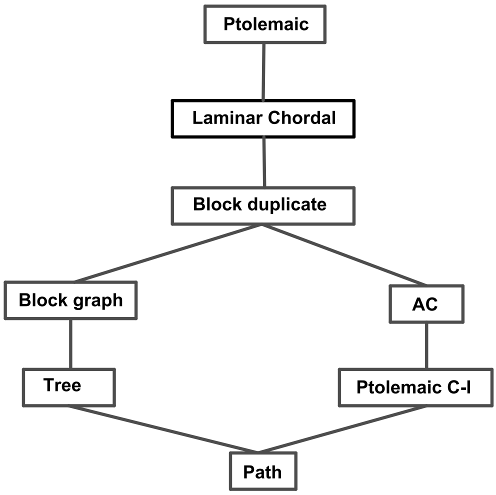

Figure 3. **A hierarchy of Ptolemaic graph**

Also, there is a forbidden subgraph characterization for the Ptolemaic C-I graph.

Theorem 7 ([1]). *A graph G is a Ptolemaic C-I graph if and only if G is a (bull, claw, gem, C_n)- free graph, for $n \geq 4$.*

3. Interval function of subclasses of Ptolemaic graphs

In this section, we provide axiomatic characterizations of the remaining subfamily of Ptolemaic graphs, namely AC -graphs, block duplicate graphs, Laminar chordal graphs and Ptolemaic C-I graphs, other than trees and block graphs, so that all the subfamily of Ptolemaic graphs in the hierarchy given in Figure 3 possess an axiomatic characterization in terms of the interval function of these graphs. Finally, we provide an axiomatic characterization of the interval function of these classes of graphs using a set of first-order axioms on an arbitrary transit function.

In [2], Balakrishnan et. al. provided the axiomatic characterization of the interval function of trees and extended the characterization to block graphs by modifying the axiom (U), which prevents cycles. The axiom (U) is the following.

(U): for all $a, b, c \in V,\ R(a, b) \cap R(b, c) = \{b\} \Rightarrow R(a, b) \cup R(b, c) = R(a, c)$.

In [2], the interval function of a tree is characterized using the axiom (U) and the axioms $(t1)$, $(t2)$, $(b1)$ and $(b2)$ as the following theorem.

Theorem 8 ([2]). *Let $R : V \times V \to 2^V$ be a function on V. Then R satisfies the axioms $(t1)$, $(t2)$, $(b1)$, $(b2)$ and (U) if and only if G_R is a tree and $R = I_{G_R}$.*

Again in [2], the axiom (U) is modified to

(U^*): for all $a, b, c \in V,\ R(a, b) \cap R(b, c) = \{b\} \Rightarrow R(a, c) \subseteq R(a, b) \cup R(b, c)$.

The axioms (U^*), $(t1)$, $(t2)$, $(b1)$ and $(b2)$ form the axiom set for the characterization of the interval function of a block graph in [2], which is stated as

Theorem 9 ([2]). *Let $R : V \times V \to 2^V$ be a function on V. Then R satisfies the axioms $(t1)$, $(t2)$, $(b1)$, $(b2)$, and (U^*) if and only if G_R is a block graph and $R = I_{G_R}$.*

For the axiomatic characterization of I_G of a Ptolemaic graph G, the essential axiom is $(J0)$.

$(J0)$: If $x \in R(u, y)$ and $y \in R(x, v)$, then $x \in R(u, v)$, for distinct $u, v, x, y \in V$.

In [8], Changat et al. characterized the graphs for which the interval function satisfies $(J0)$ as follows.

Theorem 10 ([8]). *Let G be a graph. The interval function I_G satisfies the axiom $(J0)$ if and only if G is a Ptolemaic graph.*

3.1. Laminar chordal graph

In this subsection, we first discuss the interval function for double diamond-free graphs. From Theorem 5, we can deduce that Ptolemaic graphs which are double diamond-free, are laminar chordal graphs. Using this fact, we discuss the interval function of laminar chordal graphs. We formulate the following simple first-order axiom for the purpose of the characterization.

(dd): For any pairwise distinct vertices $a, b, c, d, e, f, g \in V$, $R(a, b) = \{a, b\}$, $R(b, c) = \{b, c\}$, $R(b, d) = \{b, d\}$, $R(e, f) = \{e, f\}$, $R(f, g) = \{f, g\}$, $R(d, f) = \{d, f\}$ and for every other pair $u, v \in \{a, b, c, e, f, g\}, u \neq v$, $d \in R(u, v) \Longrightarrow$ either $b \notin R(a, c)$ or $f \notin R(e, g)$.

Proposition 1. *The interval function I of a connected graph G satisfies the axiom (dd) if and only if G is a double diamond-free graph.*

Proof. Let I be the interval function of a connected graph G. Assume that G contains a double diamond as an induced subgraph. It is easily seen that the vertices a, b, c, d, e, f, g (as shown in Fig. 1), $I(a, b) = \{a, b\}$, satisfy $I(b, c) = \{b, c\}$, $I(b, d) = \{b, d\}$,$I(e, f) = \{e, f\}$, $I(f, g) = \{f, g\}$, $I(d, f) = \{d, f\}$ and for every other pair $u, v \in \{a, b, c, e, f, g\}$, $u \neq v$, $d \in Iu, v)$, but $b \in I(a, c)$ and $f \in I(e, g)$. Therefore, if G contains a double diamond as an induced subgraph, then axiom (dd) is violated. Conversely, we need to prove that if G is double diamond-free, then it satisfies (dd). Suppose that G does not satisfy (dd). That is, there exist distinct vertices a, b, c, d, e, f, g such that $I(a, b) = \{a, b\}$, $I(b, c) = \{b, c\}$, $I(b, d) = \{b, d\}$,$I(e, f) = \{e, f\}$, $I(f, g) = \{f, g\}$, $I(d, f) = \{d, f\}$ and for every other pair $u, v \in \{a, b, c, e, f, g\}$, $u \neq v$, $d \in I(u, v)$ and $b \in I(a, c)$ and $f \in I(e, g)$. Since $ab, bc \in E(G)$ and $b \in I(a, c)$ then $d(a, c) = 2$. Also, $d \in I(a, c)$, it is clear that $ad, dc \in E(G)$. Similarly, $ef, fg \in E(G)$ and $f \in I(e, g)$ imply $d(e, g) = 2$. Also, given that $d \in I(e, g)$, then clearly $ef, fg \in E(G)$. a, b, and c are not adjacent to e, f and G. $bd, eg \in E(G)$ and $ac, eg \notin E(G)$. Therefore, we get an induced double diamond formed by the vertices $\{a, b, c, d, e, f, g\}$, a contradiction. Thus proved. □

From Theorems 5, 10, and Proposition 1 we get the following corollary.

Corollary 1. *The interval function I of a connected graph G satisfies the axioms $(J0)$ and (dd) if and only if G is a laminar chordal graph.*

3.2. Block duplicate graph

In this subsection, we first discuss the interval function for dart-free graphs. From Theorem 2, we can deduce that Ptolemaic graphs which are dart-free, are block duplicate graphs. Using this fact, we can characterize the interval function for block duplicate graphs. We formulate the following axiom for this purpose.

(dt): For any pairwise distinct vertices $a, b, c, d, e \in V$, $b \in R(a, d) \cap R(a, c) \cap R(a, e) \cap R(c, d)$ and $R(b, c) = \{b, c\}$, $R(b, d) = \{b, d\}$, $R(b, e) = \{b, e\} \Longrightarrow e \notin R(c, d)$.

Proposition 2. *The interval function I of a connected graph G satisfies the axiom (dt) if and only if G is a dart-free graph.*

Proof. Let I be the interval function of a connected graph G. Assume that G contains a dart as an induced subgraph. It is easily seen that the vertices a, b, c, d, e (as shown in Figure 1), $b \in I(a, d) \cap I(a, c) \cap I(a, e) \cap I(c, d)$ and $I(b, c) = \{b, c\}$, $I(b, d) = \{b, d\}$ and $I(b, e) = \{b, e\}$ but $e \in I(c, d)$. Hence, if G contains a dart as an induced subgraph, then the axiom (dt) is violated.

Conversely, we need to prove that if G is dart-free, then it satisfies (dt). Suppose that G does not satisfy (dt). That is, there exist distinct vertices a, b, c, d, e such that $b \in I(a,d) \cap I(a,c) \cap I(a,e) \cap I(c,d)$ and $I(b,c) = \{b,c\}$, $I(b,d) = \{b,d\}$, $I(b,e) = \{b,e\}$ and $e \in I(c,d)$. Let P be a shortest ae-path through b. Since $e \in I(c,d)$ and $ce, de \in E(G)$ implies $d(c,d) = 2$. Also, $b \in I(c,d)$, which implies that $cb, bd \in E(G)$. Hence, the vertices $\{a,b,c,d,e\}$ form an induced dart, a contradiction. Hence the result. □

The following corollary is an immediate consequence of Theorems 2 and 10, together with Proposition 2.

Corollary 2. *The interval function I of a connected graph G satisfies axiom $(J0)$ and (dt) if and only if G is a block duplicate graph.*

Let x be a vertex of a graph G. Then we denote the set consisting of x and its true twins by W_x. Note that if x does not have true twins, then $W_x = \{x\}$. Consider the following axioms for R defined on the vertex set $V(G)$ of a connected graph G.

(E): $uv \in E(G)$ if and only if $R(u,v) = \{u,v\}$.

(T): $x \in R(u,v), x \neq u,v \iff x' \in R(u,v)$, where x and x' are true twins.

(U''): $R(a,b) \cap R(b,c) = \{b\} \implies R(a,c) \subseteq R(a,b) \cup R(b,c)$ or $R(a,b) = R(a,b) \cup R(b,c) \cup W_b$, where W_b is the set of true twins of b, if there exists a true twin for b, for all $a, b, c \in V$

Note that, due to the axiom (U''), it follows that if $R(a,b) \cap R(b,c) = \{b\}$, then $b \in R(a,b) \subseteq R(a,b) \cup R(b,c) \cup W_b$, for any three vertices a, b, c.

Theorem 11. *Let $G = (V,E)$ be a connected graph and let $R : V \times V \to 2^V$ be a transit function on V. Then R satisfies the axioms $(b_1), (b_2), (E), (T)$ and (U'') if and only if G is a block duplicate graph and $R = I_G$.*

Proof. First, let us assume that R is the interval function of the block duplicate graph G. Then R being the interval function, it satisfies (b_1) and (b_2). Now assume that $R(u,x) \cap R(x,v) = \{x\}$. Then there are three possibilities:

Case (i). x lies on a shortest u,v-path and does not has a true twin.
Then $R(u,v) = R(u,x) \cup R(x,v)$.

Case (ii). x lies on a shortest u,v-path and has a true twin.
Now $R(u,v) = R(u,x) \cup W_x \cup R(x,v)$

Case (iii). x does not belong to any shortest path.
Then x is adjacent to two consecutive vertices y and z on a shortest u,v-path P. Therefore, $R(u,v) = R(u,x) \cup R(x,v) - \{x\}$. In this case, we have $R(u,v) \subset R(u,x) \cup R(x,v)$.

Conversely, let $R : V \times V \to 2^V$ be a transit function on G satisfying the axioms $(b1), (b2), (E), (T)$ and (U''). We have to prove that G is a block duplicate graph and that $R = I_G$. We will use the axiom (E) without mention.
In order to prove that G is a block duplicate graph, by Theorem 2 it is proved that G is chordal and (gem, dart)-free.

Claim 1. G is chordal.

Assume to the contrary that there is an induced cycle C of length $n \geq 4$. Let $x_1, x_2, \ldots, x_n$ be the consecutive vertices of C, so that the edges are edges x_1x_2, $x_2x_3, \ldots, x_{n-1}x_n$, x_nx_1. Note that we have $R(x_i, x_{i+1}) = \{x_i, x_{i+1}\}$ modulo n, for $i = 1, 2, \ldots, n$. Then $R(x_1, x_2) \cap R(x_2, x_3) = \{x_2\}$, so that, by (U''), we have $R(x_1, x_3) \subseteq \{x_1, x_3\} \cup W_{x_2}$. Since x_1 and x_3 are not adjacent, it follows that $x_2 \in R(x_1, x_3)$. Similarly, we have $R(x_1, x_{n-1}) \subseteq \{x_1, x_{n-1}\} \cup W_{x_n}$. Hence $R(x_1, x_{n-1}) \cap R(x_{n-1}, x_{n-2}) = \{x_{n-1}\}$, and so, by (U''), we get $R(x_1, x_{n-2}) \subseteq \{x_1, x_{n-2}\} \cup W_{x_n} \cup W_{x_{n-1}}$. Continuing in this way along C via $x_{n-2}, \ldots, x_3$, we deduce that $R(x_1, x_3) = R(x_3, x_1) \subseteq \{x_1, x_3\} \cup W_{x_n} \cup \ldots \cup W_{x_3}$, but this contradicts that we had $x_2 \in R(x_1, x_3)$. This settles Claim 1.

Claim 2. G does not contain an induced gem.
Let x be the vertex of degree 4, and let $u \to z \to v \to y$ be the induced path of length 3, so that u and y have degree 2, and z and v have degree 3. Note that, since y is adjacent to x but not to z, it follows that z is not a twin of x. So $z \notin W_x$ and $x \notin W_z$. Since $R(u, x) \cap R(x, v) = \{x\}$, we have $x \in R(u, v) \subseteq \{u, v\} \cup W_x$. Similarly, $R(u, z) \cap R(z, v) = \{z\}$ implies that $z \in R(u, v) \subseteq \{u, v\} \cup W_z$. But this is impossible, which settles Claim 2.

Claim 3. G does not contain an induced dart.
Let x be the vertex of degree 3, let u and v be vertices of degree 2, and let z be the vertex of degree 4. Now, due to the existence of the vertex of degree 1 in the dart, x and z are not twins. The same argument as in the proof of Claim 2 now settles Claim 3, and we have proved that G is a block duplicate graph.

Claim 4. $R = I_G$.
First we prove that $R(u, v) \subseteq I(u, v)$. In order to prove this, we use induction on $d(u, v)$. If $d(u, v) = 0$, then $u = v$ and $R(u, u) = \{u\} = I(u, u)$. Let $d(u, v) = 1$. Here u and v are adjacent and so we have $R(u, v) = \{u, v\} = I(u, v)$. Now let $d(u, v) = 2$ and let x be a common neighbor of u and v. Then $R(u, x) \cap R(x, v) = \{x\}$ and so by (U''), we have $R(u, v) \subseteq \{u, v\} \cup W_x = I(u, v)$. Assume that $R(y, z) \subseteq I(y, z)$, for all y, z with $d(y, z) \leq k$, where $k \geq 2$. Let $d(u, v) = k + 1$. Let x be a neighbor of v with $d(u, x) = k$. By induction, we have $R(u, x) \subseteq I(u, x)$. By $(b1)$, we have $v \notin R(u, x)$. Hence $R(u, x) \cap R(x, v) = \{x\}$. So by (U''), we have $R(u, v) \subseteq R(u, x) \cup R(x, v) \cup W_x$. Hence $R(u, v) \subseteq I(u, x) \cup \{x, v\} \cup W_x = I(u, v)$.
Now we prove that $I(u, v) = R(u, v)$. Again we use induction on $d(u, v)$. For $d(u, v) \leq 1$, we have $R(u, v) = I(u, v)$. Let $d(u, v) = 2$, and let x be a common neighbor of u and v. So $(u, x) \cap R(x, v) = \{x\}$. Then, by (U''), we have $R(u, v) = \{u, v\} \cup W_x = I(u, v)$.

Let $d(u,v) \geq 3$. Since u and v are not adjacent, there is a vertex z distinct from u and v in $R(u,v)$. We already proved that $R(u,v) \subseteq I(u,v)$, so z lies in $I(u,v)$. Assume that $I(u,v) \nsubseteq R(u,v)$. Then, there must exist a vertex y in $I(u,v) - R(u,v)$. By (T), the vertices y and z cannot be true twins. Since G is a block duplicate graph, there exists a shortest u,v-path P containing both y and z. Now P starts in $u \in R(u,v)$ and ends in $v \in R(u,v)$, but along the way, it gets out of $R(u,v)$, since it passes through $y \notin R(u,v)$. This means that we choose y and z to be adjacent on P with y between u and z on P. By $(b2)$, we have $R(u,z) \subseteq R(u,v)$. Since $y \notin R(u,v)$, it follows that $y \notin R(u,z)$ as well. Now z being an internal vertex of the shortest u,v-path P, we have $d(u,z) < d(u,v)$. Hence, by induction, $R(u,z) = I(u,z)$. But as y is on a shortest path between u and z, we have $y \in I(u,z)$. Hence $y \in R(u,v)$, a contradiction. This completes the proof. □

3.3. AC graph

In this subsection, we first characterize the interval function of claw-free graphs. From Theorem 4, we can deduce that Ptolemaic graphs which are claw-free, are AC graphs. Using this fact, we characterize the interval function of AC graphs. We require the following axiom for this purpose.

(cw): For any pairwise distinct vertices $a,b,c,d \in V$, $b \in R(a,c)$ and $b \in R(a,d) \Rightarrow b \notin R(c,d)$.

Proposition 3. *The interval function I of a connected graph G satisfies the axiom (cw) if and only if G is a claw-free graph.*

Proof. Let I be the interval function of a connected graph G. Assume that G contains a claw as an induced subgraph. It is easily seen that the vertices a,b,c,d (as shown in Figure 1), $b \in I(a,c)$ and $b \in I(a,d)$ but $b \in I(c,d)$. Therefore, if G contains claw as an induced subgraph, then the axiom (cw) violates.

Conversely, we need to prove that if G is claw-free, then it satisfies (cw). Suppose that G does not satisfy (cw). That is, there exist distinct vertices $a,b,c,d \in V$, with $b \in I(a,c)$, $b \in I(a,d)$, and $b \in I(c,d)$. That is, $b \in I(a,c) \cap I(a,d) \cap I(c,d)$. This implies that there is a shortest $a-c$ path P, shortest $a-d$ path Q, shortest $c-d$ path R, and P and Q branch out from b. Therefore, b has a degree of at least three. Let x,y,z be three neighbours of b, where x is on $a-b$ subpath of P or Q, y is on $b-c$ subpath of Q or R, and z is on $b-d$ subpath of P and R. Therefore, we get an induced claw formed by the vertices $\{b,x,y,z\}$, a contradiction. Hence, the theorem. □

From Theorems 4, 10 and Proposition 3 we get the following corollary

Corollary 3. *The interval function I of a connected graph G satisfies axiom $(J0)$ and (cw) if and only if G is an AC graph.*

It follows from Theorem 8 that the interval function of a connected graph G satisfies axiom (U) if and only if G is a tree. Again, it follows easily that, a tree graph G is a path if and only if G is claw-free. So, by Theorem 8 and axiom (cw), we have the following remark.

Remark 1. Let $R : V \times V \to 2^V$ be a function on V. Then R satisfies the axioms $(t1)$, $(t2)$, $(b1)$, $(b2)$ and (U) and (cw) if and only if G_R is a path and $R = I_{G_R}$.

3.4. Ptolemaic C-I graph

In this subsection, we first characterize the interval function for (claw, bull)-free graphs. From Theorem 7, we can deduce that Ptolemaic graphs which are (claw, bull)-free, are Ptolemaic C-I graphs. Using this, we have the required characterization of the interval function for Ptolemaic C-I graphs. The following axiom is formulated for this purpose.

(cb): For elements $a, b, c, d, e \in V$, pairwise distinct, except b and e, $b \in R(a, c)$, $e \in R(c, d)$ and $b, e \in R(a, d) \Rightarrow c \in R(a, d)$.

Proposition 4. *The interval function I of a connected graph G satisfies the axiom (cb) if and only if G is a (claw, bull)- free graph.*

Proof. Let I be the interval function of a connected graph G. Assume that G contains a claw or bull as an induced subgraph. First, let G contain a claw as an induced subgraph (as shown in Figure 1) $b \in R(a, c)$, $b \in I(c, d)$ and $b \in I(a, d)$ but $c \notin I(a, d)$, this violates the axiom (cb) when $e = b$. If G contains the bull as an induced subgraph, then $b \in I(a, c)$, $e \in I(c, d)$ and $b, e \in I(a, d)$, but $c \notin I(a, d)$. Hence, if G contains claw or bull as an induced subgraph, then the axiom (cb) is violated.
Conversely, we need to prove that if G is a graph free of (claw,bull), then it satisfies the axiom (cb).

Case 1. For $e = b$; the axiom (cb) becomes, for any pairwise distinct vertices $a, b, c, d \in V$, $b \in I(a, c)$, $b \in I(c, d)$ and $b \in I(a, d) \Rightarrow c \in I(a, d)$. Suppose that G does not satisfy (cb). That is, there exist distinct vertices $a, b, c, d \in V$, with $b \in I(a, c)$, $b \in I(a, d)$, $b \in I(c, d)$. That is, $b \in I(a, c) \cap I(a, d) \cap I(c, d)$ and $c \notin I(a, d)$. This implies that there is a shortest $a - c$ path P, shortest $a - d$ path Q, shortest $c - d$ path R, and P and Q branch out from b. Therefore, b has a degree of at least three. Let x, y, z be three neighbours of b, where x is on $a - b$ subpath of P or Q, y is on $b - c$ subpath of Q or R, and z is on $b - d$ subpath of P and R. Hence, we get an induced claw formed by the vertices $\{b, x, y, z\}$.

Case 2. For $e \neq b$; suppose G does not satisfy (cb). That is, for any pairwise distinct vertices $a, b, c, d, e \in V$, $b \in I(a, c)$, $e \in I(c, d)$ and $b, e \in I(a, d)$ and $c \notin I(a, d)$. This implies that there is a shortest $a - c$ path P through b, shortest $c - d$ path Q through e and shortest $a - d$ path R through b and e. Let P and R branch out from

b'. Therefore, b has a degree of at least three in G. Let a', c', e' be three neighbours of b', where a' is on the $b' - a$ subpath of P, c' is on the $b' - c$ subpath of P and e' is on the $b' - d$ subpath of R. Since a', b' and c' are on the shortest path R, it is clear that $ae' \notin E(G)$. Now, if $c'e' \notin E(G)$, the vertices $\{a', b', c', e'\}$ induce a claw in G. If $ae' \in E(G)$, then find the neighbour of e' in $e' - d$ subpath of R, and let d' be the neighbour. Hence, the vertices $\{a', b', c', e', d'\}$ induce a bull in G.
That is, if G does not satisfy the axiom (cb), then G contains a claw or a bull as an induced subgraph. Hence the Theorem. □

From Theorems 7, 10 and Proposition 4 we get the following corollary.

Corollary 4. *The interval function I of a connected graph G satisfies the axiom $(J0)$ and (cb) if and only if G is a Ptolemaic C-I graph.*

For the following discussions, we require the following axioms:

$(b3)$: If $x \in R(u, v)$ and $y \in R(u, x)$, then $x \in R(y, v)$, for all $u, v, x, y \in V$.

$(J2)$: If $R(u, x) = \{u, x\}, R(x, v) = \{x, v\}$ and $R(u, v) \neq \{u, v\}$, then $x \in R(u, v)$, for distinct $u, x, v \in V$.

The following theorem is proved in [10], which provides a characterization of the interval function of a Ptolemaic graph using a set of first-order axioms on an arbitrary transit function R.

Theorem 12 ([10]). *Let R be an arbitrary transit function defined on a non-empty set V. Then the underlying graph G_R is a Ptolemaic graph if and only if R satisfies the axioms $(b3), (J0)$ and $(J2)$, and $R(u, v) = I_{G_R}(u, v)$.*

Now, using this theorem and the axiom corresponding to the particular subgraph of the Ptolemaic graph in the hierarchy of Ptolemaic graphs, we have the corresponding results of the characterization of the interval function of the particular subgraph. We summarize these results as the following theorem.

Theorem 13. *Let R be an arbitrary transit function defined on a non-empty set V. Then, the following holds.*

(a) The underlying graph G_R is a laminar chordal graph if and only if R satisfies the axioms $(b3), (J0)$, $(J2)$, (dd), and $R(u, v) = I_{G_R}(u, v)$.

(b) The underlying graph G_R is a block duplicate graph if and only if R satisfies the axioms $(b3), (J0)$, $(J2)$, (dt), and $R(u, v) = I_{G_R}(u, v)$.

(c) The underlying graph G_R is an AC graph if and only if R satisfies the axioms $(b3), (J0)$, $(J2)$, (cw), and $R(u, v) = I_{G_R}(u, v)$.

(d) The underlying graph G_R is a (claw, bull)-graph if and only if R satisfies the axioms $(b3), (J0)$, $(J2)$, (cb), and $R(u, v) = I_{G_R}(u, v)$.

Example 1. (Described in Figure 4(i)) Let $V = \{a, b, c, d, e\}$ and define a transit function R on V as follows: $R(a, b) = \{a, b\}, R(a, c) = \{a, c\}, R(a, d) = \{a, b, c, d\}, R(a, e) = V, R(b, c) = \{b, c\}, R(b, d) = \{b, d\}, R(b, e) = \{b, e\}, R(c, d) = \{c, d\}, R(c, e) = \{b, c, d, e\}, R(d, e) = \{d, e\}, R(x, x) = \{x\}$ and $R(x, y) = R(y, x)$ for all $x, y \in V$.
(Since $d \in R(a, e), b \in R(a, d)$, but $d \notin R(b, e)$, this implies that R does not satisfy the (b3) axiom).

Example 2. (Described in Figure 4(ii)) Let $V = \{a, b, c, d, e\}$ and define a transit function R on V as follows: $R(a, b) = \{a, b\}, R(a, c) = \{a, c\}, R(a, d) = \{a, b, c, d\}, R(a, e) = \{a, b, e\}, R(b, c) = \{b, c\}, R(b, d) = \{b, d\}, R(b, e) = \{b, e\}, R(c, d) = \{c, d\}, R(c, e) = \{b, c, d, e\}, R(d, e) = \{d, e\}, R(x, x) = \{x\}$ and $R(x, y) = R(y, x)$ for all $x, y \in V$.
(Since $c \in R(a, d), d \in R(c, e)$ but $c \notin R(a, e)$, so R does not satisfy $(J0)$).

Example 3. (Described in Figure 4(iii)) Let $V = \{a, b, c, d, e\}$ and define a transit function R on V as follows: $R(a, e) = \{a, e\}, R(b, e) = \{b, e\}, R(a, b) = \{a, b, c\}$ and for all other pair $u, v \in V$, $R(u, v) = \{u, v\}$ and $R(x, x) = \{x\}$, $R(x, y) = R(y, x)$ for all $x, y \in V$.
(Since $R(a, e) = \{a, e\}, R(b, e) = \{b, e\}$ and $R(a, b) \neq \{a, b\}$ but $e \notin R(a, b)$, so we can see that R fails to satisfy $(J2)$).

Example 4. (Described in Figure 4(iv)) Let $V = \{a, b, c, d, e, f, g\}$ and define a transit function R on V as follows: $R(a, c) = \{a, b, c, d\}, R(a, e) = \{a, d, e\}, R(a, f) = \{a, d, f\}, R(a, g) = \{a, d, g\}, R(b, e) = \{b, d, e\}, R(b, f) = \{b, d, f\}, R(b, g) = \{b, d, g\}, R(c, e) = \{c, d, e\}, R(c, f) = \{c, d, f\}, R(c, g) = \{c, d, g\}, R(e, g) = \{d, e, f, g\}$, and for all other pair $u, v \in V$, $R(u, v) = \{u, v\}$ and $R(x, x) = \{x\}$, $R(x, y) = R(y, x)$ for all $x, y \in V$.
(Since $R(a, b) = \{a, b\}$, $R(b, c) = \{b, c\}$, $R(b, d) = \{b, d\}$, $R(e, f) = \{e, f\}$, $R(f, g) = \{f, g\}$, $R(d, f) = \{d, f\}$ and for every other pair $u, v \in \{a, b, c, e, f, g\}$,$u \neq v$, $d \in R(u, v)$ but $b \in R(a, c)$ and $f \in R(e, g)$, fails to satisfy (dd)).

Example 5. (Described in Figure 4(v)) Let $V = \{a, b, c, d, e\}$ and define a transit function R on V as follows: $R(a, c) = \{a, b, c\}, R(a, d) = \{a, b, d\}, R(a, e) = \{a, b, e\}, R(c, d) = \{c, b, d, e\}$, and for all other pair $u, v \in V$, $R(u, v) = \{u, v\}$ and $R(x, x) = \{x\}$, $R(x, y) = R(y, x)$ for all $x, y \in V$.
(Since $b \in R(a, d) \cap R(a, c) \cap R(a, e) \cap R(c, d)$ and $R(b, c) = \{b, c\}$, $R(b, d) = \{b, d\}$ and $R(b, e) = \{b, e\}$ but $e \in R(c, d)$, this implies that R fails to satisfy (dt)).

Example 6. (Described in Figure 4(vi)) Let $V = \{a, b, c, d\}$ and define a transit function R on V as follows: $R(a, b) = \{a, b\}, R(b, c) = \{b, c\}, R(b, d) = \{b, d\}, R(a, c) = \{a, b, c\}, R(a, d) = \{a, b, d\}, R(c, d) = \{c, b, d\}$, and $R(x, x) = \{x\}$, $R(x, y) = R(y, x)$ for all $x, y \in V$.
(Since $b \in R(a, c)$ and $b \in R(a, d)$ but $b \in R(c, d)$ implies R fails to satisfy (cw). Also $b \in R(a, c)$, $b \in R(c, d)$ and $b \in R(a, d)$ but $c \notin R(a, d)$ implies R fails to satisfy (cb)).

The Examples 6, 2, 3 and 4 show that the axioms $(b3), (J0)$, $(J2)$, and (dd) are independent.

- Example 6 serves as an example for $(J0)$, $(J2)$, (dd) but not $(b3)$.

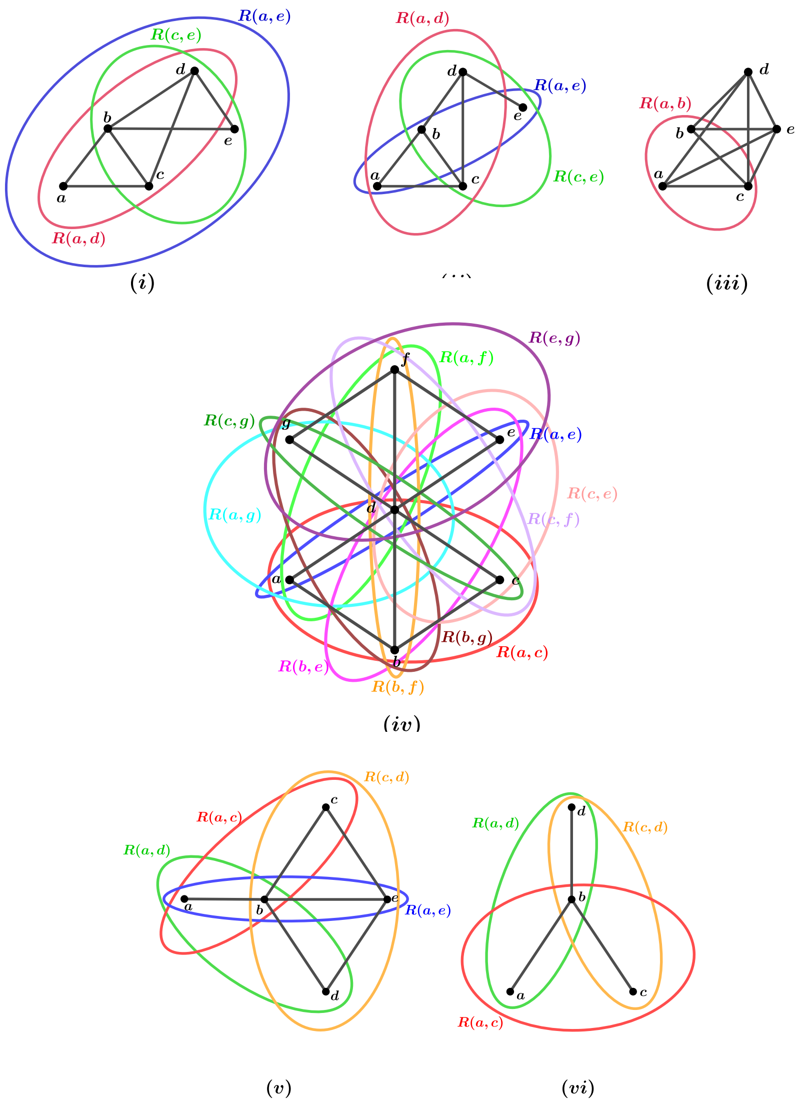

Figure 4. **Illustration of Examples 1-6 ($R(u, v) = \{u, v\}$ denoted as the edge uv)**

- Example 2 serves as an example for $(b3)$, $(J2)$, (dd) but not $(J0)$.
- If we define R as in Example 3, then R satisfy $(b3)$, $(J0)$, (dd) but not $(J2)$.
- If we define R as in Example 4, then R satisfy $(b3)$, $(J0)$, $(J2)$ but not (dd).

Examples 6, 2, 3, and 5 demonstrate the independence of the axioms $(b3)$, $(J0)$, $(J2)$, and (dt).

- Example 6 illustrates $(J0)$, $(J2)$, and (dt) but not $(b3)$.

- Example 2 exemplifies ($b3$), ($J2$), and (dt) but not ($J0$).
- In Example 3, if we define R as shown, it satisfies ($b3$), ($J0$), and (dt) but not ($J2$).
- In Example 5, if we define R as shown, it satisfies ($b3$), ($J0$), and ($J2$) but not (dt).

The examples presented in Examples 6, 2, 3, and 6 demonstrate the independence of the axioms ($b3$), ($J0$), ($J2$), and (cw).

- Example 6 forms an example for ($J0$), ($J2$), (cw) but not ($b3$).
- Example 2 form an example for ($b3$), ($J2$), (cw) but not ($J0$).
- If we define R as in Example 3, then R satisfy ($b3$), ($J0$), (cw) but not ($J2$).
- If we define R as in Example 6, then R satisfy ($b3$), ($J0$), ($J2$) but not (cw).

The same examples, Examples 6, 2, 3, and 6, show that the axioms ($b3$), ($J0$), ($J2$), and (cb) are also independent.

- Example 6 forms an example for ($J0$), ($J2$), (cb) but not ($b3$).
- Example 2 form an example for ($b3$), ($J2$), (cb) but not ($J0$).
- If we define R as in Example 3, then R satisfy ($b3$), ($J0$), (cb) but not ($J2$).
- If we define R as in Example 6, then R satisfy ($b3$), ($J0$), ($J2$) but not (cb).

4. Concluding Remarks

In this paper, by way of axiomatic characterization of the interval function of the hierarchies of subclasses of Ptolemaic graphs, namely the laminar chordal graphs, block duplicate graphs, AC graphs, Ptolemaic C-I graphs, blocks, trees and paths, we have provided another characterization of these graph families using the interval function as the main tool. It may be noted that the interval function of chordal graphs doesn't possess a first-order axiomatic characterization and is proved in [7]. The family of Ptolemaic graphs is a subclass of chordal graphs which possess a nice first-order axiomatic characterization of its interval function [9]. We have proved that similar is the case of all the graphs in the hierarchies of the subclass of Ptolemaic graphs considered in this paper. It would be an interesting problem to check whether there is any family of graphs between chordal graphs and Ptolemaic graphs which doesn't have a first-order definable interval function.

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