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Research Article

Net-degree variance and Sombor index of signed graphs

Sandeep Kumar[†], **Deepa Sinha**[*]

South Asian University, New Delhi-110068, India
[†]ssulakh.94@gmail.com
[*]deepasinha2001@gmail.com, deepasinha@sau.ac.in

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Abstract: A *signed graph* Σ is an ordered pair (Σ^u,σ), where Σ^u=(V,E) is the *underlying graph* and σ is sign mapping called *signature*, which assigns each edge in E a sign from the set $\{+,-\}$. The study of vertex-degree-based topological index: known as *Sombor index* was initiated by I. Gutman in 2021 for any graph G. He defined it as $SO(G) = \sum_{e_{ij}\in E(G)} \sqrt{d_G(v_i)^2 + d_G(v_j)^2}$. In this work, the concept of the Sombor index is extended to connected signed graphs. The Sombor index is derived mathematically for signed paths and signed cycles, and is supported by computational algorithms. Furthermore, it is proved that the Sombor index of a connected signed graph Σ is maximized if and only if the *net-degree variance* of Σ is also maximized. As an application, this study provides a solution to the net-degree variance maximization problem for certain types of signed graphs.

Keywords: signed graph, eigenvalues, index, degree variance, sombor index.

AMS Subject classification: 05C22, 05C50, 05C90

1. Introduction

Variance is a fundamental concept in probability theory and statistics, with widespread applications in science, engineering, and various practical contexts. It quantifies the extent to which the outcomes of a distribution are dispersed, thereby reflecting the inherent variability within the distribution.
In many real-world scenarios, however, probability distributions are defined over the vertices of a network-such as websites on the internet, individuals in a social network, or neurons in the brain. These vertices form the foundational elements of a network, and when analyzing distributions or signals associated with them, it becomes natural

* *Corresponding Author*

to consider the underlying network structure.
In the analysis of empirically observed graphs, the variance of the vertex degrees can serve as a measure of the graph's heterogeneity. Unfortunately, the conventional definition of variance does not account for this structure, resulting in a lack of essential methodological tools for analyzing distributions and signals on graphs.

For preliminary notation and terminology, we refer to Harary [8], Zaslavsky [13] and West [12]. Throughout this paper, all considered graphs are simple and finite. A *signed graph*, denoted by $\Sigma = (\Sigma^u,\sigma)$, is an ordered pair consisting of an underlying graph $\Sigma^u = (V, E)$, where $|V| = n$ throughout; and a *signature* mapping $\sigma : E \to \{+, -\}$ that assigns each edge in Σ^u a sign-either positive('+') or negative('$-$').
A *subsigned graph* H of a signed graph Σ is a signed graph whose vertices and edges are subsets of those in Σ, and the signs of the edges in H are preserved. In this paper, edges labelled with a '+' sign are referred to as positive edges and are represented by solid lines, whereas, edges labelled with a '$-$' sign are called negative edges and are represented by dashed lines. If all edges of Σ are assigned the same sign ('+' or '$-$'), the graph is said to be homogeneous; otherwise, it is heterogeneous. Note that an unsigned graph can be regarded as homogeneous signed graph with all edge signs being '+'.
The set of all signed graphs sharing the same underlying graph Σ^u is denoted by $\psi(\Sigma^u)$.
The adjacency matrix of Σ, with vertices $v_1, v_2, \ldots, v_n$, is the $n \times n$ matrix $A(\Sigma) = [a_{i,j}]$, where:

$$a_{i,j} = \begin{cases} 0 & \text{if } v_i \text{ and } v_j \text{ are not adjacent} \\ 1 & \text{if } \sigma(v_i, v_j) \text{ is positive} \\ -1 & \text{if } \sigma(v_i, v_j) \text{ is negative} \end{cases} \tag{1.1}$$

Let $d^+_\Sigma(v_i)$ and $d^-_\Sigma(v_i)$ denote the number of positive and negative edges incident to vertex v_i, respectively. The *net-degree* of vertex v_i is denoted by $d^\pm_\Sigma(v_i)$ [10] and is defined as:

$$d^\pm_\Sigma(v_i) = d^+_\Sigma(v_i) - d^-_\Sigma(v_i).$$

Throughout this paper, we consider only the numerical value of the net-degree, *i.e.*, $d^\pm_\Sigma(v_i) = |d^+_\Sigma(v_i) - d^-_\Sigma(v_i)|$. As usual, the total degree of vertex v_i, denoted by $d_\Sigma(v_i)$, is the total number of incident edges and is given by:

$$d_\Sigma(v_i) = d^+_\Sigma(v_i) + d^-_\Sigma(v_i).$$

By a *negative(positive) section* [4–6] of a subsigned graph Σ' of a signed graph Σ, we mean a maximal edge induced connected subsigned graph in Σ' consisting solely of the negative(positive) edges of Σ. A *signed cycle* on n vertices is a connected signed graph $\Sigma = (C_n, \sigma)$, where the number of vertices equals the number of edges, each vertex has degree two, and $\psi(C_n)$ denotes the set of all signed cycles on n vertices whose underlying graph is C_n. A *signed tree* is a connected signed graph on n vertices that contains no cycle subgraphs. A *signed path* $\Sigma = (\Sigma^u, \sigma)$ on n vertices is a specific kind of signed tree in which exactly two vertices have degree one and the remaining $n-2$ vertices have degree two. A *signed complete graph* $\Sigma = (K_n, \sigma)$ is a signed graph where every pair of distinct vertices is connected by an edge.

Let $\Pi_G : d_G(v_1), d_G(v_2), \dots, d_G(v_n)$ denote the degree sequence of a graph G. The *degree variance*, $V(\Pi_G)$, was introduced in 1981 [11] as a measure of graph heterogeneity. It is defined as the variance of the degrees' dispersion:

$$V(\Pi_G) = \frac{1}{n} \sum_{1 \le i \le n} (d_G(v_i) - d_G)^2 = \frac{1}{n} \sum_{1 \le i \le n} d_G(v_i)^2 - d_G^2 \tag{1.2}$$

where $d_G = \frac{\sum_{1 \le i \le n} d_G(v_i)}{n}$ is the mean degree of G.

The *net-degree variance* refers to the variance in the dispersion of the net-degrees in a signed graph Σ.

In chemical and mathematical literature, various vertex-degree-based graph invariants - referred to as "topological indices"-have been studied [2]. One can also refer [1, 3, 9] for related work. The general formula for a topological index TI of a graph G is given by:

$$TI = TI(G) = \sum_{e_{ij} \in E(G)} F(d_G(v_i), d_G(v_j)), \tag{1.3}$$

where $F(x, y)$ is a symmetric function, *i.e.*, $F(x, y) = F(y, x)$. Recently, Gutman [7] introduced a new vertex-degree-based topological index for a graph G, defined as:

$$SO(G) = \sum_{e_{ij} \in E(G)} \sqrt{d_G(v_i)^2 + d_G(v_j)^2}. \tag{1.4}$$

This index is called *Sombor index*. The expected values and variances of the Sombor index for general random chains have been studied in [14].

We now extend the Sombor index to signed graphs. For a signed graph Σ, we define the signed Sombor index as:

$$SO(\Sigma) = \sum_{e_{ij} \in E(\Sigma)} \sqrt{d^{\pm}_{\Sigma}(v_i)^2 + d^{\pm}_{\Sigma}(v_j)^2}. \tag{1.5}$$

Throughout this paper, when referring to the Sombor index of the *underlying graph* Σ^u of a signed graph Σ, we will use the following expression:

$$SO(\Sigma^u) = \sum_{e_{ij} \in E(\Sigma^u)} \sqrt{d_{\Sigma^u}(v_i)^2 + d_{\Sigma^u}(v_j)^2}. \tag{1.6}$$

It can be observed that, for any homogeneous signed graph Σ, the Sombor index of Σ is equal to that of its underlying graph, *i.e.*, $SO(\Sigma) = SO(\Sigma^u)$.

From the definition of the Sombor index given in Equation (1.5), we directly obtain the following result:

Theorem 1. *Let $\Sigma' = (K_n, \sigma)$ be a homogeneous signed graph on n vertices. Then for any signed graph Σ of order n, the following inequality holds:*

$$0 \leq SO(\Sigma) \leq SO(\Sigma^u) \leq SO(\Sigma'). \tag{1.7}$$

Proof. Recall that $SO(K_n) = \frac{n(n-1)^2}{\sqrt{2}}$, and that $d_{\Sigma}^{\pm}(v_i) \leq d_{\Sigma^u}(v_i)$ for all vertices v_i. $\square$

The left-hand side of inequality (1.7) is sharp if and only if Σ is a null signed graph, or if net-degree of each vertex is 0. The right-hand side of inequality (1.7) is sharp if and only if Σ is isomorphic to a homogeneous signed complete graph on n vertices.

Remark 1. If Σ is a signed tree on n vertices with $n \geq 2$, then $SO(\Sigma) \geq \sqrt{2}$.

The *negation* of a signed graph Σ, denoted η (Σ), is the signed graph obtained by reversing the sign of each edge in Σ.

Remark 2. If $\eta(\Sigma)$ is the negation of a signed graph Σ, then the Sombor index remains unchanged: $SO(\eta(\Sigma)) = SO(\Sigma)$.

The relationship introduced between the topological index and the net-degree variance is new and serves as a geometric approach to maximizing net-degree variance.
The paper is organized as follows: In Section 2, we derive the Sombor index for signed paths and signed cycles, along with computational algorithms. In Section 3, we introduce the maximum net-degree variance problem on signed graphs based on vertex-degree-based topological indices. A relationship between the maximum net-degree variance and these indices is established. In Section 4, we provide the structures of specific signed graphs that maximize net-degree variance using the newly established relationship.

2. Sombor index of signed cycles and signed paths

Let $Z \in \psi(C_n)$ be a heterogeneous signed cycle within a connected signed graph Σ. Recalling that a *negative(positive) section* in such a cycle Z is defined as a maximal all-negative(positive) path within signed cycle Z. If this path contains m vertices, then the length of the negative(positive) section is $m-1$.

Lemma 1. *If $Z \in \psi(C_n)$ be a heterogeneous signed cycle on n vertices containing a negative section of length l, then there exist m edges depending on l, such that both their* *incident vertices do not have net-degree in the* $\{-2, 2\}$*, where:*

$$m = \begin{cases} 3 & \text{if } \ l = 1 \\ 4 & \text{if } \ 2 \leq l < n-1 \\ 3 & \text{if } \ l = n-1. \end{cases}$$

Proof. Let the vertex set $V = \{v_i : i \ modulo \ n, \ i \in \mathbb{Z}\} = \{v_0, v_1, \ . \ . \ ., v_{n-1}\}$, and the edge set $E = \{e_i : i \ modulo \ n, \ i \in \mathbb{Z}\} = \{e_0, e_1, \ . \ . \ ., e_{n-1}\}$, of the signed cycle where edge e_i is incident to v_i and v_{i+1} for $0 \leq i \leq n-1$.

Case 1. $l = 1$:
Consider v_i and v_{i+1} for $0 \leq i \leq n-1$ as the vertices of the negative section. Then, $d_Z^{\pm}(v_i) = d_Z^{\pm}(v_{i+1}) = 0$. The edges e_{i-1}, e_i and e_{i+1} are incident to the vertices whose net-degrees not in $\{-2, 2\}$.

Case 2, $2 \leq l < n-1$:
Let v_k and v_l be the end vertices of the negative section, such that, $0 \leq k \leq n-1$, $0 \leq l \leq n-1$ and $l \geq k+2$. Then, $d_Z^{\pm}(v_k) = d_Z^{\pm}(v_l) = 0$, and $d_Z^{\pm}(v_m) = -2$ for $k < m < l$. The four edges e_{k-1}, e_k, e_{l-1}, and e_l all have incident vertices with net-degrees not in $\{-2, \ 2\}$.

Case 3. $l = n-1$:
Let v_i and v_{i+n-1} for $0 \leq i \leq n-1$ be the end vertices of the negative section. Then, $d_Z^{\pm}(v_i) = d_Z^{\pm}(v_{i+n-1}) = 0$. The edges $e_{i-1} = e_{i+n-1}$, e_i, and e_{i+n-2} all have incident vertices whose net-degrees are not in $\{-2, 2\}$. □

The following two theorems compute the Sombor index of a signed cycle on n vertices, $n \geq 3$.

Theorem 2. *Let* $Z \in \psi(C_n)$ *be a homogeneous signed cycle on* n *vertices,* $n \geq 3$*, then the Sombor index of* Z *is given by:*

$$SO(Z) = 2n\sqrt{2}.$$

Proof. In a homogeneous cycle Z, $d_Z^{\pm}(v_i) \in \{-2, 2\}$ for all the vertices. Thus, each term of the Sombor index is $\sqrt{2^2 + 2^2} = 2\sqrt{2}$, and there are n such terms (each edge counted once),leading to the result, $SO(Z) = 2n\sqrt{2}$. □

Theorem 3. *Let* $Z \in \psi(C_n)$ *be a heterogeneous signed cycle on* n *vertices,* $n \geq 3$*, with* x *negative sections of length one,* y *negative sections of length greater than equal to two, and* z *positive sections of length one. Then, the Sombor index of* Z *is given by:*

$$SO(Z) = 2[\sqrt{2}(n - 3x - 4y + z) + 2x + 4y - 2z]. \tag{2.1}$$

Proof. Let V and E be the vertex and edge sets of the signed cycle Z such that $V = \{v_i : i \ modulo \ n, \ i \in \mathbb{Z}\}$ and $E = \{e_i : i \ modulo \ n, \ i \in \mathbb{Z}\}$. Clearly, for $0 \leq i \leq n-1$, $d_Z^{\pm}(v_i) \in \{-2, 0, 2\}$. If v_i and v_{i+1} are incident to the positive section of length one,

then $d_Z^{\pm}(v_i) = d_Z^{\pm}(v_j) = 0$. The adjacent edges to the positive section of length one are the edges of negative sections and while counting the number of edges whose both the incident vertices do not have net-degree in the set $\{-2, 2\}$, according to Lemma 1, the positive section of length one is doubly counted in calculating edge contributions where net-degree is not in $\{-2, 2\}$. Hence, the number of edges whose both the incident vertices have net-degrees in $\{-2, 2\}$ is $n - 3x - 4y + z$, and those where one incident vertex has net-degree 0 and the other incident vertex has net-degree in $\{-2, 2\}$ is $2x + 4y - 2z$. For such edges e_i, for which both incident vertices have net-degrees in $\{-2,\ 2\}$ the corresponding Sombor index contributions are $\sqrt{d_Z^{\pm}(v_i)^2 + d_Z^{\pm}(v_{i+1})^2} = 2\sqrt{2}$, and those edges e_i whose one incident vertex has net-degree 0 the other has net-degree in $\{-2,\ 2\}$ the corresponding Sombor index contributions are $\sqrt{d_Z^{\pm}(v_i)^2 + d_Z^{\pm}(v_{i+1})^2} = 2$.
Substituting and simplifying yields:

$$SO(Z) = 2[\sqrt{2}(n - 3x - 4y + z) + 2x + 4y - 2z].$$

□

Example 1. Let Z be a heterogeneous signed cycle on 6 vertices, as shown in Figure 1. It is easy to see that Z has one negative section of length 1, *i.e.* $x = 1$, one positive section of length 1, *i.e.* $z = 1$, and one negative section of length ≥ 2, *i.e.* $y = 1$. Using Theorem 3, Sombor index of Z is:

$$\begin{aligned} SO(Z) &= 2[\sqrt{2}(n - 3x - 4y + z) + 2x + 4y - 2z] \\ &= 2[\sqrt{2}(6 - 3 - 4 + 1) + 2 + 4 - 2] \\ &= 8. \end{aligned}$$

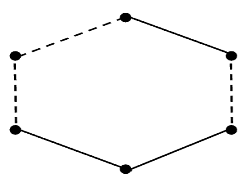

Figure 1. **Heterogeneous signed cycle**

Lemma 2. *If Σ is a signed path on 2 vertices, then Sombor index is given by:*

$$SO(\Sigma) = \sqrt{2}.$$

Proof. The path contains a single edge; thus, each vertex has net-degree 1 or -1. Therefore, $\sqrt{d_\Sigma^{\pm}(v_i)^2 + d_\Sigma^{\pm}(v_j)^2} = \sqrt{2}$. □

Lemma 3. *For a signed path Σ on 3 vertices, the Sombor index is given by:*

$$SO(\Sigma) = \begin{cases} 2 & \textit{if } \Sigma \textit{ is heterogeneous} \\ 2\sqrt{5} & \textit{if } \Sigma \textit{ is homogeneous .} \end{cases} \tag{2.2}$$

Proof. If Σ is heterogeneous signed path $v_1v_2v_3$ (edges are of different signs), the net-degrees of the three vertices are -1, 0 and 1; say $d_\Sigma^\pm(v_1) = -1$, $d_\Sigma^\pm(v_2) = 0$ and $d_\Sigma^\pm(v_3) = 1$. Therefore, $\sqrt{d_\Sigma^\pm(v_1)^2 + d_\Sigma^\pm(v_2)^2} + \sqrt{d_\Sigma^\pm(v_2)^2 + d_\Sigma^\pm(v_3)^2} = \sqrt{(-1)^2 + (0)^2} + \sqrt{(0)^2 + (1)^2} = 2$. If Σ is homogeneous (edges are of same signs), pendant vertices have net-degree either 1 or -1 and non pendant vertex will have net-degree either 2 or -2. Therefore, $SO(\Sigma) = \sqrt{d_\Sigma^\pm(v_1)^2 + d_\Sigma^\pm(v_2)^2} + \sqrt{d_\Sigma^\pm(v_2)^2 + d_\Sigma^\pm(v_3)^2} = 2\sqrt{5}$. □

Lemma 4. *For a signed path Σ on 4 vertices, the Sombor index is given by:*

$$SO(\Sigma) = \begin{cases} 2(\sqrt{2} + \sqrt{5}) & \textit{if } \Sigma \textit{ is homogeneous} \\ 2 & \textit{if } \Sigma \textit{ is heterogeneous and pendant vertices have same net degree} \\ 3 + \sqrt{5} & \textit{if } \Sigma \textit{ is heterogeneous and pendant vertices have different net degree.} \end{cases}$$

Proof. Case 1: Homogeneous signed path.
If Σ is homogeneous, then two pendant vertices will have net-degrees of either 1 or (-1), while the internal vertices will have net-degrees 2 or -2. Through straightforward calculation, it follows that $SO(\Sigma) = 2(\sqrt{2} + \sqrt{5})$.
Case 2: Heterogeneous signed path with identical pendant net-degrees.
If Σ is heterogeneous and the pendant vertices have same net-degree, then both the pendant vertices will have net-degree either 1 or -1 and the remaining $n-2$ vertices will each have net-degree 0. Therefore, $SO(\Sigma) = 2$.
Case 3: Heterogeneous signed path with different pendant net-degrees.
If Σ is heterogeneous and pendant vertices have different net-degrees, then the signed path Σ is isomorphic to one of the signed graphs S_1 or S_2, as illustrated in Figure 2 and can be easily verified that $SO(S_1) = SO(S_2) = 3 + \sqrt{5}$ □

Figure 2. **Signed graphs S_1 and S_2**

Lemma 5. *Let Σ be a heterogeneous signed path on n vertices, $n \geq 5$ with $\{v_1, v_2, \ldots, v_n\}$ vertex set and let q be the number of edges whose both the incident vertices have net-degree in the set $\{-2, 2\}$, then q is given by:*

$$q = \begin{cases} n - 3x - 4y + z - 3 & \text{if } d^{\pm}_{\Sigma}(v_1) = d^{\pm}_{\Sigma}(v_n) = 1 \\ n - 3x - 4y + z - 1 & \text{if } d^{\pm}_{\Sigma}(v_1) \neq d^{\pm}_{\Sigma}(v_n) \\ n - 3x - 4y + z + 1 & \text{if } d^{\pm}_{\Sigma}(v_1) = d^{\pm}_{\Sigma}(v_n) = -1 \end{cases}$$

here, x and y are the numbers of negative sections of length one and length greater than or equal to two, respectively, and z be the number of positive sections of length one

Proof. Let $e_i \in E(\Sigma)$ be an edge incident to the vertices v_i and v_{i+1}, for all $1 \leq i \leq n-1$.

Case 1. $d^{\pm}_{\Sigma}(v_1) = d^{\pm}_{\Sigma}(v_n) = 1$.
This case is further divided into three sub cases.

Subcase 1.1. $d^{\pm}_{\Sigma}(v_2) = d^{\pm}_{\Sigma}(v_{n-2}) = 2$, i.e. e_2 and e_{n-2} have positive signs.
Then, $d^{\pm}_{\Sigma}(v_1) = d^{\pm}_{\Sigma}(v_n) = 1$, $d^{\pm}_{\Sigma}(v_2) = d^{\pm}_{\Sigma}(v_{n-1}) = 2$ and $d^{\pm}_{\Sigma}(v_i) \in \{-2, 0, 2\}$, $3 \leq i \leq n-2$. Suppose v_i and v_{i+1}, $3 \leq i \leq n-3$ are incident vertices of a negative section of length 1. Then, $d^{\pm}_{\Sigma}(v_i) = d^{\pm}_{\Sigma}(v_{i+1}) = 0$. Consequently, e_{i-1}, e_i, and e_{i+1} are the three edges such that at least one of the end vertices does not have net-degree in the set $\{-2, 2\}$ (though one incident vertex to e_{i-1} and e_{i+1} may have net-degree in the set $\{-2, 2\}$). Thus, there are $3x$ edges for which both incident vertices have net-degrees not in $\{-2, 2\}$.
Now, suppose v_i and v_j, with $3 \leq i, j \leq n-2$, $j - i \geq 2$ are the end vertices of a negative section of length ≥ 2. In this case, $d^{\pm}_{\Sigma}(v_i) = d^{\pm}_{\Sigma}(v_j) = 0$, and $d^{\pm}_{\Sigma}(v_m) = -2$ for all $i < m < j$. Then, the edges e_{i-1}, e_i, e_{j-1}, and e_j have both incident vertices with net-degree not in the set $\{-2, 2\}$. There are $4y$ edges with net-degrees not in the set $\{-2, 2\}$ at both incident vertices.
Additionally, the net-degrees of both incident vertices of the edges e_1 and e_2 do not belong to the set $\{-2, 2\}$. In this scenario, the edges adjacent to the positive section of length 1 are the edges of negative sections. While counting the edges for which both incident vertices have net-degrees in $\{-2, 2\}$, the positive section of length 1 is considered twice. Hence, the number of edges whose both incident vertices do not have net-degree in $\{-2, 2\}$ is $3x + 4y - z + 2$. Therefore, the number of edges whose both incident vertices have net-degree in $\{-2, 2\}$ is $n - 3x - 4y + z - 3$.

Subcase 1.2. One pendant vertex has net-degree 0, other has net-degree 2 *i.e.*, e_2 and e_{n-2} have opposite signs.

Assume without loss of generality that $d^{\pm}_{\Sigma}(v_2) = 0$, $d^{\pm}_{\Sigma}(v_{n-2}) = 2$, *i.e.*, e_2 is negative edge and e_{n-2} is positive edge. Let $d^{\pm}_{\Sigma}(v_i) \in \{-2, 0, 2\}$ for $3 \leq i \leq n-2$.
For each pair v_i and v_{i+1} (with $2 \leq i \leq n-3$) forming a negative section of length 1, $d^{\pm}_{\Sigma}(v_i) = d^{\pm}_{\Sigma}(v_{i+1}) = 0$, and the edges e_{i-1}, e_i and e_{i+1} have both the incident vertices not having net-degree in $\{-2, 2\}$, (one incident vertex to e_{i-1} and e_{i+1} can have net-degree in the set $\{-2, 2\}$). Again, there are $3x$ such edges which do not

have net-degrees in the set $\{-2,\ 2\}$ at both incident vertices.

For negative sections of length ≥ 2 with end vertices v_i and v_j $(2 \leq i, j \leq n-2$, $j - i \geq 2)$, $d_{\Sigma}^{\pm}(v_i) = d_{\Sigma}^{\pm}(v_j) = 0$ and $d_{\Sigma}^{\pm}(v_m) = -2$ for $i < m < j$. Thus, e_{i-1}, e_i, e_{j-1} and e_j are the four edges whose both the incident vertices do not have net-degrees in the set $\{-2,\ 2\}$. Thus, $4y$ such edges exist not having both incident vertices with net-degree in $\{-2,\ 2\}$.
Also, net-degrees of both incident vertices of edge e_{n-1} will not be in the set $\{-2,\ 2\}$. In this case, the adjacent edges to a positive section of length 1 are the edges of negative sections. While counting edges whose both incident vertices have net-degrees in $\{-2, 2\}$, the positive section of length 1 is considered twice except e_1 *i.e.*, the number of edges that is considered twice is $z - 1$. Thus, the number of edges whose both the incident vertices do not have net-degrees in the set $\{-2,\ 2\}$ is $3x + 4y - z + 2$.
Thus, the number of edges whose both incident vertices have net-degree in the set $\{-2,\ 2\}$ is $n - 3x - 4y + z - 3$.
Subcase 1.3: $d_{\Sigma}^{\pm}(v_2) = d_{\Sigma}^{\pm}(v_{n-2}) = 0$, *i.e.*, e_2 and e_{n-2} have negative signs.

Here, $d_{\Sigma}^{\pm}(v_1) = d_{\Sigma}^{\pm}(v_n) = 1$, $d_{\Sigma}^{\pm}(v_2) = d_{\Sigma}^{\pm}(v_{n-1}) = 0$ and for $3 \leq i \leq n-2$, $d_{\Sigma}^{\pm}(v_i) \in \{-2,\ 0,\ 2\}$. Consider the vertices v_i and v_{i+1} for $2 \leq i \leq n-2$ as the incident vertices of a negative section of length 1. Then, $d_{\Sigma}^{\pm}(v_i) = d_{\Sigma}^{\pm}(v_{i+1}) = 0$. In this case, the three edges e_{i-1}, e_i, and e_{i+1} are such that none of their incident vertices have net-degrees in set $\{-2,\ 2\}$ (although one of the incident vertices of e_{i-1} or e_{i+1} may have net-degree in the set $\{-2,\ 2\}$). Therefore, there are $3x$ such edges which do not have net-degree in set $\{-2,\ 2\}$ at both the incident vertices.
Now, consider a negative section of length $\geq$2, with end vertices v_i and v_j, where, $3 \leq i, j \leq n-2$, $j - i \geq 2$. In this case, $d_{\Sigma}^{\pm}(v_i) = d_{\Sigma}^{\pm}(v_j) = 0$, and $d_{\Sigma}^{\pm}(v_m) = -2$ for $i < m < j$. The four edges e_{i-1}, e_i, e_{j-1} and e_j are such that both their incident vertices have net-degrees not belonging to $\{-2,\ 2\}$. There are $4y$ such edges which do not have net-degrees in the set $\{-2,\ 2\}$ at both the incident vertices. In this configuration, adjacent edges to a positive section of length 1 part of negative sections. While calculating the number of edges whose both incident vertices have net-degrees in the set $\{-2, 2\}$, the positive section of length 1 is counted twice, except for the two pendant edges.Therefore, the number of edges counted twice is $z - 2$. Hence, the number of edges for which both incident vertices do not have net-degrees in the set $\{-2,\ 2\}$ is $3x + 4y - z + 2$. Therefore, the number of edges whose both the incident vertices have net-degree in the set $\{-2,\ 2\}$ is $n - 3x - 4y + z - 3$.

Case 2. $d_{\Sigma}^{\pm}(v_1) \neq d_{\Sigma}^{\pm}(v_n)$ i.e. pendant edges have different signs.
Without loss of generality, consider $d_{\Sigma}^{\pm}(v_1) = 1$ and $d_{\Sigma}^{\pm}(v_n) = -1$.
In this case, $d_{\Sigma}^{\pm}(v_1) = 1$, $d_{\Sigma}^{\pm}(v_n) = -1$, and for all $2 \leq i \leq n-1$, $d_{\Sigma}^{\pm}(v_i) \in \{-2, 0, 2\}$.
Suppose v_i and v_{i+1}, where $1 \leq i \leq n-2$, are the incident vertices of a negative section of length 1. Then, $d_{\Sigma}^{\pm}(v_i) = d_{\Sigma}^{\pm}(v_{i+1}) = 0$. Hence, if $i = 1$, then $\{e_i, e_{i+1}\}$ have both incident vertices not belonging to the set $\{-2, 2\}$ and if $i \geq 2$, $\{e_{i-1}, e_i, e_{i+1}\}$ are the

edges whose both the incident vertices will not have net-degrees in set $\{-2, 2\}$.
Consider a negative section of length ≥ 2 with end vertices v_i and v_j, where $2 \leq i, j \leq n-1$, and $j - i \geq 2$. In this case, $d_\Sigma^\pm(v_i) = d_\Sigma^\pm(v_j) = 0$ and $d_\Sigma^\pm(v_m) = -2$ for all $i < m < j$. Then, if $i = 1$, $\{e_i, e_{j-1}, e_j\}$ have both incident vertices with net-degree not in the set $\{-2, 2\}$; and if $i \geq 2$, $\{e_{i-1}, e_i, e_{j-1}, e_j\}$ have both incident vertices with net-degree not in the set $\{-2, 2\}$.

Subcase 2.1. $d_\Sigma^\pm(v_{n-1}) = 2$.
In this scenario, e_n has incident vertices of net-degrees 0 and 2. The edges adjacent to any positive section of length 1 are part of negative sections.
While counting the number of edges whose both incident vertices have net-degrees in the set $\{-2, 2\}$, each positive section of length 1 is considered twice and the number of such edges counted twice is z. Therefore, the number of edges where both incident vertices do not have net-degrees in the set $\{-2,\ 2\}$ is $3x+4y-z+2$; the number of edges where both incident vertices have net-degrees in the set $\{-2,\ 2\}$ is $n-3x-4y+z-1$.

Subcase 2.2. $d_\Sigma^\pm(v_{n-1}) = 0$.
In this case, the edges adjacent to the positive section of length 1 belong to negative sections. While counting the number of edges whose both incident vertices have net-degrees in the set $\{-2, 2\}$, the positive section of length 1 is counted twice, except for the edge e_n. Therefore, the number of such edges counted twice is $z - 1$. Hence, the number of edges whose both incident vertices do not have net-degrees in the set $\{-2,\ 2\}$ is $3x+4y-z+2$; and the number of edges whose both incident vertices have net-degrees in the set $\{-2,\ 2\}$ is $n - 3x - 4y + z - 1$.

Case 3. $d_\Sigma^\pm(v_1) = d_\Sigma^\pm(v_n) = -1$ *i.e.*, both the pendant edges have negative signs.
In this case, $d_\Sigma^\pm(v_1) = d_\Sigma^\pm(v_n) = -1$, and for all $2 \leq i \leq n-1$, $d_\Sigma^\pm(v_i) \ \in \{-2, 0, 2\}$. Consider v_i and v_{i+1}, where $1 \leq i \leq n-1$, as the incident vertices of a negative section of length 1. Then $d_\Sigma^\pm(v_i) = d_\Sigma^\pm(v_{i+1}) = 0$.
In such cases, if $i = 1$, then the edges $\{e_1, e_2\}$ have both incident vertices whose net-degrees are not in the set $\{-2, 2\}$; if $2 \leq i \leq n-2$, then the edges $\{e_{i-1}, e_i, e_{i+1}\}$ have both incident vertices whose net-degrees are not in the set $\{-2, 2\}$; if $i = n-1$, the edges $\{e_{n-2}, e_{n-1}\}$ have both incident vertices whose net-degrees are not in the set $\{-2, 2\}$.
Thus, there are $3x$ such edges, where both incident vertices do not have net-degrees in $\{-2, 2\}$. Now, consider v_i and v_j, where $1 \leq i, j \leq n-1$, and $j - i \geq 2$, as the end vertices of a negative section of length ≥ 2. In this case, $d_\Sigma^\pm(v_i) = d_\Sigma^\pm(v_j) = 0$ and $d_\Sigma^\pm(v_m) = -2$ for all $i < m < j$. Then, if $i = 1$, the edges $\{e_1, e_{j-1}, e_j\}$ have both end vertices whose net-degrees are not in $\{-2, 2\}$; if $i \geq 2$, the $\{e_{i-1}, e_i, e_{j-1}, e_j\}$ have both end vertices whose net-degrees are not in $\{-2, 2\}$. Therefore, there are $4y$ such edges.

As in the previous cases, the edges adjacent to each positive section of length 1 are part of negative sections. While counting the number of edges whose both incident vertices have net-degrees in the set $\{-2, 2\}$, each positive section of length 1 is counted twice. Hence, the number of such doubly-counted edges is z. Thus, the number of edges for which both incident vertices do not have net-degrees in the set $\{-2,\ 2\}$ is $3x + 4y - z - 2$ and consequently, the number of edges whose both incident vertices have net-degree in the set $\{-2,\ 2\}$ is $n - 3x - 4y + z + 1$. $\square$

Let Σ be a signed path on n vertices and $n-1$ edges, where $n \geq 5$, with vertex set $\{v_1, v_2, \ldots, v_n\}$ and edge set $\{e_1, e_2, \ldots, e_{n-2}, e_{n-1}\}$, such that $e_i \in E(\Sigma)$ if and only if e_i is incident to both v_i and v_{i+1} for $1 \leq i \leq n-1$. Let x be the number of negative sections of length 1, and y be the number of negative sections of length ≥ 2 and let z be the number of positive sections of length 1.
In Theorems 4 through 13, the Sombor index of the signed path Σ on n vertices, where $n \geq 5$ is determined:

Theorem 4. *Let Σ be a signed path on n vertices, where $n \geq 5$ with edge set $\{e_1, e_2, \ldots, e_{n-2}, e_{n-1}\}$ such that the edges $e_1, e_2, e_{n-2}, e_{n-1}$ have positive signs. Then Sombor index of Σ is given by:*

$$SO(\Sigma) = 2[\sqrt{2}(n - 3x - 4y + z - 3) + 2x + 4y - 2z + \sqrt{5}]. \tag{2.3}$$

Proof. Given that e_1, e_2, e_{n-2} and e_{n-1} are positive, which means $d^{\pm}_{\Sigma}(v_1) = d^{\pm}_{\Sigma}(v_n) = 1$ and $d^{\pm}_{\Sigma}(v_2) = d^{\pm}_{\Sigma}(v_{n-1}) = 2$. By Lemma 5, the number of edges whose both incident vertices have net-degrees in the set $\{-2,\ 2\}$ is:

$$n - 3x - 4y + z - 3.$$

Each negative or positive section of length 1 contributes edges in which both incident vertices have net-degree 0. The two pendant edges have one incident vertex with net-degree 1 and other with net-degree in the set $\{-2, 2\}$.
Thus, the number of edges with one incident vertex of net-degree 0 and the other in the set $\{-2,\ 2\}$ is:

$$(n-1) - (n - 3x - 4y + z - 3) - (x + z) - 2 = 2x + 4y - 2z.$$

Let e_i be an edge such that:

- If both its incident vertices have net-degrees in $\{-2,\ 2\}$, then

$$\sqrt{d^{\pm}_{\Sigma}(v_i)^2 + d^{\pm}_{\Sigma}(v_{i+1})^2} = 2\sqrt{2}.$$

- If one incident vertex has net-degree 0 and other has a net-degree in the set $\{-2,\ 2\}$, then

$$\sqrt{d^{\pm}_{\Sigma}(v_i)^2 + d^{\pm}_{\Sigma}(v_{i+1})^2} = 2.$$

Therefore, the total contribution from internal edges is:

$$\sum_{2\leq i\leq n-2} \sqrt{d_{\Sigma}^{\pm}(v_i)^2 + d_{\Sigma}^{\pm}(v_{i+1})^2} = 2\sqrt{2}(n-3x-4y+z-3)+4x+8y-4z.$$

Additionally, the contributions from the pendant edges e_1 (v_1v_2) and e_{n-1} $(v_{n-1}v_n)$ are:

$$\sqrt{d_{\Sigma}^{\pm}(v_1)^2 + d_{\Sigma}^{\pm}(v_2)^2} = \sqrt{5}, \sqrt{d_{\Sigma}^{\pm}(v_{n-1})^2 + d_{\Sigma}^{\pm}(v_n)^2} = \sqrt{5}.$$

Therefore, the Sombor index of Σ is:

$$SO(\Sigma) = 2[\sqrt{2}(n-3x-4y+z-3)+2x+4y-2z+\sqrt{5}].$$

This completes the proof. □

Theorem 5. *Let Σ be a signed path on n vertices, where $n \geq 5$ with edge set $\{e_1, e_2, \ldots, e_{n-2}, e_{n-1}\}$, such that edges e_1, e_{n-2}, e_{n-1} have positive signs, and the edge e_2 has a negative sign. Then, the Sombor index of Σ is given by:*

$$SO(\Sigma) = 2\sqrt{2}(n-3x-4y+z-3)+4x+8y-4z+3+\sqrt{5}. \tag{2.4}$$

Proof. In this configuration, the net-degrees of the vertices are as follows: $d_{\Sigma}^{\pm}(v_1) = d_{\Sigma}^{\pm}(v_n) = 1$, $d_{\Sigma}^{\pm}(v_2) = 0$, $d_{\Sigma}^{\pm}(v_{n-1}) = 2$ and for $3 \leq i \leq n-2$, $d_{\Sigma}^{\pm}(v_i) \in \{-2,\ 0,\ 2\}$.
By Lemma 5, the number of edges with both incident vertices having net-degrees in the set $\{-2,\ 2\}$ is:

$$n-3x-4y+z-3.$$

Each section whether positive or negative of length 1, has both incident vertices with net-degree 0, except for edge e_1, which has one incident vertex of net-degree 0 and other of net-degree 1. The pendant edge e_{n-1} connects one incident vertex with net-degree 1 and another with net-degree in the set $\{-2,\ 2\}$. Thus, the number of edges whose one incident vertex has net-degree 0 and the other has net-degree in $\{-2,\ 2\}$ is:

$$(n-1)-(n-3x-4y+z-3)-(x+z-1)-2 = 2x+4y-2z+1.$$

Therefore, the sum over all interior edges is:

$$\sum_{2\leq i\leq n-2} \sqrt{d_{\Sigma}^{\pm}(v_i)^2 + d_{\Sigma}^{\pm}(v_{i+1})^2} = 2\sqrt{2}(n-3x-4y+z-3)+4x+8y-4z+2.$$

Additionally:
$\sqrt{d_{\Sigma}^{\pm}(v_1)^2 + d_{\Sigma}^{\pm}(v_2)^2} = 1$, $\sqrt{d_{\Sigma}^{\pm}(v_{n-1})^2 + d_{\Sigma}^{\pm}(v_n)^2} = \sqrt{5}$.
Hence, the Sombor index of Σ is:

$$SO(\Sigma) = 2\sqrt{2}(n-3x-4y+z-3)+4x+8y-4z+3+\sqrt{5}.$$

This concludes the proof. $\square$

Theorem 6. *Let Σ be a signed path on n vertices, where $n \geq 5$ with edge set $\{e_1, e_2, ..., e_{n-2}, e_{n-1}\}$, such that edges e_2, e_{n-2}, e_{n-1} have positive signs and e_1 has a negative sign. Then, the Sombor index of Σ is given by:*

$$SO(\Sigma) = 2\sqrt{2}(n - 3x - 4y + z - 1) + 4x + 8y - 4z - 1 + \sqrt{5}. \tag{2.5}$$

Proof. In this configuration, the net-degrees of the vertices are as follows: $d^{\pm}_{\Sigma}(v_1) = -1$, $d^{\pm}_{\Sigma}(v_n) = 1$, $d^{\pm}_{\Sigma}(v_2) = 0$, $d^{\pm}_{\Sigma}(v_{n-1}) = 2$ and for $3 \leq i \leq n-2$, $d^{\pm}_{\Sigma}(v_i) \in \{-2,\ 0,\ 2\}$.
By Lemma 5, the number of edges for which both incident vertices have net-degrees in the set $\{-2,\ 2\}$ is:

$$n - 3x - 4y + z - 1.$$

Each section of length 1 (positive or negative) has incident vertices of net-degrees 0 except e_1. The pendant edge e_1 connects one incident vertex of net-degree 0 and another of net-degree -1, while edge e_{n-1} connects one vertex of net-degree 1 and another of net-degree in the set $\{-2, 2\}$. Thus, the number of edges in which one incident vertex has net-degree 0 and the other has a net-degree in the set $\{-2,\ 2\}$ is:

$$(n-1) - (n - 3x - 4y + z - 1) - (x + z - 1) - 2 = 2x + 4y - 2z - 1.$$

Additionally:

$$\sum_{2 \leq i \leq n-2} \sqrt{d^{\pm}_{\Sigma}(v_i)^2 + d^{\pm}_{\Sigma}(v_{i+1})^2} = 2\sqrt{2}(n - 3x - 4y + z - 1) + 4x + 8y - 4z - 2.$$

and $\sqrt{d^{\pm}_{\Sigma}(v_1)^2 + d^{\pm}_{\Sigma}(v_2)^2} = 1$, $\sqrt{d^{\pm}_{\Sigma}(v_{n-1})^2 + d^{\pm}_{\Sigma}(v_n)^2} = \sqrt{5}$.
Therefore, the Sombor index of Σ is:

$$SO(\Sigma) = 2\sqrt{2}(n - 3x - 4y + z - 1) + 4x + 8y - 4z - 1 + \sqrt{5}.$$

This completes the proof. $\square$

Theorem 7. *Let Σ be a signed path on n vertices, where $n \geq 5$ with edge set $\{e_1, e_2, \ldots, e_{n-2}, e_{n-1}\}$, such that edges e_{n-2}, e_{n-1} have positive signs and e_1, e_2 have negative signs. Then, the Sombor index of Σ is given by:*

$$SO(\Sigma) = 2\sqrt{2}(n - 3x - 4y + z - 1) + 4x + 8y - 4z - 4 + 2\sqrt{5}. \tag{2.6}$$

Proof. In this configuration, the net-degrees of vertices are:
$d_\Sigma^\pm(v_1) = -1$, $d_\Sigma^\pm(v_n) = 1$, $d_\Sigma^\pm(v_2) = -2$, $d_\Sigma^\pm(v_{n-1}) = 2$ and for $3 \le i \le n-2$, $d_\Sigma^\pm(v_i) \in \{-2,\ 0,\ 2\}$.
By Lemma 5, the number of edges whose both incident vertices have net-degrees in the set $\{-2,\ 2\}$ is:

$$n - 3x - 4y + z - 1.$$

Each negative or positive section of length 1 has both incident vertices with net-degree 0. The pendant edge e_1 has one incident vertex with net-degree 0 and the other with net-degree in $\{-2, 2\}$. Similarly, the edge e_{n-1} connects one incident vertex with net-degree 1 and another with net-degree in $\{-2, 2\}$.
The number of edges for which one incident vertex has net-degree 0 and other has net-degree in $\{-2,\ 2\}$ is:

$$(n-1) - (n - 3x - 4y + z - 1) - (x + z) - 2 = 2x + 4y - 2z - 2.$$

Therefore, the sum over interior edges becomes:

$$\sum_{2 \le i \le n-2} \sqrt{d_\Sigma^\pm(v_i)^2 + d_\Sigma^\pm(v_{i+1})^2} = 2\sqrt{2}(n - 3x - 4y + z - 1) + 4x + 8y - 4z - 2.$$

Additionally, for the edge e_1 (between v_1 and v_2) and for the edge e_{n-1} (between v_{n-1} and v_n): $\sqrt{d_\Sigma^\pm(v_1)^2 + d_\Sigma^\pm(v_2)^2} = \sqrt{5}$, $\sqrt{d_\Sigma^\pm(v_{n-1})^2 + d_\Sigma^\pm(v_n)^2} = \sqrt{5}$.
Thus, the Sombor index of Σ is:

$$SO(\Sigma) = 2\sqrt{2}(n - 3x - 4y + z - 1) + 4x + 8y - 4z - 4 + \sqrt{5}.$$

This concludes the proof. □

Theorem 8. *Let Σ be a signed path on n vertices, where $n \ge 5$ with edge set $\{e_1, e_2, \ldots, e_{n-2}, e_{n-1}\}$, such that edges e_1, e_2, e_{n-2} have positive signs and e_{n-1} have negative sign. Then, the Sombor index of Σ is given by:*

$$SO(\Sigma) = 2\sqrt{2}(n - 3x - 4y + z - 1) + 4x + 8y - 4z - 1 + \sqrt{5}. \tag{2.7}$$

Proof. In this configuration, the net-degrees of the vertices are as follows:
$d_\Sigma^\pm(v_1) = -1$, $d_\Sigma^\pm(v_n) = 1$, $d_\Sigma^\pm(v_2) = -2$, $d_\Sigma^\pm(v_{n-1}) = 0$ and for $3 \le i \le n-2$, $d_\Sigma^\pm(v_i) \in \{-2,\ 0,\ 2\}$.
By Lemma 5, the number of edges whose both incident vertices have net-degrees in the set $\{-2,\ 2\}$ is:

$$n - 3x - 4y + z - 1.$$

Each negative or positive section of length 1 has both incident vertices with net-degree 0 except e_{n-1}. The pendant edge e_1 connects one incident vertex of net-degree -1 and another of net-degree in $\{-2, 2\}$, while edge e_{n-1} connects one vertex of net-degree 1

and another of net-degree in the set $\{-2, 2\}$.
The number of edges for which one incident vertex has net-degree 0 and other has net-degree in $\{-2,\ 2\}$ is:

$$(n-1)-(n-3x-4y+z-1)-(x+z-1)-2=2x+4y-2z-1.$$

Additionally:

$$\sum_{2\leq i\leq n-2}\sqrt{d_\Sigma^{\pm}(v_i)^2+d_\Sigma^{\pm}(v_{i+1})^2}=2\sqrt{2}(n-3x-4y+z-1)+4x+8y-4z-2,$$

and $\sqrt{d_\Sigma^{\pm}(v_1)^2+d_\Sigma^{\pm}(v_2)^2}=\sqrt{5}$, $\sqrt{d_\Sigma^{\pm}(v_{n-1})^2+d_\Sigma^{\pm}(v_n)^2}=1$.
Therefore, the Sombor index of Σ is:

$$SO(\Sigma)=2\sqrt{2}(n-3x-4y+z-1)+4x+8y-4z-1+\sqrt{5}.$$

This concludes the proof. □

Theorem 9. *Let Σ be a signed path on n vertices, where $n \geq 5$ with edge set $\{e_1, e_2, \ldots, e_{n-2}, e_{n-1}\}$, such that edges $e_1, e_2, e_{n-2}, e_{n-1}$ have negative signs. Then, the Sombor index of Σ is given by:*

$$SO(\Sigma)=2\sqrt{2}(n-3-3x-4y+z+4)+4x+8y+4z-8+2\sqrt{5}. \tag{2.8}$$

Proof. In this configuration, the net-degrees of the vertices are as follows:
$d_\Sigma^{\pm}(v_1)=d_\Sigma^{\pm}(v_n)=-1$ and $d_\Sigma^{\pm}(v_2)=d_\Sigma^{\pm}(v_{n-1})=-2$.
By Lemma 5, the number of edges whose both incident vertices have net-degrees in the set $\{-2,\ 2\}$ is:

$$n-3x-4y+z+1.$$

Each negative or positive section of length 1 has both incident vertices with net-degree 0. The two pendant edges have one incident vertex with net-degree 1 and other with net-degree in the set $\{-2, 2\}$.
The number of edges for which one incident vertex has net-degree 0 and other has net-degree in $\{-2,\ 2\}$ is:

$$(n-1)-(n-3x-4y+z+1)-(x+z)-2=2x+4y-2z-4.$$

Additionally:

$$\sum_{2\leq i\leq n-2}\sqrt{d_\Sigma^{\pm}(v_i)^2+d_\Sigma^{\pm}(v_{i+1})^2}=2\sqrt{2}(n-3x-4y+z+1)+4x+8y-4z-4,$$

and $\sqrt{d_\Sigma^{\pm}(v_1)^2+d_\Sigma^{\pm}(v_2)^2}=\sqrt{5}$, $\sqrt{d_\Sigma^{\pm}(v_{n-1})^2+d_\Sigma^{\pm}(v_n)^2}=\sqrt{5}$.
Therefore, the Sombor index of Σ is:

$$SO(\Sigma) = 2\sqrt{2}(n - 3x - 4y + z + 1) + 4x + 8y - 4z - 8 + 2\sqrt{5}.$$

This concludes the proof. □

Theorem 10. *Let Σ be a signed path on n vertices, where $n \geq 5$ with edge set $\{e_1, e_2, \dots, e_{n-2}, e_{n-1}\}$, such that edges e_1, e_2, e_{n-1} have negative signs and e_{n-2} has positive sign. Then, the Sombor index of Σ is given by:*

$$SO(\Sigma) = 2\sqrt{2}(n - 3x - 4y + z + 1) + 4x + 8y + 4z - 5 + \sqrt{5}. \tag{2.9}$$

Proof. In this configuration, the net-degrees of the vertices are as follows: $d^{\pm}_{\Sigma}(v_1) = d^{\pm}_{\Sigma}(v_n) = -1$, $d^{\pm}_{\Sigma}(v_2) = -2$, $d^{\pm}_{\Sigma}(v_{n-1}) = 0$ and for $3 \leq i \leq n-2$ $d^{\pm}_{\Sigma}(v_i) \in \{-2,\ 0,\ 2\}$.
By Lemma 5, the number of edges with both incident vertices have net-degrees in the set $\{-2,\ 2\}$ is:

$$n - 3x - 4y + z + 1.$$

Each negative or positive section of length 1 has both incident vertices with net-degree 0 except e_{n-1}. The pendant edge e_1 connects one incident vertex of net-degree -1 and another of net-degree in $\{-2, 2\}$, while edge e_{n-1} connects one vertex of net-degree 0 and another of net-degree -1.
The number of edges for which one incident vertex has net-degree 0 and other has net-degree in $\{-2,\ 2\}$ is:

$$(n-1) - (n - 3x - 4y + z + 1) - (x + z) - 2 = 2x + 4y - 2z - 3.$$

Additionally:

$$\sum_{2 \leq i \leq n-2} \sqrt{d^{\pm}_{\Sigma}(v_i)^2 + d^{\pm}_{\Sigma}(v_{i+1})^2} = 2\sqrt{2}(n - 3x - 4y + z + 1) + 4x + 8y - 4z - 3,$$

and $\sqrt{d^{\pm}_{\Sigma}(v_1)^2 + d^{\pm}_{\Sigma}(v_2)^2} = \sqrt{5}$, $\sqrt{d^{\pm}_{\Sigma}(v_{n-1})^2 + d^{\pm}_{\Sigma}(v_n)^2} = 1$.
Therefore, the Sombor index of Σ is:

$$SO(\Sigma) = 2\sqrt{2}(n - 3x - 4y + z + 1) + 4x + 8y - 4z - 5 + \sqrt{5}.$$

This concludes the proof. □

Theorem 11. *Let Σ be a signed path on n vertices, where $n \geq 5$ with edge set $\{e_1, e_2, \dots, e_{n-2}, e_{n-1}\}$, such that edges e_1, e_{n-2} have negative signs and e_2, e_{n-1} have positive signs. Then, the Sombor index of Σ is given by:*

$$SO(\Sigma) = 2\sqrt{2}(n - 3x - 4y + z - 1) + 4x + 8y - 4z + 2. \tag{2.10}$$

Proof. In this configuration, the net-degrees of the vertices are as follows: $d^{\pm}_{\Sigma}(v_1) = -1$, $d^{\pm}_{\Sigma}(v_n) = 1$, $d^{\pm}_{\Sigma}(v_2) = d^{\pm}_{\Sigma}(v_{n-1}) = 0$ and for $3 \le i \le n-2$, $d^{\pm}_{\Sigma}(v_i) \in \{-2,\ 0,\ 2\}$. By Lemma 5, the number of edges with both incident vertices have net-degrees in the set $\{-2,\ 2\}$ is:

$$n - 3x - 4y + z - 1.$$

Each negative or positive section of length 1 has both incident vertices with net-degree 0 except e_1 and e_{n-1}. The pendant edge e_1 connects one incident vertex of net-degree -1 and another of net-degree 0, while edge e_{n-1} connects one vertex of net-degree 0 and another of net-degree 1.
The number of edges for which one incident vertex has net-degree 0 and other has net-degree in $\{-2,\ 2\}$ is:

$$(n-1) - (n - 3x - 4y + z - 1) - (x + z - 2) - 2 = 2x + 4y - 2z.$$

Additionally:

$$\sum_{2 \le i \le n-2} \sqrt{d^{\pm}_{\Sigma}(v_i)^2 + d^{\pm}_{\Sigma}(v_{i+1})^2} = 2\sqrt{2}(n - 3x - 4y + z - 1) + 4x + 8y - 4z,$$

and $\sqrt{d^{\pm}_{\Sigma}(v_1)^2 + d^{\pm}_{\Sigma}(v_2)^2} = 1$, $\sqrt{d^{\pm}_{\Sigma}(v_{n-1})^2 + d^{\pm}_{\Sigma}(v_n)^2} = 1$.
Therefore, the Sombor index of Σ is:

$$SO(\Sigma) = 2\sqrt{2}(n - 3x - 4y + z - 1) + 4x + 8y - 4z + 2.$$

This concludes the proof. □

Theorem 12. *Let Σ be a signed path on n vertices, where $n \ge 5$ with edge set $\{e_1, e_2, \ldots, e_{n-2}, e_{n-1}\}$, such that edges e_1, e_{n-1} have negative signs and e_2, e_{n-2} have positive signs. Then, the Sombor index of Σ is given by:*

$$SO(\Sigma) = 2\sqrt{2}(n - 3x - 4y + z + 1) + 4x + 8y - 4z - 2. \tag{2.11}$$

Proof. In this configuration, the net-degrees of the vertices are as follows: $d^{\pm}_{\Sigma}(v_1) = d^{\pm}_{\Sigma}(v_n) = -1$, $d^{\pm}_{\Sigma}(v_2) = d^{\pm}_{\Sigma}(v_{n-1}) = 0$ and for $3 \le i \le n-2$, $d^{\pm}_{\Sigma}(v_i) \in \{-2,\ 0,\ 2\}$. By Lemma 5, the number of edges with both incident vertices have net-degrees in the set $\{-2,\ 2\}$ is:

$$n - 3x - 4y + z + 1.$$

Each negative or positive section of length 1 has both incident vertices with net-degree 0 except e_1 and e_{n-1}.The pendant edge e_1 connects one incident vertex of net-degree -1 and another of net-degree 0, while edge e_{n-1} connects one vertex of net-degree -1 and another of net-degree 0.
The pendant edges, e_1 and e_{n-1} have their one incident vertex with net-degree -1 other with net-degree 0.

The number of edges for which one incident vertex has net-degree 0 and other has net-degree in $\{-2,\ 2\}$ is:

$$(n-1)-(n-3x-4y+z+1)-(x+z-2)-2=2x+4y-2z-2.$$

Additionally:

$$\sum_{2\leq i\leq n-2}\sqrt{d^{\pm}_{\Sigma}(v_i)^2+d^{\pm}_{\Sigma}(v_{i+1})^2}=2\sqrt{2}(n-3x-4y+z+1)+4x+8y-4z-4,$$

and $\sqrt{d^{\pm}_{\Sigma}(v_1)^2+d^{\pm}_{\Sigma}(v_2)^2}=1, \sqrt{d^{\pm}_{\Sigma}(v_{n-1})^2+d^{\pm}_{\Sigma}(v_n)^2}=1.$
Therefore, the Sombor index of Σ is:

$$SO(\Sigma)=2\sqrt{2}(n-3x-4y+z+1)+4x+8y-4z-2$$

This concludes the proof. □

Theorem 13. *Let Σ be a signed path on n vertices, where $n\geq 5$ with edge set $\{e_1,e_2,...,e_{n-2},e_{n-1}\}$, such that edges e_2,e_{n-2} have negative signs and e_1,e_{n-1} have positive signs. Then, the Sombor index of Σ is given by:*

$$SO(\Sigma)=2\sqrt{2}(n-3-3x-4y+z)+4x+8y-4(z-2)-2 \tag{2.12}$$

Proof. In this configuration, the net-degrees of the vertices are as follows:
$d^{\pm}_{\Sigma}(v_1)=d^{\pm}_{\Sigma}(v_n)=1,\ d^{\pm}_{\Sigma}(v_2)=d^{\pm}_{\Sigma}(v_{n-1})=0$
and for $3\leq i\leq n-2,\ d^{\pm}_{\Sigma}(v_i)\ \in\{-2,\ 0,\ 2\}$.
By Lemma 5, the number of edges with both incident vertices have net-degrees in the set $\{-2,\ 2\}$ is:

$$n-3x-4y+z-3.$$

Each negative or positive section of length 1 has both incident vertices with net-degree 0 except e_1 and e_{n-1}. The pendant edge e_1 connects one incident vertex of net-degree 0 and another of net-degree 1, while edge e_{n-1} connects one vertex of net-degree 0 and another of net-degree 1.
The number of edges for which one incident vertex has net-degree 0 and other has net-degree in $\{-2,\ 2\}$ is:

$$(n-1)-(n-3x-4y+z-3)-(x+z-2)-2=2x+4y-2z+2.$$

Additionally:

$$\sum_{2\leq i\leq n-2}\sqrt{d^{\pm}_{\Sigma}(v_i)^2+d^{\pm}_{\Sigma}(v_{i+1})^2}=2\sqrt{2}(n-3x-4y+z+1)+4x+8y-4z+4,$$

and $\sqrt{d^{\pm}_{\Sigma}(v_1)^2+d^{\pm}_{\Sigma}(v_2)^2}=1, \sqrt{d^{\pm}_{\Sigma}(v_{n-1})^2+d^{\pm}_{\Sigma}(v_n)^2}=1.$ Therefore, the Sombor index of Σ is:

$$SO(\Sigma) = 2\sqrt{2}(n - 3x - 4y + z + 1) + 4x + 8y - 4z + 6,$$

This concludes the proof. □

Example 2. Let Σ be a signed path on 6 vertices, as shown in Figure 3. It is easy to see that first edge has negative sign, *i.e.* e_1 has negative sign, e_2 has positive sign, e_{n-1} has positive sign, and e_{n-2} has negative sign. Here, the signs of these four edges matches the conditions of the Theorem **??**. Using Theorem **??**, Sombor index of Σ is:

$$\begin{aligned} SO(\Sigma) &= 2\sqrt{2}(n - 3x - 4y + z - 1) + 4x + 8y - 4z + 2 \\ &= 2\sqrt{2}(6 - 3 \times 2 - 4 + 1 - 1) + 4 \times 2 + 8 - 4 + 2 \\ &= 14 - 8\sqrt{2}. \end{aligned}$$

Figure 3. **Signed path on 6 vertices**

2.1. Algorithm

Now we give the algorithms to find out the Sombor index of any signed cycle in $\psi(C_n)$ and signed path, which may be used computationally.

Algorithm 1 Sombor index of random signed cycle of a given order

Input: order of cycle (N)

for $j = 1 : N$ **do**
    for $j = 1 : N$ **do**
        Generate upper triangular adjacency matrix using (1.1)
    end for
end for
for $l = 1 : N$ **do**
    if $l == N$ **then**
        Identify the upper right most corner, f(l)
    else
        Identify f(l) = super diagonal entries $a_{l,l+1}$
    end if
end for
if $f(1) == f(N)$ **then**
    L = count the number of consecutive negative signs
    if $f(1) == -1$ **then**
        for $h = 1 : numel(L) - 1$ **do**
            if $h == 1$ **then**
                calculate m2(h) = sum of first and last element of L
            else

```
20:                 assign m2(h) = L(h)
21:             end if
22:         end for
23:     else
24:         assign m2 = L
25:     end if
26:     for m = 1 : length(m2) do
27:         if m2(m) == 1 then
28:             x=x+1
29:         end if
30:     end for
31:     calculate y = length(m2)-x
32:     M = count the number of consecutive positive signs
33:     if f(1) == 1 then
34:         for e = 1 : numel(M) − 1 do
35:             if e == 1 then
36:                 calculate m1(e) = sum of first and last element of M
37:             elseassign m1(e) = M(e)
38:             end if
39:         end for
40:     else
41:         assign m1 = M
42:     end if
43:     for O = 1 : length(m1) do
44:         if m1(o) == 1 then
45:             z=z+1
46:         end if
47:     end for
48: else
49:     repeat step 15
50:     for m = 1 : length(L) do
51:         if L(m) == 1 then
52:             repeat step 29
53:         end if
54:     end for
55:     calculate y = length(L)−x
56:     repeat step 33
57:     for o = 1 : length(M) do
58:         if M(o) == 1 then
59:             repeat step 46
60:         end if
61:     end for
62: end if
63: calculate SO using (2.1)
```

Algorithm 2 Sombor index of random signed path of a given order

1: **Input:** order of path (N)
2: **Initialisation** x=0, y=0, z=0
3: **for** $i = 1 : N$ **do**
4:     **for** $i = 1 : N$ **do**
5:         Generate upper triangular adjacency matrix using (1.1)
6:     **end for**
7: **end for**
8: **for** $l = 1 : N - 1$ **do**
9:     **Identify**, e(l) = super diagonal entries $a_{l,l+1}$
10: **end for**
11: **if** $e(1) == 1$ && $e(2) == 2$ && $e(n-2) == 1$ && $e(n-1) == 1$ **then**
12:     count the number of consecutive negative signs, L
13:     **for** $m = 1 : length(L)$ **do**
14:         **if** $L(m) == 1$ **then**
15:             x=x+1
16:         **end if**
17:     **end for**
18:     **Calculate** y = length(L) − x
19:     count the number of consecutive positive signs, M
20:     **for** $o = 1 : length(M)$ **do**
21:         **if** $M(o) == 1$ **then**
22:             z=z+1
23:         **end if**
24:     **end for**
25:     **Calculate** SO using (2.3)
26: **else**
27:     **if** $e(1) == 1$ && $e(2) == 2$ && $e(n-2) == 1$ && $e(n-1) == 1$ **then**
28:         **Repeat** step 12 to step 24
29:         **Calculate** SO using (2.4)
30:     **end if**
31:     **if** $e(1) == 1$ && $e(2) == 2$ && $e(n-2) == 1$ && $e(n-1) == 1$ **then**
32:         **Repeat** step 12 to step 24
33:         **Calculate** SO using (2.5)
34:     **end if**
35:     **if Check** the edge signs for e_1 e_2 e_{n-2} *and* e_{n-1} **then**
36:         **Repeat** step 12 to step 24
37:         **Calculate** SO using (2.6-2.12) according to edge signs
38:     **end if**
39: **end if**

3. Relation between the Sombor index and net-degree variance of a signed graph

Let $\Pi_\Sigma : d^{\pm}_{\Sigma}(v_1), d^{\pm}_{\Sigma}(v_2), ..., d^{\pm}_{\Sigma}(v_n)$ represent the net-degree sequence of a signed graph Σ with n vertices. Consider Π_Σ as a distribution of net-degrees, then using Equation (1.2) the variance of this distribution is given by:

$$V(\Pi_\Sigma) = \frac{1}{n} \sum_{1 \le i \le n} (d^{\pm}_{\Sigma}(v_i) - \bar{d}^{\pm}_{\Sigma})^2 = \frac{1}{n} \sum_{1 \le i \le n} d^{\pm}_{\Sigma}(v_i)^2 - (\bar{d}^{\pm}_{\Sigma})^2 \qquad (3.1)$$

where $d^{\pm}_{\Sigma} = \dfrac{\sum_{1\leq i\leq n} d^{\pm}_{\Sigma}(v_i)}{n}$ is the mean net-degree.
Let Σ be a connected signed graph with n vertices and q edges. In such signed graphs, multiple graphical net-degree sequences can exist. This raises the natural question: which graphical net-degree sequence, maximizes the net-degree variance?
To address this, the following theorem establishes a relationship between the Sombor index and the net-degree variance of a signed graph a new geometric prespective for studying variance.

Theorem 14. *The Sombor index of connected signed graph Σ is maximized if and only if the net-degree variance of its degree distribution is maximized.*

Proof. Let $\Pi_{\Sigma_j} : d^{\pm}_{\Sigma_j}(v_1), d^{\pm}_{\Sigma_j}(v_2), \ldots, d^{\pm}_{\Sigma_j}(v_n)$, for $1 \leq j \leq k$, where k is number of possible graphical net-degree sequences, represent a graphical net-degree sequence of Σ. From Equation (3.1), the net-degree variance with respect to net-degree distribution is:

$$V(\Pi_{\Sigma_j}) = \frac{1}{n} \sum_{1\leq i\leq n} (d^{\pm}_{\Sigma_j}(v_i) - d^{\pm}_{\Sigma_j})^2 = \frac{1}{n} \sum_{1\leq i\leq n} d^{\pm}_{\Sigma_j}(v_i)^2 - (d^{\pm}_{\Sigma_j})^2 \tag{3.2}$$

where $d^{\pm}_{\Sigma_j} = \dfrac{\sum_{1\leq i\leq n} d^{\pm}_{\Sigma_j}(v_i)}{n}$ is the mean net-degree. Here, $d^{\pm}_{\Sigma_j}$ remains constant for all j and also n is a fixed number. Therefore,

$$V(\Pi_{\Sigma_j}) \propto \sum_{1\leq i\leq n} d^{\pm}_{\Sigma_j}(v_i)^2. \tag{3.3}$$

Now the Sombor index for Σ_j is:

$$\begin{aligned} SO(\Sigma_j) &= \sum_{e_{rs}\in E(\Sigma_j)} \sqrt{d^{\pm}_{\Sigma_j}(v_r)^2 + d^{\pm}_{\Sigma_j}(v_s)^2} \\ &\propto \sum_{e_{rs}\in E(\Sigma_j)} (d^{\pm}_{\Sigma_j}(v_r)^2 + d^{\pm}_{\Sigma_j}(v_s)^2) \\ &= \sum_{1\leq i\leq n} (d_{\Sigma_j}(v_i))(d^{\pm}_{\Sigma_j}(v_i)^2) \\ &\propto \sum_{1\leq i\leq n} d^{\pm}_{\Sigma_j}(v_i)^3. \end{aligned}$$

This gives us,

$$SO(\Sigma_j) \propto \sum_{1\leq i\leq n} d^{\pm}_{\Sigma_j}(v_i)^3. \tag{3.4}$$

From Equations (3.3) and (3.4), we get:

$$V(\Pi_{\Sigma_j}) \propto SO(\Sigma_j).$$

Thus, the higher the Sombor index, the greater net-degree variance, proving the theorem. □

On considering all the edges of the signed graph with positive sign, the remark can be concluded as:

Remark 3. For a connected graph G on n vertices and q edges, the Sombor index is maximized if and only if the degree variance of G is also maximized.

Proposition 1. *If the Sombor index of a connected signed graph Σ with n vertices, p positive edges and q negative edges, is maximized then the largest eigenvalue of the adjacency matrix of Σ is also maximized.*

The proof is provided elsewhere.

4. Applications

As an application of Somber index of signed graph, Theorem 16 and Theorem 17 are presented.

Remark 4. Let x_1, x_2 and x_3 be three points in $\mathbb{R}^2$, where all three lie either in the set $\{(a, n-1), (a, n), (a, n+1)\}$ or in the set $\{(n-1, a), (n, a), (n+1, a)\}$, with a, $n \in \mathbb{I}$. Then:

$$\sqrt{(n-1)^2 + a^2} + \sqrt{(n+1)^2 + a^2} \geq 2\sqrt{n^2 + a^2}.$$

Theorem 15. *Let $\Sigma = (K_n, \sigma)$ be a signed graph on n vertices with p negative edges where $p < n-1$. If the Sombor index is maximized, then the negative edges induce a signed star on $p+1$ vertices.*

Proof. The result is proved by induction.
Base case: $p = 1$. The signed graph has single negative edge-a trivial star.
Inductive hypothesis: Assume the result holds for $p-1$ negative edges. The subgraph induced by negative edges form $K_{1,p-1}$. Then, there exists a vertex v in Σ with net-degree $n-1-2(p-1)$. Additionally, there are $p-1$ vertices $\{v_1, v_2, \ldots, v_i\}$, $1 \leq i \leq p-1$, each having net-degree $n-3$, and the remaining $n-p$ vertices, $\{v_{i+1}, v_{i+2}, \ldots, v_n\}$, with net-degree $n-1$.
Inductive step: Now consider the case when Σ has p negative edges. More precisely, suppose the sign of one positive edge is changed to negative in inductive hypothesis. There are three scenarios, depending on the incident vertices of the new negative edge.

Case 1. The edge has both incident vertices in the set $\{v_1, v_2, \ldots, v_i\}$, $1 \leq i \leq p-1$. In this case the Sombor index of Σ is given by:

$$\begin{aligned} SO(\Sigma) = A(say) =& 2\sqrt{(n-1-2(p-1))^2 + (n-5)^2} \\ &+ (p-3)\{\sqrt{(n-1-2(p-1))^2 + (n-3)^2} \\ &+ 2\sqrt{(n-3)^2 + (n-5)^2}\} + \sqrt{2}(n-5) \\ &+ (n-p)\{\sqrt{(n-1-2(p-1))^2 + (n-1)^2} \\ &+ 2\sqrt{(n-1)^2 + (n-5)^2} + (p-3)\sqrt{(n-1)^2 + (n-3)^2}\} \\ &+ \frac{(n-p)(n-p-1)(n-1)}{\sqrt{2}} + \frac{(p-3)(p-4)(n-3)}{\sqrt{2}}. \end{aligned}$$

Case 2. The edge has both incident vertices in the set $\{v_{i+1}, v_{i+2}, \ldots, v_n\}$, $1 \leq i \leq p-1$.
In this scenario the Sombor index of Σ is given by:

$$\begin{aligned} SO(\Sigma) = B(say) =& 2\sqrt{(n-1-2(p-1))^2 + (n-3)^2} \\ &+ (n-p-2)\{\sqrt{(n-1-2(p-1))^2 + (n-1)^2} \\ &+ 2\sqrt{(n-3)^2 + (n-1)^2} + \frac{(n-p-3)(n-1)}{\sqrt{2}} \\ &+ (p-1)\sqrt{(n-1)^2 + (n-3)^2}\} + \sqrt{2}(n-3) \\ &+ (p-1)\{\sqrt{(n-1-2(p-1))^2 + (n-3)^2} \\ &+ 2\sqrt{2}(n-3) + \frac{(p-2)(n-3)}{\sqrt{2}}\}. \end{aligned}$$

Case 3. One incident vertex of edge is central vertex v and other incident vertex is in set $\{v_{i+1}, v_{i+2}, \ldots, v_n\}$, $1 \leq i \leq p-1$.
In this configuration the Sombor index of Σ is given by:

$$\begin{aligned} SO(\Sigma) = C(say) =& p\sqrt{(n-1-2p)^2 + (n-3)^2} + \frac{(p)(p-1)(n-3)}{\sqrt{2}} \\ &+ (n-1-p)\{\sqrt{(n-1-2p)^2 + (n-1)^2} \\ &+ p\sqrt{(n-3)^2 + (n-1)^2} + \frac{(n-p-2)(n-1)}{\sqrt{2}}\}. \end{aligned}$$

From Remark 4, it follows that $C \geq A \geq B$. Hence, the configuration in Case 3 yields the maximum Sombor index. Therefore, the negative edges of Σ must induce a signed star with $p+1$ vertices. □

Theorem 16. *Let $\Sigma = (K_n, \sigma)$ be a signed graph on n vertices with p negative edges, such that $p < n-1$. If the net-degree variance is maximized, then the negative edges induce a signed star on $p+1$ vertices.*

Proof. The proof directly follows from Theorem 15 and Theorem 14. □

Theorem 17. *Let G be a tree, on n vertices and $n-1$ edges. If G maximizes the degree variance then $G \cong K_{1,n-1}$.*

Proof. Let $d(v_i)$ be the degree of the vertex v_i. For any edge $v_iv_j \in E(G)$, $d(v_i) + d(v_j) \leq n$. The maximum value of $\sqrt{d(v_i)^2 + d(v_j)^2}$ is achieved when $d(v_i) + d(v_j) = n$, and through comparison, it is evident that:

$$\sqrt{1^2 + (n-1)^2} > \sqrt{2^2 + (n-2)^2} > \cdots > \sqrt{(\lfloor \frac{n}{2} \rfloor)^2 + (\lceil \frac{n}{2} \rceil)^2}.$$

Since each edge in $K_{1,n-1}$ is incident to a vertex of degree 1 and $n-1$, this configuration gives the maximum possible Sombor index for $K_{1,n-1}$ and thus, maximum variance by using Theorem 14. □

5. Conclusion

This chapter explored the relationship between the Sombor index and the net-degree variance in signed graphs, contributing significantly to both graph theory and mathematical chemistry. By deriving bounds and analyzing signed graphs for given net-degree variances, the work enhances how the understanding of signed graph topology influences the Sombor index. The results not only generalize known inequalities but also introduce a novel geometric approach to measuring variance.
These insights offer a foundation for inverse problems, network design and real world applications where structural heterogeneity play a crucial role. The findings deepen theoretical knowledge while offering practical applications.
Through this work, the Sombor index emerges not merely as a mathematical construct, but as a lens through which the hidden order of complex systems may be discerned.

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