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Research Article

Some classes of non-induced star-perfect graphs

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Abstract: For a given graph G, let $\theta_f(G)$ denote the minimum number of stars (not necessarily induced) needed to cover the vertices of G, and let $\alpha_f(G)$ denote the maximum number of vertices in a set $S \subseteq V(G)$ such that no two distinct vertices $u, v \in S$ belong to the same subgraph of G that is a star. Clearly, $\theta_f(G) \geq \alpha_f(G)$. A graph G is said to be *non-induced star-perfect* if $\theta_f(H) = \alpha_f(H)$ for every induced subgraph H of G. A graph G is a *domination graph* if every induced subgraph H of G contains a pair of vertices x, y such that $N_H(x) \subseteq N_H[y]$. In this paper, we investigate domination graphs that are non-induced star-perfect and explore well-known subclasses within this category. Additionally, we present an integer linear programming formulation that characterizes a polytope associated with the star-covering set and star-independence set of a graph.

Keywords: star-perfect, star-covering number, star-independence number, chordal graphs, domination graphs.

AMS Subject classification: 05C15, 05C17, 05C69, 05C70

1. Introduction

In this paper, all graphs are finite and simple, and our notations are from [7], [22], and [35].

The study of non-induced star-perfect graphs is motivated by Ravindra's introduction of F-perfect graphs in 2011 [32]. For a comprehensive understanding of F-perfect graphs and induced star-perfect graphs, we refer the reader to [1] and [35]. However, the results in this paper focus specifically on non-induced star-perfect graphs. Herein, we use F and f to represent induced and non-induced stars, respectively.

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A *non-induced star* of a graph G is a set $f' \subseteq V(G)$ such that the subgraph induced by f' contains a spanning star $K_{1,r}$, where $r \geq 0$. A partition $\mathscr{C}$ of the vertex set $V(G)$ of a graph G is called a *non-induced star-covering set* if each $C_i \in \mathscr{C}$ is a non-induced star. The *non-induced star-covering number*, $\theta_f(G)$, of a graph G is defined as the minimum cardinality of a non-induced star-covering set in G. A subset $S \subseteq V(G)$ is called a *star-independent set* in G if no two distinct vertices $u, v \in S$ lie in the same star. The *star-independence number*, $\alpha_f(G)$, is the maximum cardinality of a star-independent set in G.

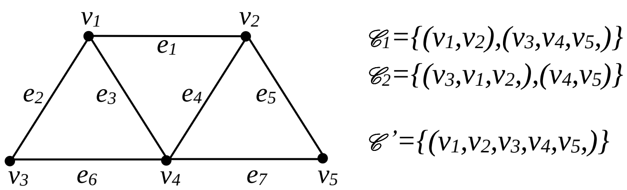

Figure 1. **Illustration of non-induced star-covering set.**

Figure 1 illustrates the distinction between induced and non-induced star-covering sets. Notice that $\mathscr{C}_1$ and $\mathscr{C}_2$ are both induced and non-induced star-covering sets since each element in these sets induces a star in G. On the other hand, $\mathscr{C}'$ is a non-induced star-covering set, as the removal of the edges e_1, e_2, and e_5 forms a star $K_{1,4}$. We remark, however, that the definition of a star-independent set remains the same for both induced and non-induced star-perfect graphs.

The definition of the non-induced star-covering goes parallel with the notion of star partitions introduced by Andreatta et al. in [3]. They established that the minimum cardinality of these invariants is equal to the domination number $\gamma(G)$. Similarly, the star-independent set corresponds to the closed neighborhood packing of a graph. We refer the reader to [4] for applications of these invariants.

The non-induced star-covering number θ_f is never smaller than the star-independence number α_f, and a graph G is said to be *non-induced star-perfect* if for every induced subgraph H of G, $\theta_f(H) = \alpha_f(H)$. These graphs are also sometimes referred to as θ_f-perfect graphs. A graph G is *non-induced star-critical* if G is not non-induced star-perfect, but all of its proper induced subgraphs are.

The class of star-perfect graphs satisfies hereditary property, and therefore, this class admits a characterization in terms of forbidden induced subgraphs. In the case of induced star-perfect graphs, it was conjectured by Ravindra that there are three types of minimal forbidden induced subgraphs: C_3, C_{3n+1}, and C_{3n+2} for $n \geq 1$. This conjecture was settled by Ravindra [35], with an alternative approach provided by Alex and Caccetta [1]. The effort to establish this conjecture stimulated the study

on non-induced star-perfect graphs. In [1], they also characterized that, if χ_F[1] and ω_F are, respectively, the star-chromatic number and the size of a maximum induced star of a graph G, then G is induced star-perfect if and only if $\chi_F(H) = \omega_F(H)$ (also referred to as χ_F-perfect) or $\alpha_F(H)\omega_F(H) \geq |H|$ for every induced subgraph H of G. Here, $\alpha_F(G)$ is defined in a manner analogous to $\alpha_f(G)$. The alternative proof to Lovász's characterization of perfect graphs [2] was the motivation for this characterization.
The simplest graph known to be non-induced star-perfect but not induced star-perfect is a complete graph on 3-vertices. In fact, the only minimally known graphs that are not non-induced star-perfect include C_{3n+1}, C_{3n+2} for $n \geq 1$, and the graphs in the class $\mathscr{G}(n,p)$ (defined in Figure 2) with $p = 2k+1$ for $k \geq 1$. To this end, Ravindra postulated the following conjectures[2]:

Conjecture 1. A graph G is non-induced star-perfect if and only if G is C_{3n+1}-free and C_{3n+2}-free for $n \geq 1$, and does not contain any graph from the class $\mathscr{G}(n,p)$ with $p = 2k+1$ for $k \geq 1$ as a proper induced subgraph.

We define a graph $G(n,p)$ as follows: We begin with a complete graph $H = K_p$, with $V(H) = \{v_1, v_2, \ldots, v_p\}$ and label the edges of a Hamilton cycle of H as $\{e_1, e_2, \ldots, e_p\}$ such that $e_i = v_i v_{i+1}$, $i = 1, 2, \ldots, p$ (note that $v_{p+1} = v_1$). We form $G(n,p)$ by adding p disjoint paths, P_{n_i} from v_i to v_{i+1} such that $P_{n_i} \cup e_i$ is a cycle C_{n_i} of length $n_i \equiv 0(\text{mod } 3)$, $V(C_{n_i}) \cap V(H) = \{v_i, v_{i+1}\}$ and $\sum_{i=1}^{p} n_i = n + p$. Figure 2 illustrates our construction of $G(n,3)$ and $G(n,4)$. Finally, we define the class $\mathscr{G}(n,p)$ to be the set of all such graphs $G(n,p)$ formed under the above construction.

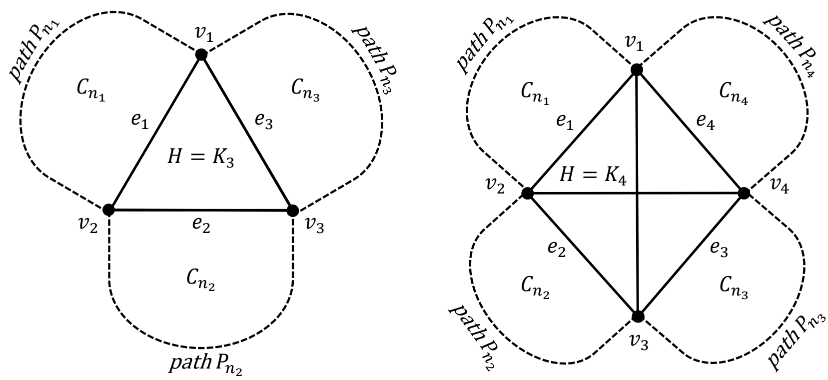

Figure 2. $G(n,3)$ **and** $G(n,4)$ **graphs.**

[1] *χ_F is the minimum number of color needed to color G such that no two vertices in the same star of G receive the same color.*
[2] *Personal Communication*

A consequence established in [1] is that a graph is induced star-perfect if and only if it is χ_F-perfect. In contrast to this result, χ_f-perfect graphs (where every induced subgraph H of G satisfies $\chi_f(H)$[3] $= \omega_f(H)$[4]) are not necessarily non-induced star-perfect. For example, the graphs in $\mathscr{G}(n,p)$, different from n-suns, are χ_f-perfect but not θ_f-perfect. However, one can identify several results concerning induced star-perfect graphs that hold for non-induced star-perfect graphs. Graphs that are simultaneously induced and non-induced star-perfect were characterized to some extent by Ravindra [32, 33] and Ghosh [21].

The primary objective of this paper is to identify several classes of non-induced star-perfect graphs. Our inclusion criterion specifically focuses on classes that contribute to the progress of identifying forbidden induced subgraphs for the class of non-induced star-perfect graphs, as stated in Conjecture 1. As a result, we exclude classes that are non-induced star-perfect by definition, such as subclasses of known non-induced star-perfect graphs or the union of two or more non-induced star-perfect graph classes. However, some exceptions are made for fundamental classes like trees and paths. While our criterion covers a broad range of classes, there may exist other classes that satisfy our criterion but are not included in this paper.

This paper is organized as follows: In Section 2, we introduce some terminology and notation concerning non-induced star-perfect graphs that will be used throughout this paper. In Section 3, we identify the classes of domination graphs that are non-induced star-perfect and prove that a domination graph with no C_4 or odd-sun is non-induced star-perfect. In Section 4, we explore well-known hereditary classes of graphs with equal non-induced star-covering number and star-independence number. In Section 5, we use a similar phenomenon to [3] to introduce an integer programming formulation to describe the invariants of non-induced star-perfect graphs. Additionally, we use the min-max equality of these invariants to establish that strongly chordal graphs are non-induced star-perfect.

2. Preliminaries

Let $G = (V, E)$ be a graph with vertex set $V(G)$ and edge set $E(G)$. For any vertex $v \in V(G)$, $N_G(v)$ and $N_G[v]$ denote the *neighborhood* and the *closed neighborhood* of v, respectively; thus, $N_G(v) = \{u \in V(G) : uv \in E(G)\}$ and $N_G[v] = N_G(v) \cup \{v\}$. The subscript indicating the graph will be omitted if only one graph is under discussion. Let $G[X]$ denote the induced subgraph of G with vertex set X, and let $G \setminus X$ denote the induced subgraph $G[V(G) \setminus X]$. A *lobe* of a graph G is an induced subgraph formed by the vertex set X along with the vertices from a single connected component of the graph obtained by removing X from G. Specifically, it consists of the vertices in X and all vertices that belong to one component of $G \setminus X$.

[3] *$\chi_f(G)$ is defined in a manner analogous to $\chi_F(G)$*

[4] *$\omega_f(G)$ is the size of a maximum non-induced star in G*

A *clique* is a set of pairwise adjacent vertices, and a *stable set* is a set of pairwise non-adjacent vertices. If $A, B \subseteq V(G)$ are disjoint, we say that A is *complete* to B if every vertex in A is adjacent to every vertex in B, and *anticomplete* to B if there is no edge between A and B. A graph is said to be *regular* if every vertex has the same number of neighbors.

A *tree* is a connected cycle-free graph. A *star* $K_{1,n}$ is a tree with at most one vertex of degree greater than one. A *non-induced star-clique* is a maximal non-induced star of the graph. A graph is *strongly star-perfect* if every induced subgraph H of G contains a star-independent set S which meets every maximal star of H.

A set $X \subseteq V(G)$ is considered *homogeneous* if every vertex in $V(G) \setminus X$ is either complete or anticomplete to X. A homogeneous set X is deemed *proper* if it contains at least two vertices ($|X| \geq 2$) and if it is not equal to the entire vertex set of G. Considering a graph G that admits a proper homogeneous set X, let x be any vertex in X. We can decompose G into two graphs: $G[X]$ and $G \setminus (X \setminus x)$; notably, the latter graph remains unchanged regardless of the choice of x, assuming X is a homogeneous set. Furthermore, both $G[X]$ and $G \setminus (X \setminus x)$ are induced subgraphs of G.

A subset $B \subseteq V(G)$ is a *packing* in G if for every two distinct vertices $u, v \in B$, we have $N[u] \cap N[v] = \emptyset$. The *packing number* $\rho(G)$ is defined as the maximum cardinality of a packing in G. For a subset $D \subseteq V(G)$, we say that D is a *dominating set* of G if every vertex in G is either in D or adjacent to a vertex in D. An *efficient closed domination set* is defined as a dominating set $D = \{v_1, v_2, \ldots, v_\gamma\}$ such that $V = \bigcup_{i=1}^{\gamma} N[v_i]$ and $N[v_i] \cap N[v_j] = \emptyset$ for all distinct indices i, j. The cardinality of the smallest dominating set is called the *domination number*, denoted as $\gamma(G)$. An *efficient closed domination graph* refers to those graphs where the minimum dominating sets coincide with maximum packing sets. The *D-partition* of G refers to a partition of $V(G)$ into dominating sets. The maximum order of a D-partition of G is known as the *domatic number*, denoted by $d(G)$. A graph for which $d(G) = \delta(G) + 1$ is termed as being *domatically full*. A graph G is called *domatically critical* if removing any edge from G results in a smaller domatic number than G.

A vertex v is said to be *simple* if for any two vertices $x, y \in N(v)$, either $N[x] \subseteq N[y]$ or $N[y] \subseteq N[x]$. An ordering $v_1, v_2, \ldots, v_n$ is called a *simple elimination ordering* if for each $1 \leq t \leq n$, the vertex v_t is simple in $G[\{v_t, \ldots, v_n\}]$. For $n \geq 3$, an n-sun is defined as a graph on $2n$ vertices whose vertex set can be partitioned into $X = \{x_1, x_2, \ldots, x_n\}$ and $Y = \{y_1, y_2, \ldots, y_n\}$ such that X forms a maximum set of pairwise non-adjacent vertices, while Y induces a clique. Moreover, for each $i \in \{1, \ldots, n\}$, vertex x_i is adjacent to exactly y_i and y_{i+1}.

A graph is called *brittle* [27] if for every induced subgraph H of G, there exists at least one vertex in H that is neither an endpoint nor a midpoint of any induced path of length four in H. In simpler terms, every subgraph contains at least one vertex that cannot be part of any induced path P_4. The term *bull* refers to the graph with vertex set $\{v_1, v_2, v_3, u, w\}$ and edge set $\{v_1v_2, v_2v_3, v_1v_3, v_1u, v_2w\}$.

A graph is *chordal* if every cycle of length four or more contains a chord. A graph is *strongly chordal* if it is chordal and every cycle of length six or more contains a chord splitting the cycle into two odd-length paths. A graph G is *weakly chordal* if every

cycle of length greater than four in G and its complement contains a chord.
A *paw-free* graph is a graph that does not contain $(\overline{P}_2 \vee K_1)$ as an induced subgraph.
A graph is a *tolerance graph* [23] if there exists a collection $\mathscr{I} = \{I_v\}_{v\in V}$ of closed intervals on the real line and an assignment of positive numbers $t = \{t_v\}_{v\in V}$ such that $vw \in E \Leftrightarrow |I_v \cap I_w| \geq \min\{t_v, t_w\}$. Here $|I_u|$ denotes the length of the interval I_u. The *tolerance chain graphs* are defined to be the tolerance graphs that have a representation consisting of a nested family of intervals (i.e., a set of intervals totally ordered by inclusion). A graph is called a *threshold graph* if it does not contain C_4, $\overline{C}_4$, or P_4 as an induced subgraph.
A graph is a *domination graph* if every induced subgraph H of G contains a pair of vertices x, y such that $N_H(x) \subseteq N_H[y]$ in H (in this case, x is said to be dominated by y in H). Domination graphs contain many important classes of graphs, which we will discuss in the next section, such as chordal graphs, strongly chordal graphs, trapezoid graphs, tolerance graphs, and brittle graphs [13, 37]. Finally, a connected graph G is termed *$\gamma\beta$-perfect* if the domination number $\gamma(H)$ equals the covering number $\beta(H)$ for every induced connected subgraph H of G.
Next, we summarize some well-known results from [1] that hold for more general situations of induced and non-induced star-perfect graphs, which allows us to structure our study on non-induced star-perfect graphs.

Observation 2. [1] If G is non-induced star-critical, then $\theta_f(G) \geq \alpha_f(G) + 1$.

Observation 3. [1] A graph G is non-induced star-critical if and only if $\alpha_f(G - f') = \alpha_f(G)$ for all maximal non-induced star-cliques f' in G.

Observation 4. [1] For every maximal non-induced star-clique f' of a non-induced star-critical graph G, $\alpha_f(G - f') = \alpha_f(G)$.

Observation 5. [1] If f' is a maximal star of a graph G, then $\theta_f(G) = \theta(G - f') + 1$.

Observation 6. [1] For every vertex $v \in V(G)$ of a non-induced star-critical graph G, there exists a maximum star-independent set of G that does not contain v, and there are at least $\alpha_f(G)$ distinct maximum star-independent sets, each containing v.

Observation 7. For any graph G, $\theta_f(G) = \gamma(G)$ and $\alpha_f(G) = \rho(G)$.

Next, we also state a couple of well-known (and easy-to-prove) lemmas.

Lemma 1. *[31] Every tree is non-induced star-perfect.*

Lemma 2. *[21] P_4-free graphs are non-induced star-perfect if and only if the graph is a tree.*

Lemma 3. *[21] A graph is non-induced star-perfect if each lobe of G with respect to a vertex v is non-induced star-perfect.*

Lemma 4. *Induced star-perfect graphs are non-induced star-perfect graphs.*

More generally, it can be shown that all K_3-free non-induced star-perfect graphs are induced star-perfect.

Fact 1. *K_3-free non-induced star-perfect graphs are induced star-perfect.*

3. Which domination graphs are non-induced star-perfect

In this section, we aim to identify various families of domination graphs with equal non-induced star-covering number and star-independence number for every induced subgraph.
The first class of domination graphs that is interesting in this context is the class of strongly chordal graphs.

Theorem 8. *Strongly chordal graphs are non-induced star-perfect.*

For the proof of Theorem 8, we use the following result due to Faber [17].

Lemma 5. *[17] [see Theorem 1 in [38]] A graph G is strongly chordal if and only if it has a simple elimination ordering.*

Proof of Theorem 8. We proceed by induction on the number of vertices. The claim is trivial for graphs with few vertices. Let us assume that the argument is true for any strongly chordal graph with fewer than n vertices, and let G be a strongly chordal graph on n vertices. Since the property of being strongly chordal is hereditary, it suffices to prove that $\theta_f(G) = \alpha_f(G)$. First, by Lemma 5, the vertices of G admit a simple elimination ordering $v_1, v_2, \ldots, v_n$ such that v_1 is a simple vertex. Next, let $x \in N(v_1)$ be a vertex in G that has the maximum degree among all the neighbors of v_1, and let $f' = N[x]$ be the maximum non-induced star obtained with center vertex x. Now, the graph $G \setminus f'$ is strongly chordal and has fewer than n vertices and hence, by the induction hypothesis, $\theta_f(G \setminus f') = \alpha_f(G \setminus f')$. Also, since f' is maximal, by Observation 5 we have $\theta_f(G) = \theta_f(G \setminus f') + 1$. Finally, observe that for any $y \in N(v_1)$ with $y \neq x$, $N[y] \subseteq f'$. By construction, every maximum star-independent set S of $G \setminus f'$ of size $\alpha_f(G \setminus f')$ misses $N[x]$ in G. Therefore, $S \cup \{x\}$ is a maximum star-independent set in G, and $\alpha_f(G) = \alpha_f(G \setminus f') + 1$. This proves that $\theta_f(G) = \alpha_f(G)$. □

Lemma 6. *Odd-sun graphs are non-induced star-critical graphs.*

Proof. Consider an n-sun of order $n = 2m+1$, $m \geq 1$, whose vertex set V can be partitioned into inner vertices with degree more than two and outer vertices with degree equal to two. Let $X = \{x_1, x_2, \ldots, x_n\}$ denote the set of all outer vertices. We know that any non-induced star-cover $\mathscr{C}$ of size θ_f should meet every vertex in X. Since each $C_i \in \mathscr{C}$ can contain at most two vertices of X, $\theta_f \geq \frac{n}{2} > m$ follows immediately. Now, for any maximal star-independent set S of size α_f, $S \subset \{x_1, x_2, \ldots, x_n\}$, and for any $x_i \in S$, $x_{i-1}, x_{i+1} \notin S$. Hence, $\alpha_f = \frac{n-1}{2} = m$. Therefore, $\theta_f > \alpha_f$. $\square$

Theorem 9. *[34] Chordal graphs are non-induced star-perfect graphs if and only if it is odd-sun-free.*

We remark that the 'if' part of Theorem 9 is also a generalization of Theorem 2.2 in [15] since all trees are strongly chordal. In [29], Lehel and Tuza proved that a chordal graph is neighborhood perfect if and only if it contains no odd-sun. This proves the following corollary.

Corollary 1. *Chordal graphs are non-induced star-perfect if and only if it is neighborhood perfect.*

It is known that C_4-free trapezoid graphs are interval graphs and all interval graphs are strongly chordal (see page 595, Proposition 12 in [5]). Therefore, we have the following corollary.

Corollary 2. *C_4-free trapezoid graphs are non-induced star-perfect.*

Our next result establishes the relationship between an efficient closed domination set and a maximum star-independent set for an efficient closed domination graph. We also present a method to construct a minimum non-induced star-covering set from a given efficient closed domination set of a graph.

Theorem 10. *If every induced subgraph H of G is an efficient closed domination graph, then G is non-induced star-perfect.*

Proof. Since every induced subgraph H of G is an efficient closed domination graph, there exists an efficient closed domination set $D_H = \{v_1, v_2, \ldots, v_\gamma\}$ for each H such that $V(H) = \bigcup_{i=1}^{\gamma} N_H[v_i]$ and $N_H[v_i] \cap N_H[v_j] = \emptyset$ for all $i \neq j$. It is straightforward to verify that for every vertex $v \in V(H)$, $|N[v] \cap D_H| = 1$. Thus, D_H is a star-independent set of H, and $|D_H| = \gamma$. Next, construct a set $\mathscr{C}_H = \{C_1, C_2, \ldots, C_\gamma\}$, where $C_i = N_H[v_i]$ for each $v_i \in D_H$ and $1 \leq i \leq \gamma$. Since $V(H) = \bigcup_{i=1}^{\gamma} N_H[v_i]$, $\mathscr{C}_H$ is a non-induced star-covering set of H, and therefore $\theta_f(H) \leq \gamma$. On the other hand, $\alpha_f(H) \leq \theta_f(H)$. Thus, $\alpha_f(H) = \theta_f(H)$, and G is non-induced star-perfect. $\square$

We now turn to our main theorem on domination graphs. A graph G is a domination graph if every induced subgraph H of G is a domination graph. Additionally, these graphs are self-complementary, and therefore, if x is said to be dominated by y in G, then y is also dominated by x in $\overline{G}$, and vice versa.

Theorem 11. *A domination graph is non-induced star-perfect if and only if it is C_4-free and odd-sun-free.*

Proof. Observe that chordless cycles of length greater than four do not contain dominated vertices. Therefore, the class of domination graphs is either chordal or weakly chordal. However, since G is C_4-free, it cannot be weakly chordal. Thus, G must be chordal, and the theorem follows from Theorem 9. □

Recall that every subgraph of a brittle graph contains a vertex that cannot be part of a P_4 within that subgraph. Chordal graphs are brittle (see [25]). This follows a result from Dirac [14]. In particular, there is a result of Dahihaus [13] that implies that brittle graphs are domination graphs.
We conclude this section with the following result.

Corollary 3. *C_4-free brittle graphs are non-induced star-perfect.*

Proof. By Theorem 11, it suffices to show that brittle graphs are n-sun-free. To this end, consider an n-sun graph. It is easy to verify that every n-sun contains a bull graph H as an induced subgraph. Let $x, y \in V(H)$ be the two end vertices of H. It can be verified that $G[N_H[x] \cup N_H[y]]$ is isomorphic to P_4, and therefore H is not brittle. □

4. Miscellaneous classes

In Section 3, we presented a forbidden induced subgraph characterization for the class of domination graphs that is non-induced star-perfect. In this section, we broaden our survey to various distinct classes of non-induced star-perfect graphs (sometimes with added structural restrictions) that contribute to the progress of proving Conjecture 1.

Lemma 7. *Every strongly star-perfect graph is non-induced star-perfect.*

Proof. Let G be a strongly star-perfect graph. Suppose G is non-induced star-critical. By Observation 3, for every maximal star f', we have $\alpha_f(G \setminus f') = \alpha_f(G)$. This implies that for each maximal star-independent set S, S misses at least one maximal star f'. However, this contradicts the assumption that G is strongly star-perfect. Therefore, G must be non-induced star-perfect. □

The next class of graphs of interest is the class of regular domatically full graphs. Recall that these are graphs with domatic number $d(G) = \delta(G) + 1$.

Theorem 12. *Regular domatically full graphs are non-induced star-perfect.*

For the proof of Theorem 12, we use the following result due to Zelinka [42].

Lemma 8. *[42] A regular domatically full graph G with n vertices and domatic number d exists if and only if d divides n; such a graph is also domatically critical.*
Its structure is as follows:
The vertex set $V(G) = \bigcup_{i=1}^{d} V_i$, where $V_i \cap V_j = \emptyset$, $|V_i| = \frac{n}{d}$, and the subgraph G_{ij} of G induced by $V_i \cup V_j$ is regular of degree 1 (for $i, j = 1, \ldots, d; i \neq j$).

Proof of Theorem 12. Let G be a regular domatically full graph with n vertices and domatic number d. Since G is regular and domatically full, each vertex of G has degree $d-1$, and thus $\omega_f(G) = d$. Also, G has the structure described in Lemma 8 such that the subgraph G_{ij} of G induced by $V_i \cup V_j$ is 1-regular for $i, j = 1, \ldots, d$ with $i \neq j$. Let $v_1^i, v_2^i, \ldots, v_t^i$ be the vertices of V_i. Since these sets are pairwise disjoint, for each i, we can construct a set $\mathscr{C}_i = \{N[v_1^i], N[v_2^i], \ldots, N[v_t^i]\}$ such that each element of $\mathscr{C}_i$ induces a non-induced star in G of size $\omega_f(G)$, and all of whose center vertices are in V_i. Clearly, each $\mathscr{C}_i$ is a minimal non-induced star-covering set, and thus $\theta_f(G) \leq |V_i|$. Combining this with the property that $\omega_f(G)\theta_f(G) \geq n$, we obtain $\theta_f(G) = |V_i|$. Furthermore, since each $G_{i,j}$ is a 1-regular bipartite graph (i.e., K_3-free), no two vertices in V_i are contained in the same maximal star, and each V_i itself is a maximal star-independent set. This proves $\theta_f(G) = \alpha_f(G)$. Finally, to show that $\theta_f(H) = \alpha_f(H)$ for every induced subgraph H of G, observe that for any induced subgraph H, $\theta_f(H) = \max |\mathscr{C}_i \cap V(H)|$. Similarly, each partition V_i is the collection of all center vertices of each $\mathscr{C}_i$, and thus $\alpha_f(H) = \max |V_i \cap V(H)|$. Therefore, $\theta_f(H) = \alpha_f(H)$ for every induced subgraph H of G, and the theorem follows. □

Next, we consider threshold graphs. These graphs, which are C_4-free, $\overline{C}_4$-free, and P_4-free, belong to the class of chordal graphs. This follows as an immediate consequence of a result due to Erdős [16], which states that all threshold graphs are split graphs (their vertex set can be partitioned as the disjoint union of an independent set and a clique either of which may be empty). Furthermore, it is known that all split graphs are chordal.
The following fact easily follows from the definition of threshold graphs.

Fact 2. *The property of being threshold is hereditary.*

Theorem 13. *Threshold graphs are non-induced star-perfect.*

For the proof of Theorem 13, we use the following result due to Chvátal (see Corollary 1B in [12]).

Lemma 9. *[12] A graph is a threshold graph if and only if there is a partition of the vertex set $V(G)$ into disjoint sets X and Y, and an ordering $x_1, x_2, \ldots, x_r$, $r \leq |V(G)|$ of X such that:*

1. *Every two vertices in Y are adjacent.*
2. *No two vertices in X are adjacent.*
3. *$N(x_1) \supseteq N(x_2) \supseteq \cdots \supseteq N(x_r)$.*

Proof of Theorem 13. Let G be a threshold graph. By Fact 2, every induced subgraph of G is also a threshold graph. Thus, it suffices to prove that $\theta_f(G) = \alpha_f(G)$. Let X and Y be the partitions of G as described in Lemma 9, and satisfying properties 1, 2, and 3. If $N(x_r) \neq \emptyset$, then $\theta_f(G) = \alpha_f(G) = 1$. Otherwise, suppose $N(x_r) = N(x_{r-1}) = \cdots = N(x_{r-\{k-1\}}) = \emptyset$ for some $k \geq 1$. By 2, we have $N(x_{r-k}) \subseteq Y$, and by 1 and 3, there exists a vertex $y \in N(x_{r-k})$ such that $N[y] = V(G) \setminus \{x_r, x_{r-1}, \ldots, x_{r-\{k-1\}}\}$. It follows that $\theta_f(G) = k+1$ and $\alpha_f(G) = \alpha_f(N[y]) + k = 1 + k$. Therefore, $\theta_f(G) = \alpha_f(G)$, which completes the proof. □

The following corollary follows from the result due to [12] that compliment of a threshold graph is threshold.

Corollary 4. *If G is a threshold graph, then G and $\overline{G}$ are non-induced star-perfect.*

In [28], the following was shown (see also [9], page 473).

Lemma 10. *[28] A graph G is a min-tolerance chain graph if and only if G is a threshold graph.*

Therefore, the following result follows from Lemma 10.

Corollary 5. *Min-tolerance chain graphs are non-induced star-perfect.*

Next, we characterize paw-free graphs that are non-induced star-perfect. Recall that these are $(\overline{P}_2 \vee K_1)$-free graphs.
First, we prove the following useful theorem on a proper homogeneous set of a graph.

Theorem 14. *Let X be a proper homogeneous set of a graph G and $x \in X$. Then G is non-induced star-perfect if and only if both $G \setminus (X \setminus \{x\})$ and $G[X]$ are non-induced star-perfect, and either $G[X]$ or $G[N[x] \setminus X]$ is a clique.*

Proof. Let X be a proper homogeneous set in G and let $x \in X$.
The 'only if' direction follows from the fact that both $G \setminus (X \setminus \{x\})$ and $G[X]$ are induced subgraphs of the non-induced star-perfect graph G and therefore are non-induced star-perfect. If neither $G[X]$ nor $G[N[x] \setminus X]$ is a clique, then (since X is a proper homogeneous set) there exist vertices $u, v \in V(G) \setminus X$ and $u', v' \in X$ such that $uv, u'v' \notin E(G)$. This leads to the induced subgraph $G[u, u', v, v']$, which is isomorphic to the cycle C_4, contradicting the fact that G is C_4-free.
Conversely, suppose that both $H = G \setminus (X \setminus \{x\})$ and $G[X]$ are non-induced star-perfect, and either $G[X]$ or $G[N[x] \setminus X]$ is a clique. To prove that G is non-induced star-perfect, it suffices to show that for every subset $X' \subseteq X \setminus \{x\}$ and $G' = H \cup X'$, we have $\theta_f(G') = \alpha_f(G')$. To this end, we construct a non-induced star-covering set $\mathscr{C}_{G'}$ for G' as follows:

- Let $\mathscr{C}_H$ be a minimum non-induced star-covering set of H. Choose an element $c_i \in \mathscr{C}_H$ that meets vertex x.
- If $|c_i| = 1$ and $G[X]$ is a clique, then set $\mathscr{C}_{G'} = (\mathscr{C}_H \setminus c_i) \cup (c_i \cup X)$. If $|c_i| = 1$ but $G[N[x] \setminus X]$ is a clique, select a vertex $v \in N[x] \setminus X$ such that $c_j \in \mathscr{C}_H$ meets vertex v, with v being an endpoint of c_j. This selection is valid; otherwise, if we define $\mathscr{C}'_H = (\mathscr{C}_H \setminus c_i) \cup (c_j \cup x)$, it would contradict the minimality of $\mathscr{C}_H$ since we would have $|\mathscr{C}'_H| < |\mathscr{C}_H|$. Thus, we set $\mathscr{C}_{G'} = (\mathscr{C}_H \setminus \{c_i, c_j\}) \cup (c_j \setminus \{v\}) \cup (v \cup X)$.
- If $|c_i| \geq 2$, then since one of the graphs $G[X]$ or $G[N[x] \setminus X]$ is a clique, the set $c_i \cup X$ contains a non-induced star. Therefore, we have $\mathscr{C}_{G'} = (\mathscr{C}_H \setminus c_i) \cup (c_i \cup X)$.

It can be easily verified that $\mathscr{C}_{G'}$ is indeed a minimum non-induced star-covering set for G', leading to the conclusion that $\theta_f(G') = |\mathscr{C}_{G'}| = |\mathscr{C}_H| = |\theta_f(H)|$. Next, let S be a maximum star-independent set of H. Since the adjacency relations of vertices in $V(H)$ remain unchanged when constructing G' from H, it follows that for all $v \in V(G')$, we have $|N[v] \cap S| = 1$. This implies that the set S remains a maximum star-independent set in G', and $\alpha_f(G') = |S| = \alpha_f(H)$. Consequently, we conclude that $\theta_f(G') = \theta_f(H) = \alpha_f(H) = \alpha_f(G')$, which completes the proof. $\square$

Theorem 15. *Paw-free graphs are non-induced star-perfect if and only if it is C_{3n+1}-free and C_{3n+2}-free.*

For the proof of Theorem 15, we use the following result due to Olariu [39].

Lemma 11. *[39] A graph G is paw-free if and only if each component of G is K_3-free or complete multipartite.*

Proof of Theorem 15. The 'only if' part is trivial.
For the converse, consider an arbitrary connected (paw, C_{3n+1}, C_{3n+2})-free graph G. If G is K_3-free, then by Lemma 4, G is non-induced star-perfect. Therefore, assume G contains a triangle. By Lemma 11, G is complete multipartite with partitions

$X_1, X_2, \ldots, X_k,\ k \geq 3$. Since G is C_4-free, there exists at most one X_i such that $G \setminus X_i$ is a clique, and $2 \leq |V(G \setminus X_i)| < |V(G)|$. This means that $V(G \setminus X_i)$ is a proper homogeneous set in G. Now, since $G \setminus X_i$ is a clique, and for each $v \in X_i$, v is anticomplete to $X_i \setminus v$, it follows that $G \setminus (X_i \setminus x)$ and $G[X_i]$ are non-induced star-perfect. Then by Theorem 14, G is non-induced star-perfect. □

We next turn our attention to bull-free graphs.

Theorem 16. *[36] Any bull-free graph has one of the following five properties:*

(i) G contains a C_5.
(ii) G is K_3-free.
(iii) $\overline{G}$ is K_3-free.
(iv) G has a proper homogeneous set.
(v) G or $\overline{G}$ contains a G_0 (see Figure 3) as an induced subgraph.

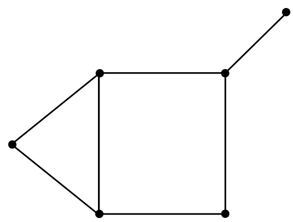

Figure 3. **Graph G_0.**

Theorem 17. *Bull-free graphs are non-induced star-perfect if and only if it is C_{3n+1}-free and C_{3n+2}-free.*

Proof. The 'if' part is trivial. To prove the 'only if' direction, consider a (bull, C_{3n+1}, C_{3n+2})-free graph G. Let H be any induced subgraph of G, and denote by $\mathscr{H}(H)$ the set of all maximal proper homogeneous sets of H. We will use induction on $h = |\mathscr{H}(H)|$. If $h = 0$, then by Theorem 16, either H or its complement $\overline{H}$ is K_3-free. If H is K_3-free, then by Lemma 4, G is non-induced star-perfect. On the other hand, if $\overline{H}$ is K_3-free, then H is P_5-free. In particular, this means that H is a bull-free graph that is also free from cycles of lengths C_{3n+1} and C_{3n+2}. Since H is bull-free and (C_4, C_5, P_5)-free, it implies that G is sun-free and chordal, respectively. Therefore, we conclude that H is sun-free chordal and hence strongly chordal (see [17]). By Theorem 8, it follows that H is non-induced star-perfect.
Now, suppose the statement holds for all graphs with $0 \leq h \leq m-1$, and let H be an induced subgraph of G with $h = m$. Let X be a proper homogeneous set in H of maximal order, and let $x \in X$. Then both $G[X]$ and $G \setminus (X \setminus \{x\})$ are induced subgraphs of H with at most $m-1$ maximal proper homogeneous sets. Thus, both

$G[X]$ and $G \setminus (X \setminus \{x\})$ are non-induced star-perfect. Next, since G is C_{3n+1}-free and C_{3n+2}-free, it follows that either $G[X]$ or $G[N[x] \setminus X]$ must be a clique. Therefore, by Theorem 14, we conclude that H is non-induced star-perfect. □

Finally, we focus on $\gamma\beta$-perfect graphs. In [40], it was observed that $\gamma\beta$-perfect graphs are not isomorphic to C_5 and do not necessarily contain C_3 or P_6 as induced subgraphs. Consequently, all C_4-free $\gamma\beta$-perfect graphs are also C_3-free, C_{3n+1}-free, and C_{3n+2}-free. These graphs represent the precise class of induced star-perfect graphs. Since all induced star-perfect graphs are included within the broader category of non-induced star-perfect graphs, we can derive the following result.

Theorem 18. *C_4-free $\gamma\beta$-perfect graphs are non-induced star-perfect.*

5. Formulating non-induced star-perfect as an integer program

A significant part of combinatorics involves optimizing problems, which are often expressed as linear forms of variables subject to integer or binary constraints. Consequently, these problems can be effectively transformed into linear and integer programming formulations. Linear and integer programming are widely used techniques for finding optimal solutions (either maximizing or minimizing) by optimizing an objective function subject to a given set of constraints. These techniques have numerous applications across various domains, including resource allocation, scheduling, and network flow optimization.

To formulate linear and integer programming problems, we consider a $(0,1)$-matrix M and column vectors $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{x}$, and $\boldsymbol{y}$. A linear program (the primal) and its dual can be expressed as follows:

$$\begin{array}{rrcl} \text{minimize} & \boldsymbol{a}^T\boldsymbol{x} & & \\ \text{subject to} & M\boldsymbol{x} & \geq & \boldsymbol{b} \\ & \boldsymbol{x} & \geq & 0 \end{array} \tag{5.1}$$

$$\begin{array}{rrcl} \text{maximize} & \boldsymbol{b}^T\boldsymbol{y} & & \\ \text{subject to} & M^T\boldsymbol{y} & \leq & \boldsymbol{a} \\ & \boldsymbol{y} & \geq & 0 \end{array} \tag{5.2}$$

In these formulations, $\boldsymbol{a}^T\boldsymbol{x}$ and $\boldsymbol{b}^T\boldsymbol{y}$ are referred to as the objective functions, while the constraints $M\boldsymbol{x} \geq \boldsymbol{b}$, $\boldsymbol{x} \geq 0$, $M^T\boldsymbol{y} \leq \boldsymbol{a}$, and $\boldsymbol{y} \geq 0$ define the feasible regions for the primal and dual programs.

The integration of integer programming techniques is now commonly used in graph theory. They provide a strong framework for solving optimization problems specific to graph invariants, such as finding minimal spanning trees, determining maximum flows, and identifying optimal matchings.

The Marriage Theorem [24] was the first problem in graph theory that illustrated integer programming-like characteristics. Notably, linear programming did not exist at the time. Ford and Fulkerson's [18] significant work, which presented one of the first applications of linear programming to graph theory, did not appear until fifty years later. Gallai's [20] seminal study also provides an early indication of this relationship. Tucker [41] investigated the significance of integer programming techniques on perfect graphs to solve optimization problems pertaining to the efficient provision of municipal services. Additionally, when the undominated rows of M correspond to the incidence vectors of maximal cliques and $\boldsymbol{a} = \boldsymbol{b} = \mathbf{1}$, it can be observed that equations 5.1 and 5.2 are indeed the covering number $\theta(G)$ and independence number $\alpha(G)$, respectively [10, 11].

Here, we expand this foundation by introducing integer programming formulation for the non-induced star-perfect invariants. The objective of this section is to explore the linear programming method for the minimum non-induced star-covering number and maximum star-independent number problems.

5.1. Preliminaries

We begin with some definitions. Let I_n denote an identity matrix of order $n \times n$. The *adjacency matrix* of a graph is a square matrix A where each entry in the $(i, j)^{th}$ cell is 1 if there is an edge between vertices i and j, and 0 otherwise. The *closed neighborhood matrix*, denoted as N, is obtained by adding the identity matrix to the adjacency matrix of a graph, i.e., $A + I_n$. To distinguish vectors from other variables, we use bold notation, and therefore the expression $N\boldsymbol{x}$ denotes the standard matrix multiplication between the $n \times n$ matrix N and the $n \times 1$ vector $\boldsymbol{x}$.

5.2. θ_f and α_f as an integer program

Given a non-induced star-covering set $\mathscr{C} = \{C_1, \ldots, C_k\}$ of a graph G, we define a set $J \subseteq V(G)$ by selecting the center vertex of each $C_i \in \mathscr{C}$. Note that if C_i is isomorphic to K_2, we choose any one vertex, and if C_i is isomorphic to a single vertex v, we choose v itself.

The characteristic function for the set J is defined such that it assigns the value 1 to each vertex in J and 0 to the vertices outside of J. A key property of this characteristic function is that it satisfies the condition that the sum of the function values assigned to any vertex and its neighbors in G is at least 1. This motivates us to define a *non-induced star-covering function* as a function $l : V(G) \to \{0, 1\}$ such that the sum of the function values over all vertices in the closed neighborhood $N_G[v]$ of v satisfies $l(N[v]) = \sum_{u \in N[v]} l(u) \geq 1$ for every vertex $v \in V(G)$. A non-induced star-covering function is considered minimal if there does not exist another non-induced star-covering function $l' : V(G) \to \{0, 1\}$, distinct from l, that satisfies $l'(v) \leq l(v)$ for every $v \in V(G)$. A minimum non-induced star-covering function on a graph G is a non-induced star-covering function l which attains the minimum value of $|l| = \sum_{v \in V(G)} l(v)$, denoted by $\theta_f(G)$, the *non-induced star-covering number* of G.

Figure 4 depicts the non-induced star-covering function $\boldsymbol{x} = [1\,0\,0\,1\,0\,0]^T$ of a 3-sun graph. It is important to note that $\boldsymbol{x} = [100100]^T$ is also a non-induced star-covering function for the covers $\{(v_1, v_2), (v_5, v_6, v_4, v_3)\}$ and $\{(v_6, v_1, v_2), (v_5, v_4, v_3)\}$.

$$N\boldsymbol{x} = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 1 \\ 1 \\ 1 \\ 2 \end{bmatrix} \geq \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

$\mathcal{C} = \{(v_1, v_6), (v_4, v_2, v_3, v_5,)\}$, $J = \{v_1, v_4\}$

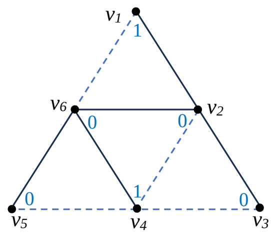

Figure 4. **A non-induced star-covering function.**

A *star-independent function* on a graph G is a function $m : V(G) \to \{0, 1\}$ that satisfies $m(N[v]) = \sum_{u \in N[v]} m(u) \leq 1$ for all vertices $v \in V(G)$. A maximum star-independent function is a star-independent function m which attains the maximum value of $|m| = \sum_{v \in V(G)} m(v)$, denoted by $\alpha_f(G)$, the *star-independent number* of G. The $\{0, 1\}$-vector $\boldsymbol{y}$ of any star-independent function y satisfies the matrix inequality $N\boldsymbol{y} \leq \mathbf{1}$. As an example, the non-induced star-covering function $\boldsymbol{y} = [1\,0\,0\,0\,0\,0]^T$ of a 3-sun graph is given in Figure 5..

$$N\boldsymbol{y} = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \leq \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

$S = \{v_1\}$

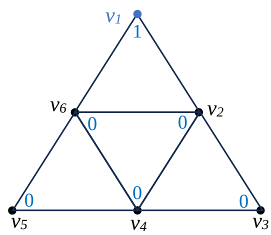

Figure 5. **A non-induced star-independent function.**

To determine the non-induced star-covering number in graphs, a common technique is to formulate the problem as an integer program using the neighborhood matrix $N = A + I_n$. Now, θ_f is the value of the integer program in Equation 5.1.

$$\begin{aligned} \theta_f = \text{minimize} \quad & \mathbf{1}^T \boldsymbol{x} \\ \text{subject to} \quad N\boldsymbol{x} \ & \geq \ \mathbf{1} \\ \boldsymbol{x} \ & \geq \ 0, \ x_i \in (0,1) \end{aligned} \tag{5.3}$$

This formulation allows for the determination of an optimal solution that minimizes the cardinality of non-induced star coverings in the graph G.
Similarly, the dual problem of determining the star-independence number can also be formulated in integer program terms; $\alpha_f(G)$ is the value of the integer program in Equation 5.2.

$$\begin{aligned} \alpha_f = \text{maximize} \ & \mathbf{1}^T \boldsymbol{y} \\ \text{subject to} \ & N\boldsymbol{y} \ \leq \ \mathbf{1} \\ & \boldsymbol{y} \ \geq \ 0, \ \ y_i \in (0,1) \end{aligned} \tag{5.4}$$

The min-max inequality tells us that

$$\theta_f(G) = \min \mathbf{1}^T \boldsymbol{x} \geq \max \mathbf{1}^T \boldsymbol{y} = \alpha_f(G)$$

Under certain conditions, the strong min-max equality holds:

$$\theta_f(G) = \min \mathbf{1}^T \boldsymbol{x} = \max \mathbf{1}^T \boldsymbol{y} = \alpha_f(G) \tag{5.5}$$

Our first goal is to show that odd-sun-free graphs are non-induced star-perfect. We first need the following terminology.
A hypergraph $\mathscr{H} = (X, \mathscr{E})$ is a mathematical structure that consists of a non-empty set of vertices X and edges $\mathscr{E} \subseteq P(X)$, where $P(X)$ is the power set of X. For a graph G, a hypergraph $\mathcal{N}(G) = (X, \mathscr{E})$ having the vertices of G as X and the closed neighborhoods $N[v]$ of each $v \in V(G)$ as $\mathscr{E}$ is called the closed neighborhood hypergraph of G. A balanced matrix is a $(0,1)$-matrix that does not contain an odd square submatrix with all row and column sums equal to two. A totally balanced matrix is a $(0,1)$-matrix that does not contain a square submatrix with no identical columns and its row and column sums equal to two.
Balanced matrices have been studied extensively by Berge [6] and Fulkerson et al. [19]. Notably, Berge studied the balancedness of hypergraphs and in his work [6], he established that if the incident matrix $\mathscr{N}$ of a closed neighborhood hypergraph is balanced, then the hypergraph is balanced. We observe that $\mathscr{N}$ is essentially the closed neighborhood matrix N of G. Consequently, N is balanced as well.

Theorem 19. *[19] If N is a balanced matrix, then the strong min-max equality (5.5) holds for Equations (5.3) and (5.4).*

Theorem 20. *[8] A graph is odd-sun-free chordal if and only if its closed neighborhood hypergraph $\mathscr{N}(G)$ is balanced.*

Theorem 9. (Restated Theorem). *Odd-sun-free chordal graphs are non-induced star-perfect.*

Our next goal is to show that strongly chordal graphs are non-induced star-perfect (see Theorem 8). To this end, we will consider a more restrictive class of matrices called totally balanced [30].

Theorem 21. *[26] If N is a totally balanced matrix, then the strong min-max equality (5.5) holds for Equations (5.3) and (5.4).*

Theorem 22. *[17] A graph is strongly chordal if and only if its closed neighborhood matrix N is totally balanced.*

Theorem 8. (Restated Theorem). *Strongly chordal graphs are non-induced star-perfect.*

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