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Research Article

Finite groups whose commuting graphs are line graphs

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Abstract: The commuting graph $\Gamma(G)$ of a group G is the simple undirected graph with group elements as a vertex set and two elements x and y are adjacent if and only if $xy = yx$ in G. By eliminating the identity element of G and all the dominant vertices of $\Gamma(G)$, the resulting subgraphs of $\Gamma(G)$ are $\Gamma^*(G)$ and $\Gamma^{**}(G)$, respectively. In this paper, we classify all the finite groups G such that the graph $\Delta(G) \in \{\Gamma(G), \Gamma^*(G), \Gamma^{**}(G)\}$ is the line graph of some graph. We also classify all the finite groups G whose graph $\Delta(G) \in \{\Gamma(G), \Gamma^*(G), \Gamma^{**}(G)\}$ is the complement of line graph.

Keywords: commuting graph, line graph, complement of line graph.

AMS Subject classification: 05C25, 05C76

1. Introduction

In the last few decades the interest of people in the study of algebraic objects using graph theoretic concepts is growing which is an interesting research topic leading to several important results and questions. The study of graphs over many algebraic structures is very important as graphs of this type have numerous applications ([8], [14]). The commuting graph on a group G was introduced by Brauer and Fowler [5] with vertex set $G \setminus \{e\}$. The commuting graphs for different non-abelian groups have been studied by many authors (see [6], [1] [12]). The graph theoretic properties such as detour distance, metric dimension and resolving polynomial properties of the commuting graph on the dihedral group D_n were studied by Faisal *et al.* [1]. Authors in [10], also studied the detour distance properties, resolving polynomial and spectral

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properties of the commuting graph of non-abelian groups of order p^4 with center having p elements. Recently, Carleton et al. [7], studied the commuting graph for A-solvable groups and Ashrafi et al. [15], studied the commuting graph of CA-groups. The line graph $L(\Gamma)$ of a graph Γ is the graph whose vertex set consists of all edges of Γ; two vertices of $L(\Gamma)$ are adjacent if and only if they are incident in Γ. All finite nilpotent groups whose power graphs and proper power graphs are line graphs were characterized by Bera [4]. Parveen et al. characterized all finite groups whose enhanced power graphs are line graphs in [13]. Furthermore, [13] determines all finite nilpotent groups whose proper enhanced power graphs are line graphs of certain graphs. In [11], Manisha et al. characterized all the finite groups whose order supergraph is the line graph. Throughout this paper, G is a finite group and e is the identity element of G.

In this paper, we aim to study the line graph of commuting graph associated to finite groups. By eliminating the identity element of G and all the dominant vertices of $\Gamma(G)$, the resulting subgraphs of $\Gamma(G)$ are $\Gamma^*(G)$ and $\Gamma^{**}(G)$, respectively. We characterize all the finite group G such that $\Delta(G) \in \{\Gamma(G), \Gamma^*(G), \Gamma^{**}(G)\}$ is a line graph of some graph. Also, we classify all finite groups G such that $\Delta(G) \in \{\Gamma(G), \Gamma^*(G), \Gamma^{**}(G)\}$ is the complement of a line graph.

2. Preliminaries

The vertex set $V(\Gamma)$ and the edge set $E(\Gamma) \subseteq V(\Gamma) \times V(\Gamma)$ form an ordered pair that constitutes a graph Γ. If $\{u, v\} \in E(\Gamma)$, then two vertices, u and v are adjacent; if so, we denote them as $u \sim v$ and if not, $u \nsim v$. When a pair of edges e_1 and e_2 have a similar endpoint, then they are referred to as incident edges. If a graph has no loops or multiple edges, it is referred to as a simple graph. In this study, we just take into consideration simple graphs. A graph Γ' such that $V(\Gamma') \subseteq V(\Gamma)$ and $E(\Gamma') \subseteq E(\Gamma)$ is called a subgraph of a graph Γ.

Suppose that $X \subseteq V(\Gamma)$. Then the subgraph Γ' induced by the set X is a graph such that $V(\Gamma') = X$ and $u, v \in X$ are adjacent if and only if they are adjacent in Γ. A vertex u of a graph Γ is referred to as a dominating vertex of Γ if it is adjacent to every other vertex of Γ. We refer to the set of all dominating vertices of Γ as $\mathrm{Dom}(\Gamma)$. A graph Γ is considered complete if every pair of vertices is adjacent to one another. K_n represents a complete graph with n vertices. The graph $\overline{\Gamma}$ such that $V(\Gamma) = V(\overline{\Gamma})$ and two vertices u and v are adjacent in $\overline{\Gamma}$ if and only if u is not adjacent to v in Γ is the complement of a graph Γ.

Throughout this paper, $\mathbb{Z}_n, D_n, S_n, A_n$ and Q_8 denotes the cyclic group of order n, dihedral group of order $2n$, symmetric group on n symbols, alternating group on n symbols and the quaternion group of order 8 respectively. The centralizer of an element x in the group G is denoted by $C_G(x)$ and the center of the group G is denoted by $Z(G)$.

A characterization of line graph and its complement are described in the next two lemmas, both of which are helpful in the sequel.

Lemma 1. [3] *A graph* Γ *is the line graph of some graph if and only if none of the nine graphs in* Figure 1 *is an induced subgraph of* Γ*.*

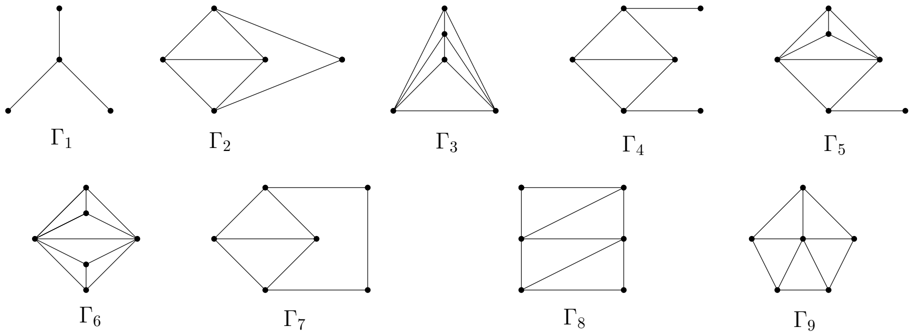

Figure 1. **Forbidden induced subgraphs of line graphs.**

Lemma 2. [2, Theorem 3.1] *A graph* Γ *is the complement of a line graph if and only if none of the nine graphs* $\overline{\Gamma_i}$ *in* Figure 2 *is an induced subgraph of* Γ*.*

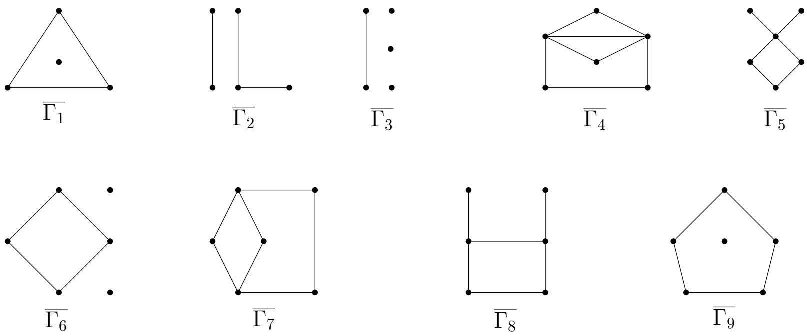

Figure 2. **Forbidden induced subgraphs of the complement of line graphs.**

3. Line graph characterization of $\Gamma(G)$

All the finite groups G such that $\Gamma(G)$ is a line graph of some graphs are classified in this section. Afterwards, we identify all the finite groups that have $\Gamma^*(G)$ and $\Gamma^{**}(G)$ as line graphs. Lastly, we characterize all the groups G that have $\Gamma(G)$, $\Gamma^*(G)$, and $\Gamma^{**}(G)$ as the complement of the line graph of some graph. One can easily observe the following.

Lemma 3. *If a graph is a complete graph, then it is the line graph and complement of line graph of some graph.*

Lemma 4. *The commuting graph $\Gamma(G)$ of a group G is complete if and only if G is abelian.*

Lemma 5. *For the commuting graph $\Gamma(G)$, the dominating set $Dom(\Gamma)$ is the center $Z(G)$ of the group G.*

Lemma 6. *[16] The maximum number of edges in an n-vertex triangle-free graph is $\lfloor \frac{n^2}{4} \rfloor$.*

Theorem 1. *The commuting graph $\Gamma(G)$ is the line graph of some graph if and only if G is abelian.*

Proof. If G is an abelian group, then $\Gamma(G)$ is the complete graph. Hence it is a line graph of some graph. Now, we show that no non-abelian group can be a line graph. Let G be a non-abelian group.
Let $\mid Z(G) \mid \geq 3$. Let three distinct elements in the center be e, x and y. Since G is a non-abelian group, there exist elements a and b such that $a \nsim b$. Then the set $\{e, x, y, a, b\}$ will make the structure of Γ_3 in Figure 1. Therefore, $\mid Z(G) \mid \leq 2$. Now, suppose that $Z(G) = \{e, x\}$. Note that there exist a and b such that $a \nsim b$. Note that $a \sim ax$ and $b \sim bx$. The set $\{e, x, a, ax, b, bx\}$ will make the structure of Γ_6 in Figure 1. Therefore, the group G has the trivial center.
Note that the probability of any two elements in G to commute is

$$P_2(G) = \frac{\text{Nubmer of conjugacy classes in } G}{\text{Total number of elements in } G} \text{(see [9])}.$$

One can easily note that if a group G has trivial center, then $P_2(G) \leq \frac{1}{2}$. Let $P_2(G) = \frac{1}{2}$. Then the maximum of half pairs of elements can commute. There is total ${}^nC_2$ pairs of n elements in which $\frac{n(n-1)}{4}$ pairs can commute where $n = \mid G \mid$. This implies that there exist $\frac{n(n-1)}{4}$ pairs that do not commute.
Let $\overline{\Gamma(G)}$ be the complement of the commuting graph $\Gamma(G)$. By Lemma 6, the maximum number of edges in $(n-1)$ non-central vertex triangle free graph in $\overline{\Gamma(G)}$ is $\lfloor \frac{(n-1)^2}{4} \rfloor$. Note that there are $\frac{n(n-1)}{4}$ pairs which do not commute and

$$\frac{n(n-1)}{4} > \left\lfloor \frac{(n-1)^2}{4} \right\rfloor. \tag{3.1}$$

Hence, there always exists three distinct elements x, y and z such that x does not commute with y, y does not commute with z and z does not commute with x. As the identity element always commute with all other elements, so the set $\{e, x, y, z\}$ will make Γ_1 in Figure 1.
If $P_2(G) < \frac{1}{2}$, then there are more choices of pairs which do not commute. One can easily check that the inequality (3.1) is satisfied in this case. Therefore, we get

an induced subgraph Γ_1 of Figure 1 in this case. Therefore, there does not exist a non-abelian group for that the commuting graph $\Gamma(G)$ is the line graph of some graph. □

Theorem 2. *The commuting graph $\Gamma^*(G)$ is the line graph of some graph if and only if one of the following conditions holds:*

1. *The group G is abelian.*
2. *The group G is a non-abelian group with trivial center and the centralizer of any non-central element is abelian.*

Proof. If G is abelian group, then $\Gamma^*(G)$ is the complete graph. Hence it is a line graph of some graph. Also, if the centralizer of any non-central element is abelian, then G is partitioned into disjoint commuting classes. Therefore, $\Gamma^*(G)$ is the line graph. For the converse, let G be a non-abelian group such that $\Gamma^*(G)$ is the line graph.
By the similar argument as in the proof of Theorem 1, $\mid Z(G) \mid \leq 2$. Now, suppose $Z(G) = \{e, x\}$. Since the conjugacy class of each non-central element contains more than one element, one can observe that $P_2(G) > \frac{1}{2}$ if and only if G is either isomorphic to D_4 or Q_8, but $\Gamma^*(D_4)$ and $\Gamma^*(Q_8)$ are not line graphs. If $P_2(G) \leq \frac{1}{2}$, then by the similar argument as in Theorem 1, we get three distinct element y, z and w such that $y \nsim z, z \nsim w$ and $w \nsim y$. Now the set $\{x, y, z, w\}$ will make Γ_1 of Figure 1. Therefore, the group G has the trivial center.
Let $x \in G$ be a non-central element. Suppose that the centralizer $C_G(x) = \{e, x, y_1, \ldots, y_{m-2}\}$ of x in G is non-abelian. By the similar argument as in Theorem 1, we can suppose $\mid Z(C_G(x)) \mid = 2$. By the similar argument for $C_G(x)$ as above, one can show that $\Gamma^*(G)$ is not a line graph. This implies that the centralizer of each non-central element is abelian. One can easily observe that $\Gamma^*(G)$ is a line graph in this case. □

Example 1. The commuting graph $\Gamma^*(D_n)$ is the line graph for dihedral groups D_n when n is odd.

If G is abelian group, then the center consists all the elements of group. In this case there is no vertex left for $\Gamma^{**}(G)$. So, we will find when the commuting graph $\Gamma^{**}(G)$ is the line graph of some graph for non-abelian groups.

Theorem 3. *Let G be a non-abelian group. Then the commuting graph $\Gamma^{**}(G)$ is the line graph of some graph if and only if centralizer of any non-central element is abelian.*

Proof. If the centralizer of each element is abelian, then G is partitioned in commuting classes. One can easily check that $\Gamma^{**}(G)$ is a line graph in this case.
If the centralizer of an element in a group G is non-abelian, then by the similar argument as in Theorem 2, one can show that $\Gamma^{**}(G)$ is not a line graph. □

Example 2. The commuting graph $\Gamma^{**}(D_n)$ is the line graph for dihedral groups D_n.

Example 3. The commuting graph $\Gamma^{**}(Q_8)$ is the line graph for quaternion groups Q_8.

Theorem 4. *Let $\Delta(G) \in \{\Gamma(G), \Gamma^*(G), \Gamma^{**}(G)\}$. Then $\Delta(G)$ is the complement of the line graph of some graph if and only if one of the following conditions holds:*

1. *The group G is abelian.*
2. *If G is a non-abelian group, then $G \cong D_4$ or $G \cong Q_8$.*

Proof. If G is abelian group, then $\Delta(G)$ is the complete graph. Hence it is the complement of a line graph of some graph. Also, if $G \cong D_4$ or $G \cong Q_8$, then one can check that $\Delta(G)$ is the complement of the line graph. For the converse, let G be a non-abelian group such that $\Delta(G)$ is the complement of the line graph.
Let $\mid Z(G) \mid > 2$. Then there exists two elements in center besides the identity. Let x and y to be two distinct elements of center other than the identity. As G is non-abelian group, there exists a and b in G such that $a \nsim b$. Now, a, ax and ay commute with each other. But b does not commute with all three of them. So the set $\{a, ax, ay, b\}$ will make $\overline{\Gamma_1}$ of Figure 2. Hence, for the complement of line graph, $\mid Z(G) \mid \leq 2$.
Let $Z(G) = \{e, x\}$. First, suppose that the centralizer of every non-central element a as $\{e, x, a, ax\}$. Then the set of non-central elements is partitioned into the commuting classes consists of only two elements. If the number of commuting classes of non-central elements is greater than three, then we get $\overline{\Gamma_3}$ of Figure 2. This implies that there are three or less commuting classes. The only possibility for such group are either D_4 or Q_8. One can easily check that $\Delta(D_4)$ and $\Delta(Q_8)$ are complement of a line graph.
Now, suppose that the centralizer $C_G(a)$ of every non-central element $a \in G$ is abelian and some centralizer $C_G(a)$ consists of more than four elements. This implies that $C_G(a) \backslash Z(G)$ consists of at least three distinct elements a_1, a_2 and a_3. Since, G is non-abelian, there exists y in $G \backslash C_G(a)$ which does not commute with any of a_1, a_2 and a_3. Therefore, we get an induced subgraph $\overline{\Gamma_1}$ of Figure 2.
Now, suppose that the centralizer of $C_G(a)$ of a non-central element $a \in G$ is non-abelian. Then the centralizer $C_G(a)$ consists of at least two elements c and d such that $c \nsim d$. Note that $c \sim cx, cx \sim ca, c \sim ca, c \nsim d, d \nsim cx$ and $d \nsim ca$. Therefore, we get an induced subgraph $\overline{\Gamma_1}$ of Figure 2. Hence, $\mid Z(G) \mid = 1$.
Now, suppose an odd prime $p \geq 5$ divide the order $\mid G \mid$ of G. Then there exists an element $x \in G$ such that $o(x) = p$. Therefore, we get a cyclic subgroup

$$H = \langle x \rangle = \{e, x, x^2, \ldots x^{p-1}\} \cong \mathbb{Z}_p.$$

Note that $C_G(x) \subseteq C_G(x^2) \subseteq C_G(x^4)$. Since G is non-abelian, we get an element $y \in G \backslash C_G(x^4)$ such that $y \nsim x, y \nsim x^2, y \nsim x^4$. Therefore, we get an induced subgraph $\overline{\Gamma_1}$ of Figure 2. This implies that only divisors of the order $\mid G \mid$ of G are 2 or 3 such that the order of each element is less than five. Therefore, $\mid G \mid = 2^\alpha 3^\beta, \alpha, \beta \geq 1$. Then, there exists a subgroup H of G generated by an involution and an element of order 3. Hence, by [17], G is isomorphic to either A_4, S_3 or S_4. One can easily check that $\Delta(A_4), \Delta(S_3)$ and $\Delta(S_4)$ are not complement of a line graph. $\square$

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