Communications in Combinatorics and Optimization
Vol. xx, No. x (xxxx), pp. 1-11
https://doi.org/10.22049/cco.2025.29422.1988

CCO
Commun. Comb. Optim.

Research Article

Generalized subdivisions in digraphs spanned by subdivision of smaller digraphs and the chromatic number

Salman Ghazal

College of Engineering and Technology, American University of the Middle East,
Egaila, 54200, Kuwait
salman.ghazal@aum.edu.kw

Received: 1 February 2024; Accepted: 30 May 2025
Published Online: 6 June 2025

Abstract: A generalized subdivision H' of a digraph H is obtained by replacing each arc $e = (x, y) \in E(H)$ with tail x and head y, by an oriented path P_e whose first arc has tail x and whose last arc has head y, all these new paths being internally disjoint. If all these new paths are directed ones, then H' is simply a subdivision of H. The number of blocks (which turns out to have the same parity of $|E(H)|$) of the generalized subdivision H' of H is the sum of all the number of blocks of the new paths P_e. In this paper, we prove that if D is spanned by a subdivision of a digraph H such that $\chi(D)$ is at least $2n + |V(H)| + |E(H)|$, then D contains a generalized subdivision of H with n blocks. This bound is simplified when H is an oriented tree. If H is an oriented cycle, then our results assert a special case of a conjecture of Cohen et al. Moreover, the bound is improved to $2n + 1$ if H is an oriented cycle with two blocks or H is a directed cycle.

Keywords: oriented cycle, Hamiltonian, chromatic number, subdivision.

AMS Subject classification: 05C15, 05C20

1. Introduction

In this paper, graphs are finite and simple, that is they have no loop nor multiple edges, while digraphs are oriented graphs. Let D be a digraph obtained by assigning to each edge $e = xy$ of G an orientation (x, y) or (y, x), but not both. In this case, we say that G is the underlying graph of D. The set of vertices of G (resp. D) is denoted by $V(G)$ (resp. $V(D)$). The set of edges of G (resp. arcs of D) is denoted by $E(G)$ (resp. $E(D)$). If $e = xy$ is an edge of G, then we say that x and y are neighbors. Also we say that y is a neighbor of x. The degree $d_G(x)$ of a vertex x of G is the number of its neighbors. A path $P = x_1x_2 \dots x_n$ is a graph on n distinct vertices $x_1, x_2, \dots, x_n$ and whose edges are x_ix_{i+1}, for $1 \le i < n$. An oriented path

$P = x_1x_2 \dots x_n$ is an orientation of a path and it is a directed path if its arcs are (x_i, x_{i+1}), for $1 \le i < n$. A cycle $C = x_1x_2 \dots x_nx_1$ is a graph on n distinct vertices $x_1, x_2, \dots, x_n$ and whose edges are x_ix_{i+1}, for $1 \le i < n$ and the edge x_nx_1. An oriented cycle $C = x_1x_2...x_nx_1$ is an orientation of a cycle and it is a directed cycle if its arcs are (x_i, x_{i+1}), for $1 \le i < n$ and (x_n, x_1). The length of a path or a cycle is the number of its edges. The length of an oriented path or cycle is the number of its arcs. The girth of a graph (resp. oriented graph) is the length of a shortest cycle (oriented cycle) it contains. A block of an oriented cycle or path is a maximal directed path in the given oriented cycle or path. If C is an oriented cycle which is not a directed cycle, then the number of its blocks is always even. If C is an oriented cycle (resp. path) that has exactly p blocks, then we say that C is an oriented cycle (reps. path) with p blocks. If each block of an oriented cycle C is of length 1, then it is called an antidirected cycle. A digraph is said to be strongly connected if between any two of its vertices x and y there is a directed path from x to y. A tree is a connected graph with no cycle. An oriented tree is an orientation of a tree.

A Hamiltonian path (resp. cycle) is a path (resp. cycle) passing through all the vertices of a graph G. A Hamiltonian directed path (resp. cycle) is a directed path (resp. cycle) passing through all the vertices of a digraph D. A digraph is Hamiltonian if it has a Hamiltonian directed cycle. A Hamiltonian oriented cycle is any oriented (not necessarily directed) cycle passing through all the vertices of D. If A is a subset of the set of vertices of D, then $D[A]$ denotes the sub-digraph of D induced by A. If H is a sub-digraph (resp. subgraph) of a digraph (resp. graph) D such that $V(D) = V(H)$, then we say that D is spanned by H and also we say that D is spanned by $V(H)$.

Let H be a digraph. A subdivision H' of H is a digraph obtained from H by replacing each arc $e = (x, y)$ by a directed path $P_e = xx_1 \cdots x_ly$ from x to y, all these new paths being internally disjoint. A generalized subdivision H' of H is a digraph obtained from H by replacing each arc $e = (x, y)$ by an oriented path $P_e = xx_1 \cdots x_ly$ such that (x, x_1) and (x_l, y) are arcs of P_e, all these new paths being internally disjoint. Note that, in the previous two definitions, the number l may vary from one arc to another. For $e \in E(H)$, we define $block(P_e)$ to be the number of blocks of the oriented path P_e. We say that H' is a generalized subdivision of H with n blocks if $\sum_{e \in E(H)} block(P_e) = n$. Note that every subdivision of H is a generalized subdivision of H with $|E(H)|$ blocks, because in this case we have $block(P_e) = 1$, for every $e \in E(H)$.

The chromatic number $\chi(G)$ of a graph G is the smallest integer n such that all the vertices can be colored using n colors in a way that any two neighbor vertices receive distinct colors. The chromatic number of a digraph D is that of its underlying graph and is denoted by $\chi(D)$. We say that D is n-chromatic if its chromatic number is n. A graph G is n-degenerate if every subgraph G' of G has a vertex v such that $d_{G'}(v) \le n$. It is well known that if G is n-degenerate graph, then $\chi(G) \le n + 1$.

A classical result of Gallai and Roy is the following:

Theorem 1. *(Roy-Galli [9], [6]) Every digraph with chromatic number at least $n+1$ contains a directed path of length at least n.*

In other words, the directed path of length n is contained in every digraph with chromatic number at least $n+1$. This prompts the following question: Which digraphs H are sub-digraphs of all digraphs with sufficiently large chromatic number?

In 1959, Erdős proved that there are graphs with arbitrarily large chromatic number and arbitrarily large girth:

Theorem 2. *(Erdős [5]) For every $k \geq 3$ and $g \geq 3$, there is a graph with chromatic number at least k and girth at least g.*

Suppose that H is a digraph that contains an oriented cycle C of length n. Then by Theorem 2, for every $k \geq 3$, there is an oriented graph D with chromatic number at least k and girth at least $n+1$. Hence, H is not a sub-digraph of D.

Thus, the only connected oriented graphs H that are possibly candidates to generalize theorem 1 are oriented trees.

However, the following celebrated theorem of Bondy shows that the story does not stop here.

Theorem 3. *(Bondy [2]) Every strongly connected digraph of chromatic number at least n contains a directed cycle of length at least n.*

By interpreting directed cycles of length at least n as subdivisions of the directed cycle of length n, the preceding theorem establishes a special case of a conjecture proposed by Cohen et al.:

Conjecture 1. ([3]) For every oriented cycle C, there is a constant $f(C)$ such that every strongly connected digraph with chromatic number at least $f(C)$ contains a subdivision of C.

Let $C(k,l)$ denote the cycle with two blocks of lengths k and l respectively and let $n = k+l$. Kim et al. [8] proved that $f(C(k,l))$ is $O(n^4)$. Ghazal and Al-Mniny [1] proved that if D is a digraph (not necessarily strongly connected) that has a Hamiltonian directed path and $\chi(D) > 3.\max\{k,l\}$, then D contains a subdivision of $C(k,l)$. Furthermore, El Joubbeh [4] proved that if D is digraph that has a Hamiltonian directed cycle and $\chi(D) \geq 3n$, then D contains a subdivision of any oriented cycle on n vertices. Recently, Ghazal and Tfaili [7] refined El-Joubbeh's proof and showed that the bound of $2n$ is sufficient.

Hamiltonian directed cycles can be viewed as spanning sub-digraphs and subdivisions of the antidirected cycle on n vertices can be viewed as an oriented cycle with n blocks. Motivated by this remark and by previous results, in this paper we consider digraphs D spanned by a subdivision of any digraph H. We prove that D contains a generalized subdivision of H with n blocks, if $\chi(D)$ is at least $2n+|V(H)|+|E(H)|$. When H is a tree on m vertices, then this bound reduces to $2n+2m-1$. If H is an oriented cycle with two blocks or a directed cycle, then a stronger bound of $2n+1$ is achieved.

Thus, our findings enhance prior results in the literature, offering extensions to existing work. Moreover, we note that our results do not assume strong connectivity. In Section 2, we present essential technical definitions and preliminary lemmas to streamline the proofs of our main results, which are detailed in Section 3, ensuring brevity and enhanced clarity.

2. Preliminary lemmas and definitions

In this section, we formally introduce the concept of generating sequences in graphs and digraphs and establish key properties governing their behavior. These sequences are employed to systematically construct oriented paths within digraphs, ensuring a specified number of blocks.

Suppose that $L = x_0x_1x_2...x_N$ is a linear ordering of the vertices of a graph G, that is L is a list of all the vertices of G. We define the intervals $[x_i, x_j] = \{x_s; i \leq s \leq j\}$, $[x_i, x_j[= \{x_s; i \leq s < j\}$ and $]x_i, x_j] = \{x_s; i < s \leq j\}$. Let $e = x_ix_j$ and $e' = x_px_q$ be two edges of G. We say that e and e' are secant edges of G with respect to L if $i < p < j < q$. Ghazal et al. [1] proved that if G has no secant edges with respect to L, then $\chi(G) \leq 3$.

Lemma 1. *([1], Lemma 8) Let L be a linear ordering of the vertices of a graph G. If G has no secant edges with respect to L, then $\chi(G) \leq 3$.*

In what follows and without loss of generality, we will suppose that $L = 0, 1, \cdots, N$ is a linear ordering of the vertices of a graph G. A *generating sequence* of length s of G with respect to L, denoted by B_s, is a sequence of vertices of G and intervals of L of the form:

$$b_0 = 0,\ [b_0 + 1, b_1[,\ b_1,\ [b_1 + 1, b_2[,\ b_2,\ \ldots,\ b_{s-1},\ [b_{s-1} + 1, b_s[,\ b_s$$

such that for all $0 \leq i < s$, we have:

- b_{i+1} is maximum with the property that $[b_i + 1, b_{i+1}[$ has no secant edges.
- $[b_i + 1, b_{i+1}]$ has secant edges.

In the proofs of some of our results, we will utilize the following slightly different sequence to streamline our argument. A modified generating sequence of length s of G with respect to L, denoted by B'_s, is a sequence of intervals and vertices of the form:

$$[b_0+1, b_1[,\ b_1,\ [b_1+1, b_2[,\ b_2,\ \ldots,\ b_{s-1},\ [b_{s-1}+1, b_s[,\ b_s$$

such that $b_0+1=0$ and for all $0 \leq i < s$, we have:

- b_{i+1} is maximum with the property that $[b_i+1, b_{i+1}[$ has no secant edges.
- $[b_i+1, b_{i+1}]$ has secant edges.

Suppose that B_s is a generating sequence of length s and let $r \leq s$. Then the subsequence B_r:

$$b_0=0, [b_0+1, b_1[,\ b_1,\ [b_1+1, b_2[,\ b_2,\ \ldots,\ b_{r-1},\ [b_{r-1}+1, b_r[,\ b_r$$

is the generating sequence of length r. B_s is said to be a maximum generating sequence if s is maximum, that is B_s is not a subsequence of another generating sequence B_t with $s<t$. Analogously, we define a subsequence of a modified generating sequence as well as a maximum modified generating sequence. A generating sequence (also a modified generating sequence) of a digraph is that of its underlying graph.

Lemma 2. *Suppose that B_s: b_0, $[b_0+1, b_1[$, b_1, $[b_1+1, b_2[$, b_2, $\ldots$, b_{s-1}, $[b_{s-1}+1, b_s[$, b_s is a maximum generating sequence of G with respect to a linear ordering L. Then:*

- $\chi(G[b_0, b_s]) \leq 4s+1$.
- $\chi(G) \leq 4s+4$.

Proof. Assume that $[b_s+1, N]$ has secant edges. Let b_{s+1} be the minimum such that $[b_s+1, b_{s+1}]$ has secant edges. Then $[b_s+1, b_{s+1}[$ has no secant edges. By adding the terms $[b_s+1, b_{s+1}[$, b_{s+1} to the sequence B_s, we get a generating sequence of length $s+1$, which contradicts the maximality of B_s. Therefore, $[b_s+1, N]$ has no secant edges. Thus $\chi(G[b_s+1, N]) \leq 3$ by Lemma 1.
Since

$$[b_0, b_s] = \{b_0, b_1, \ldots, b_s\} \bigcup (\bigcup_{i=0}^{s-1} [b_i+1, b_{i+1}[),$$

we have that

$$\chi(G[b_0, b_s]) \leq (s+1) + \sum_{i=0}^{s-1} 3 = 4s+1.$$

Finally, $\chi(G) \leq (4s+1)+3 = 4s+4$ because $V(G) = [b_0, b_s] \cup [b_s+1, N]$. □

In a similar way, we can prove the following:

Lemma 3. *Suppose that B'_s: $[b_0+1, b_1[$, b_1, $[b_1+1, b_2[$, b_2, $\ldots$, b_{s-1}, $[b_{s-1}+1, b_s[$, b_s is a maximum modified generating sequence of G with respect to a linear ordering L. Then $\chi(G) \leq 4s+3$.*

Lemma 4. *Suppose that $\chi(G) \geq 4t+1$. Then G has a generating sequence of length t.*

Proof. Let B_s be a maximum generating sequence of G. Then, $4t+1 \leq \chi(G) \leq 4s+4$. Hence, $t \leq s+\frac{3}{4}$. But s and t are integers, therefore, $t \leq s$. We conclude that the sequence B_t is a generating sequence of length t.

□

Lemma 5. *Suppose that $\chi(G) \geq 4t$. Then G has a modified generating sequence of length t.*

Proof. Let B'_s be a maximum modified generating sequence of G. Then, $4t \leq \chi(G) \leq 4s+3$. Hence, $t \leq s+\frac{3}{4}$. But s and t are integers, therefore $t \leq s$. We conclude that the sequence B'_t is a modified generating sequence of length t.

□

From now on we, suppose that $L = 0, 1, \ldots, N$ is a directed Hamiltonian path of a digraph D. Clearly, L can be considered as a linear ordering of the vertices of D. Suppose that

$$B_s\colon\ b_0,\ [b_0+1, b_1[,\ b_1,\ [b_1+1, b_2[,\ b_2,\ \ldots,\ b_{s-1},\ [b_{s-1}+1, b_s[,\ b_s$$

is a generating sequence of a digraph D with respect to L. For each $0 \leq i < s$, let $e_i = u_i v_i$ and $e'_i = u'_i v'_i$ be secant edges in $[b_i+1, b_{i+1}]$ with $u_i < u'_i < v_i < v'_i$. Let $\vec{e_i}$ and $\vec{e'_i}$ denote the orientations in D of the edges e_i and e'_i. Note that by definition of b_{i+1}, we have $b_{i+1} = v'_i$. For $0 \leq i \leq j \leq N$, we denote by $L[i,j]$ the sub-path $i, i+1, \ldots, j$ of L directed from i to j.

Using some sub-paths of L and the orientations $\vec{e_i}$ and $\vec{e'_i}$ of the secant edges, we will generate an oriented path, denoted by $P(B_s)$. Formally, we define:

$$P(B_s) = (L[b_0,u_0] \cup \vec{e_0} \cup L[u'_0,v_0] \cup \vec{e'_0}) \bigcup (L[b_1,u_1] \cup \vec{e_1} \cup L[u'_1,v_1] \cup \vec{e'_1}) \bigcup (L[b_2,u_2] \cup \vec{e_2} \cup L[u'_2,v_2] \cup \vec{e'_2}) \bigcup \cdots \bigcup (L[b_{s-1},u_{s-1}] \cup \vec{e}_{s-1} \cup L[u'_{s-1},v_{s-1}] \cup \vec{e'}_{s-1}) \bigcup L[b_s,N]$$

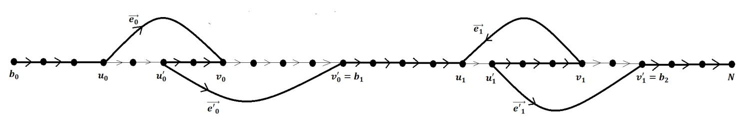

Figure 1. **Example of $P(B_s)$ drawn in bold with $s=2$.**

These generated paths $P(B_s)$ will be helpful in the next section to form our desired subdivisions with certain number of blocks, because the number of blocks of the oriented path $P(B_s)$ is known. In fact, we have:

Remark 1. Suppose that B_s is a generating sequence of a digraph D with respect to a directed path $L = 0, 1, \dots, N$. We have:

- If $b_s < N$, then $P(B_s)$ consists of $2s + 1$ blocks.
- If $b_s = N$ and $\vec{e'}_{s-1} = (u'_{s-1}, v'_{s-1})$, then $P(B_s)$ consists of $2s + 1$ blocks.
- If $b_s = N$ and $\vec{e'}_{s-1} = (v'_{s-1}, u'_{s-1})$, then $P(B_s)$ consists of $2s$ blocks.

3. Main results

In this section, we demonstrate that if a digraph D is spanned by a subdivision of H and satisfies $\chi(D) \geq 2n + |E(H)| + |V(H)|$, then D necessarily contains a generalized subdivision of H with n blocks. We subsequently derive several corollaries from this result. Prior to this, we begin by establishing better bound for the cases where H is an oriented cycle.

Recall that the number of blocks of any oriented cycle is always even. Also, oriented cycles with n blocks are the subdivisions of the antidirected cycle on n vertices.

Theorem 4. *Let n be an even positive integer and D be a digraph spanned by a directed cycle. If $\chi(D) \geq 2n + 1$, then D contains an oriented cycle with n blocks.*

Proof. Set $n = 2t$ and let D be a digraph such that $\chi(D) \geq 2n + 1$. Suppose that it is spanned by a directed cycle $C = 0, 1, \dots, N, 0$. Since $\chi(D) \geq 4t + 1$, by Lemma 4 D contains B_t, a generating sequence of D with respect to the directed path $L = 0, 1, \dots, N$. By Remark 1, the path $P(B_t)$ consists either of $2t + 1$ blocks or $2t$ blocks. In both cases, we observe that $C' = P(B_t) \cup (N, 0)$ forms an oriented cycle with $2t = n$ blocks. In fact, in the first case, the last block of $P(B_t)$, $(N, 0)$ and the first block of $P(B_t)$ combine together to form one block of C', and that is why the number of blocks decreases from $2t + 1$ to $2t$. In the second case, the arc $(N, 0)$ combines with the first block of $P(B_t)$ to form one block of C' and that is why the number of blocks remains $2t$. □

Theorem 5. *Let n be an even positive integer and D be a digraph spanned by an oriented cycle with two blocks. If $\chi(D) \geq 2n + 1$, then D contains an oriented cycle with n blocks.*

Proof. Set $n = 2t$ and let D be a digraph such that $\chi(D) \geq 2n + 1$ and suppose that it is spanned by an oriented cycle $C = Q \cup Q'$ with two blocks Q and Q'. For simplicity we denote these two internally disjoint paths by $Q = 0, 1, \dots, N$ and $Q' = 0, 1, \dots, M$. Let B_s be a maximum generating sequence of $D[Q]$ with respect to Q.

Case 1: Suppose that $b_s < N$ or ($b_s = N$ and $\overrightarrow{e'_{s-1}} = (u'_{s-1}, v'_{s-1})$).
By Lemma 2 and by Remark 1, we have $\chi(D[Q]) \leq 4s + 4$ and $P(B_s)$ consists

of $2s+1$ blocks. If $s \geq t-1$, then the oriented cycle $P(B_{t-1}) \cup Q'$ consists of $(2(t-1)+1)+1 = 2t = n$ blocks. Else, $s \leq t-2$. Hence

$$\chi(D[1, M-1]) \geq (2n+1) - (4s+4) \geq 4t - 4s - 4 = 4r,$$

where $r = t - s - 1$.By Lemma 5, $D[1, M-1]$ has a modified generating sequence A'_r with respect to the ordering $Q' - \{0, M\} = 1, 2, \ldots, M-1$. Consider the sequence $A_r : 0, A'_r$, obtained from A'_r by adding the term 0 before its first term. Then A_r is a generating of $D[Q']$ with respect to Q' and $a_r < M$. By Remark 1, $P(A_r)$ has $2r+1$ blocks. Therefore, the oriented cycle $P(B_s) \cup P(A_r)$ consists of $(2s+1)+(2r+1) = 2s+1+2t-2s-2+1 = 2t = n$ blocks.

Case 2: Suppose that $b_s = N$ and $\overrightarrow{e'_{s-1}} = (v'_{s-1}, u'_{s-1})$.
By the first point of Lemma 2 and by Remark 1, we have $\chi(D[Q]) \leq 4s+1$ and $P(B_s)$ consists of $2s$ blocks. Hence,

$$\chi(D[1, M-1]) \geq (2n+1) - (4s+1) = 4t - 4s = 4r,$$

where $r = t - s$. By Lemma 5, $D[1, M-1]$ has a modified generating sequence A'_r with respect to the ordering $Q' - \{0, M\} = 1, 2, \ldots, M-1$. Consider the sequence $A_r : 0, A'_r$, obtained from A'_r by adding the term 0 before its first term. Then A_r is a generating sequence of $D[Q']$ with respect to Q' and $a_r < M$. By Remark 1, $P(A_r)$ has $2r+1$ blocks. Therefore, the oriented cycle $C' = P(B_s) \cup P(A_r)$ consists of $(2s)+(2r+1)-1 = 2t = n$ blocks. Note that the union of the last block of $P(B_s)$ and the first block of $P(A_r)$ is a directed path that forms one block of C', which explains the -1 in the last equality. □

Theorem 6. *Let n be an even positive integer and D be a digraph spanned by a non-directed cycle with at most n blocks. If $\chi(D) \geq 4n$, then D contains an oriented cycle with n blocks.*

Proof. Set $n = 2t$ and let D be a digraph such that $\chi(D) \geq 4n$. Suppose that it is spanned by an oriented cycle $C = Q_1 \cup Q_2 \cup \cdots \cup Q_{2t'}$ with $2t'$ blocks, where $2t' \leq n$ and the Q_i's being listed consecutively. For each $1 \leq i \leq 2t'$, let u_i and v_i denote the initial vertex and the terminal vertex of Q_i, respectively, $H_i = D[Q_i - \{u_i, v_i\}]$ and B'_{s_i} be a maximum modified generating sequence of H_i with respect to the ordering $Q_i - \{u_i, v_i\}$. By Lemma 3, we get $\chi(H_i) \leq 4s_i + 3$, for all i. Since $V(D) = (\bigcup_{i=1}^{2t'} V(H_i)) \bigcup (\bigcup_{i=1}^{2t'} \{u_i, v_i\})$ and $\bigcup_{i=1}^{2t'} \{u_i, v_i\}$ consists of $2t'$ vertices, we have that:

$$4n \leq \chi(D) \leq \sum_{i=1}^{2t'} \chi(H_i) + 2t' \leq \sum_{i=1}^{2t'} (4s_i+3) + 2t' = \sum_{i=1}^{2t'} (4s_i+2) + 2t' + 2t' = 4t' + 2\sum_{i=1}^{2t'} (2s_i+1).$$

Hence, $\sum_{i=1}^{2t'} (2s_i+1) \geq (4n - 4t')/2 \geq (4n-2n)/2 = n$. Then there are non-negative integers $r_1 \leq s_1, \ldots, r_{2t'} \leq s_{2t'}$ such that $\sum_{i=1}^{2t'} (2r_i+1) = n$. For each i, consider the generating sequence $B_{s_i} : u_i, B'_{s_i}$ of $D[Q_i]$ with respect to Q_i and let B_{r_i} be its subsequence of length r_i. By Lemma 1, each $P(B_{r_i})$ consists of $2r_i + 1$ blocks. Therefore, the oriented cycle formed by $\bigcup_{i=1}^{2t'} P(B_{r_i})$ consists of $\sum_{i=1}^{2t'}(2r_i + 1) = n$ blocks. □

Remark 2. Suppose that H' is a generalized subdivision of a digraph H with n blocks. Then:

- For every $e \in E(H)$, $block(P_e)$ is odd.
- n and $|E(H)|$ have the same parity.
- $n \geq |E(H)|$.
- $n = |E(H)|$ if and only if H' is a subdivision of H.

Proof. Since $P_e = xx_1 \ldots x_l y$ such that (x, x_1) and (x_l, y) are arcs of P_e, then it starts by a forward block and also ends by a forward block. Hence the number of it blocks must be odd. Therefore, the parity of $\sum_{e\in E(H)} block(P_e) = n$ and the parity of $|E(H)|$ are the same.
Since for every $e \in E(H)$, $block(P_e) \geq 1$, then $n = \sum_{e\in E(H)} block(P_e) \geq |E(H)|$. For the last point we have: H' is a subdivision of $H \Leftrightarrow$ for every $e \in E(H)$, P_e is a directed path $\Leftrightarrow$ for every $e \in E(H)$, $block(P_e) = 1 \Leftrightarrow \sum_{e\in E(H)} block(P_e) = |E(H)|$ $\Leftrightarrow n = |E(H)|$. □

Theorem 7. *Let H be a digraph and $n \geq |E(H)|$ such that n and $|E(H)|$ have the same parity. Let D be a digraph spanned by a subdivision of H. If $\chi(D) \geq 2n + |E(H)| + |V(H)|$, then D contains a generalized subdivision of H with n blocks.*

Proof. Let D be a digraph spanned by a subdivision of H and $|E(H)|$ and n have the same parity. Suppose that D is a digraph such that $\chi(D) \geq 2n+|E(H)|+|V(H)|$. For all $e = (x(e), y(e)) \in E(H)$, let P_e denote the directed path in D that replaces e, $D_e = D[P_e - \{x(e), y(e)\}]$ and $B'_{s(e)}$ be a maximum modified generating sequence of D_e with respect to the ordering $P_e - \{x(e), y(e)\}$. Since $V(D) = (\bigcup_{e\in E(H)} V(D_e)) \bigcup (\bigcup_{e\in E(H)}\{x(e), y(e)\})$, we obtain that:

$$2n+|E(H)|+|V(H)| \leq \chi(D) \leq \sum_{e\in E(H)} \chi(D_e)+|V(H)| \leq \sum_{e\in E(H)} (4s(e)+3)+|V(H)|$$

$$= \sum_{e\in E(H)} (4s(e) + 2) + |E(H)| + |V(H)| = 2\sum_{e\in E(H)} (2s(e) + 1) + |E(H)| + |V(H)|.$$

Hence $\sum_{e\in E(H)}(2s(e) + 1) \geq n$. Since n and $|E(H)|$ have the same parity, we get that for all $e \in E(H)$, there is $0 \leq r(e) \leq s(e)$ such that $\sum_{e\in E(H)}(2r(e)+1) = n$. For all $e \in E(H)$, consider the generating sequence $B_{s(e)} : x(e), B'_{s(e)}$ and its generating subsequence $B_{r(e)}$ of length $r(e)$. Note that $B_{r(e)}$ can be viewed as a generating sequence of $D[P_e]$ with respect to P_e. Then by Remark 1, we deduce that the path

$P(B_{r(e)})$ consists of $2r(e)+1$ blocks. Therefore $\bigcup_{e\in E(H)} P(B_{r(e)})$ forms a generalized subdivision of H consisting of $\sum_{e\in E(H)}(2r(e)+1) = n$ blocks. □

Corollary 1. *Let H be any tree on m vertices and $n \geq m-1$ such that n and $m-1$ have the same parity. Let D be a digraph spanned by any subdivision of H. If $\chi(D) \geq 2n+2m-1$, then D contains a generalized subdivision of H with n blocks.*

Proof. It is enough to note that $|E(H)| = |V(H)| = m-1$ and apply Theorem 7. □

Corollary 2. *Let H be a digraph on m vertices and $n \geq |E(H)|$ such that n and $|E(H)|$ have the same parity. Let D be a digraph spanned by any subdivision of H. If $\chi(D) \geq 2n + m(m+1)/2$, then D contains a generalized subdivision of H with n blocks.*

Proof. It is enough to note that $|V(H)| = m$, $|E(H)| \leq m(m-1)/2$ and apply Theorem 7. □

Remark 3. Let $n \geq 2t'$ and let C' be the antidirected cycle on $2t'$ vertices. We have the following easy observation:

- C is a subdivision of C' $\Leftrightarrow$ C is an oriented cycle with $2t'$ blocks.
- C is a generalized subdivision with n blocks of C' $\Leftrightarrow$ C is an oriented cycle with n blocks.

Theorem 8. *Let n be an even integer and suppose that D is spanned by an oriented cycle with $2t'$ blocks, where $n \geq 2t'$. If $\chi(D) \geq 2n + 4t'$, then D contains an oriented cycle with exactly n blocks.*

Proof. Note that an oriented cycle with $2t'$ blocks is a subdivision of the antidirected cycle C' on $2t'$ vertices. Since $|V(C')| = |E(C')| = 2t'$ and $\chi(D) \geq 2n + 4t'$, by applying Theorem 7, we conclude that D contains a generalized subdivision C of C' with n blocks. By the previous remark, C is an oriented cycle with n blocks. □

Conflict of Interest: The authors declare that they have no conflict of interest.

Data Availability: Data sharing is not applicable to this article as no data sets were generated or analyzed during the current study.

References

[1] D. Al-Mniny and S. Ghazal, *Secant edges: a tool for Cohen et al.'s conjectures about subdivisions of oriented cycles and bispindles in hamiltonian digraphs with large chromatic number*, Graphs Combin. **41** (2025), no. 1, Article ID: 13. https://doi.org/10.1007/s00373-024-02865-7.

[2] J.A. Bondy, *Diconnected orientations and a conjecture of las vergnas*, J. Lond. Math. Soc. **2** (1976), no. 2, 277–282.
https://doi.org/10.1112/jlms/s2-14.2.277.

[3] N. Cohen, F. Havet, W. Lochet, and N. Nisse, *Subdivisions of oriented cycles in digraphs with large chromatic number*, Cycles with two blocks in k-chromatic digraphs **89** (2018), no. 4, 439–456.
https://doi.org/10.1002/jgt.22360.

[4] M. El Joubbeh, *Subdivisions of oriented cycles in Hamiltonian digraphs with small chromatic number*, Discrete Math. **346** (2023), no. 1, Article ID: 113209.
https://doi.org/10.1016/j.disc.2022.113209.

[5] P. Erdös, *Graph theory and probability*, Can. J. Math. **11** (1959), 34–38.
https://doi.org/10.4153/CJM-1959-003-9.

[6] T. Gallai, *Theory of Graphs*, ch. On directed graphs and circuits, pp. 115–118, Academic Press, New York, 1968.

[7] S. Ghazal and S. Tfaili, *On subdivisions of oriented cycles in Hamiltonian digraphs with small chromatic number*, Commun. Comb. Optim., In press.
https://doi.org/10.22049/cco.2024.29517.2036.

[8] R. Kim, S.J. Kim, J. Ma, and B. Park, *Cycles with two blocks in k-chromatic digraphs*, J. Graph Theory **88** (2018), no. 4, 592–605.
https://doi.org/10.1002/jgt.22232.

[9] B. Roy, *Nombre chromatique et plus longs chemins d'un graphe*, Rev. Franç. Inform. Rech. Opér. **1** (1967), no. 5, 129–132.