Communications in Combinatorics and Optimization
Vol. xx, No. x (xxxx), pp. 1-18
https://doi.org/10.22049/cco.2025.30199.2355

CCO
Commun. Comb. Optim.

Research Article

Spectral properties of eccentricity sum matrix of graphs

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Received: 16 December 2024; Accepted: 31 May 2025
Published Online: 10 June 2025

Abstract: The spectral properties of extended adjacency matrices possess high discriminating power and correlate well with various physicochemical properties and biological activities of organic compounds. In the current article, a detailed investigation of one of the extended adjacency matrices called the eccentricity sum matrix is undertaken. The eccentricity sum matrix of a graph G, denoted by $A_{\varepsilon^c}(G)$ is a real symmetric matrix that if $i \neq j$ and $v_iv_j \in E(G)$, then the $(i,j)^{th}-$entry is $e(v_i)+e(v_j)$ and zero otherwise, where $e(v_i)$ is the eccentricity of vertex v_i. The properties like trace, principle minors, and eigenvalues of the eccentricity sum matrix are explored. Moreover, we present some bounds for spectral radius and energy. Also, the energy and spectrum of some classes of graphs like fan graphs, bi-star graphs, etc., and their complements are obtained.

Keywords: eccentricity, spectral radius, cocktail party graph, bi-star graph, crown graph.

AMS Subject classification: 05C10, 05C30 , 05C90

1. Introduction

In this work, by a graph $G(V,E)$, we mean a simple, finite and connected one with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ and edge set $E(G)$. The number of elements in $V(G)$ is the order and that of $E(G)$ is the size of G. If the vertices v_i and v_j are adjacent, we write $v_i \sim v_j$, otherwise $v_i \nsim v_j$. The distance $d(v_i, v_j)$ between v_i and v_j is the length of any shortest path connecting them. The eccentricity of a vertex v_i

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is the distance of vertex v_i from the farthest vertex. That is, $e(v_i) = \max_{v_j \in V(G)} d(v_i, v_j)$. Further, $R(G) = \min_{v_j \in V(G)} e(v_j)$ and $D(G) = \max_{v_j \in V(G)} e(v_j)$ are called the radius and diameter of G, respectively. For more graph theoretic terms and notations, readers can refer to [4].

The adjacency matrix $A(G)$ of the graph G is defined such that its $(i,j)^{th}$- entry for $i \neq j$ is equal to 1 if $v_iv_j \in E(G)$ and 0 otherwise. Suppose $\lambda_1 > \lambda_2 \geq \cdots \geq \lambda_n$ denote the eigenvalues of $A(G)$, then the largest eigenvalue λ_1 is usually referred to as the spectral radius of $A(G)$. Studies on graph spectrum can be found in [6], [21]. The energy of the graph was introduced in 1978 by Ivan Gutman and is defined as $E(G) = \sum_{i=1}^{n} |\lambda_i|$ [11]. Nowadays, research on graph energy is very popular leading to several important conclusions, which can be found in recent articles [8],[7],[5],[20] and the references cited therein.

The concept of topological indices has evolved over several decades through the intersection of graph theory and chemical sciences. Initially, topological indices were developed to describe molecular structures mathematically, but over time, they became essential tools in quantitative structure-property relationships (QSPR) and quantitative structure-activity relationships (QSAR) studies. In 1947, Harold Wiener introduced the Wiener index [19], marking the first topological index ever proposed. After that several topological indices were discovered based on degree, distance, and eccentricity. In fact, in the last few years, degree-based topological molecular descriptors have become increasingly important in chemical, biological, and physical research. Researchers on the verge of refining the existing topological indices replaced degrees with eccentricities in well-known degree-based topological indices such as Zagreb indices, Harmonic index, Atom bond connectivity index, Geometric arithmetic index, Forgotten index, etc. One of the relevant eccentricity-based topological indices is the eccentricity connectivity index denoted by $\varepsilon^c(G)$ and is defined as

$$\varepsilon^c(G) = \sum_{v_i \sim v_j} e(v_i) + e(v_j)$$

There is an extensive body of literature reporting several mathematical properties of the eccentricity connectivity index, which are given in [16], [22], [14], [13], [2] and [10]. Motivated by this, the extended adjacency matrix corresponding to the eccentricity connectivity index was proposed and studied in [18] and [17]. The extended adjacency matrix corresponding to the eccentricity connectivity index is denoted by A_{ε^c} has the entries a_{ij} given by

$$A_{\varepsilon^c} = a_{ij} = \begin{cases} e(v_i) + e(v_j) & if\ v_i \sim v_j \\ 0 & otherwise \end{cases}$$

This matrix is also known as the eccentricity sum matrix. The characteristic polynomial of A_{ε^c} is denoted by $\phi_{A_\varepsilon}(G,\mu) = det(\mu I - A_{\varepsilon^c})$. Suppose $\mu_1 > \mu_2 > \cdots > \mu_k$ are the k distinct eigenvalues of A_{ε^c} with respective multiplicities $c_1, c_2, \ldots, c_k$, then the set of these eigenvalues is known as ε^c- spectrum of G. The ε^c- spectrum of G is often represented in the form of an array given by ε^c- $spec(G) = \begin{pmatrix} \mu_1 & \mu_2 & \mu_3 & \ldots & \mu_k \\ c_1 & c_2 & c_3 & \ldots & c_k \end{pmatrix}$. The largest eigenvalue μ_1 is called ε^c- spectral radius. The absolute sum of all eigenvalues of A_{ε^c} denoted by $EA_{\varepsilon^c}(G) = \sum_{i=1}^{k} |\mu_i|$ is called the ε^c- energy. The specification G can be omitted if the graph under consideration is understood in the above-defined parameters. After the notion of parameters associated with extended adjacency matrices, the ones associated with classical adjacency matrices are often preceded by $A(G)$ or A, with A representing the adjacency matrix. That is, A-eigenvalues, A-spectral radius and A-energy are the ones associated with the adjacency matrices. The abbreviations $tr(A)$, $det(A)$ denote the trace and determinant functions associated with the matrix A. Given below are the necessary existing results required to develop some of our main results.

Lemma 1. [1] *The Cauchy–Schwarz inequality: If* $(a_1, a_2, \ldots, a_p)$ *and* $(b_1, b_2, \ldots, b_p)$ *are real p-vectors then,*

$$\sum_{i=1}^{p} a_i b_i \leq \sum_{i=1}^{p} a_i^2 \sum_{i=1}^{p} b_i^2.$$

Lemma 2. [3] *The Radon's inequality: If* $n \in N, x_k \geq 0, y_k \geq 0, k \in \{1, 2, \ldots, n\}$ *and* $m \geq 0$, *then*

$$\frac{x_1^{m+1}}{y_1^m} + \frac{x_2^{m+1}}{y_2^m} + \frac{x_3^{m+1}}{y_3^m} + \cdots + \frac{x_n^{m+1}}{y_n^m} \geq \frac{(x_1 + x_2 + x_3 + \cdots + x_n)^{m+1}}{(y_1 + y_2 + y_3 + \cdots + y_n)^m}.$$

Lemma 3. [15] *Let* $A = \begin{pmatrix} A_0 & A_1 \\ A_1 & A_0 \end{pmatrix}$ *be a symmetric matrix partitioned into blocks. Then the eigenvalues of* A *are the eigenvalues of the matrices* $A_0 + A_1$ *and* $A_0 - A_1$.

The Kronecker product of a matrix $A = (a_{ij})_{p\times q}$ and $B = (bij)_{r\times s}$ is defined as

$$A \otimes B = \begin{pmatrix} a_{11}B & \ldots & a_{1q}B \\ \vdots & \ddots & \vdots \\ a_{p1}B & \ldots & a_{pq}B \end{pmatrix}.$$

Lemma 4. [12] *Let* A *be a square matrix of order* m *with spectrum* $\sigma(A) = (\mu_i)$, $1 \leq i \leq m$ *and* B *be a square matrix of order* n *with* $\sigma(B) = (\lambda_j)$, $1 \leq j \leq n$. *Then* $\sigma(A \otimes B) = (\mu_i \lambda_j), 1 \leq i \leq m, \ 1 \leq j \leq n$.

A partition of a square matrix A is said to be equitable if all the blocks of the partitioned matrix have constant row sums and each of the diagonal blocks is of square order. A quotient matrix Q of a square matrix A corresponding to an equitable partition is a matrix whose entries are the constant row sums of the corresponding blocks of A. The quotient matrices are useful in finding some eigenvalues of block matrices, which are comparatively larger in size. In the theory of graph spectra, equitable partitions play an important role, mostly because of the following result.

Lemma 5. [9] *Let A be a real symmetric matrix with a quotient matrix Q. Then the characteristic polynomial of Q divides the characteristic polynomial of A.*

This article is organized as follows. Section 2 gives some elementary properties of the eccentricity sum matrix. In Section 3, the upper and lower bounds for ε^c- energy and ε^c- spectral radius in terms of the graph parameters like radius, size, and other well-known topological indices are derived. Finally, we conclude with the last section where the ε^c-energy of some classes of graphs and their complement are derived.

2. Some properties of eccentricity sum matrix

This section includes some elementary properties of the eccentricity sum matrix of G. In the matrix A_{ε^c}, for $i \neq j$, the principal submatrix formed by i^{th} row and j^{th} column of order 2×2 is the zero matrix if $v_i \nsim v_j$ and otherwise it equals

$$\begin{pmatrix} 0 & e(v_i)+e(v_j) \\ e(v_i)+e(v_j) & 0 \end{pmatrix}.$$

Similarly, the principal submatrix formed by any three distinct rows and columns i, j, k of order 3×3 is non-singular if and only if the corresponding vertices v_i, v_j, v_k constitute a triangle in G. In that case, the submatrix is given by

$$\begin{pmatrix} 0 & e(v_i)+e(v_j) & e(v_i)+e(v_k) \\ e(v_i)+e(v_j) & 0 & e(v_j)+e(v_k) \\ e(v_i)+e(v_k) & e(v_k)+e(v_j) & 0 \end{pmatrix}.$$

The following remarks give the bounds for the sum of all principal minors of orders 2 and 3.

Remark 1. Let G be a connected graph whose diameter is D and radius is R. Let S_2 and S_3 represent the sum of all principal minors of A_{ε^c} of order 2 and 3, respectively. Then

$$2mR^2 \leq |S_2| \leq 2mD^2$$

and

$$16TR^3 \leq S_3 \leq 16TD^3$$

where m, T are the number of edges and triangles in G, respectively.

Proof. The principal submatrix formed by i^{th} row and j^{th} column of order 2 is the matrix $\begin{pmatrix} 0 & e(v_i)+e(v_j) \\ e(v_i)+e(v_j) & 0 \end{pmatrix}$ whenever $v_i \sim v_j$. Since $R \leq e(v_i) \leq D$ for all v_i in G, it is true that

$$\sum_{v_i \sim v_j} det \begin{pmatrix} 0 & 2R \\ 2R & 0 \end{pmatrix} \leq S_2 \leq \sum_{v_i \sim v_j} det \begin{pmatrix} 0 & 2D \\ 2D & 0 \end{pmatrix}.$$

As the summation runs over all the edges in G twice, $2mR^2 \leq |S_2| \leq 2mD^2$. Similarly, on considering a principal submatrix of order 3, which is nonzero only when the corresponding three vertices are mutually adjacent, we get

$$\sum_{v_i v_j v_k \in T} det \begin{pmatrix} 0 & 2R & 2R \\ 2R & 0 & 2R \\ 2R & 2R & 0 \end{pmatrix} \leq S_3 \leq \sum_{v_i v_j v_k \in T} det \begin{pmatrix} 0 & 2D & 2D \\ 2D & 0 & 2D \\ 2D & 2D & 0 \end{pmatrix}.$$

This implies $16TR^3 \leq S_3 \leq 16TD^3$. □

Proposition 1. *Let $\phi_{A_\varepsilon}(G,\mu) = c_0\mu^n + c_1\mu^{n-1} + \cdots + c_{n-1}\mu + c_n$ be the characteristic polynomial of A_{ε^c}. Then*

i. $c_0 = 1$

ii. $c_1 = 0$

iii. $c_2 = -P$ *where* $P = \sum_{v_i \sim v_j} [e(v_i) + e(v_j)]^2$

iv. $c_3 = -2 \sum_{v_i v_j v_k \in T} [e(v_i)+e(v_j)]\,[e(v_j)+e(v_k)]\,[e(v_i)+e(v_k)]$, *where T is the set of all triangles in G.*

Proof. By definition $\phi_{A_\varepsilon}(G,\mu) = det(\mu I - A_{\varepsilon^c})$, we have $c_0 = 1$. It is true that $(-1)^i c_i$ is the sum of all principal minors of A_{ε^c} of order i. Given this and since the trace of A_{ε^c} is zero, $c_1 = 0$. Similarly, $(-1)^2 c_2$ gives the sum of all 2×2 principal submatrices, which can be obtained as follows:

$$\begin{aligned}
(-1)^2 c_2 &= \sum_{1 \leq i \leq j \leq n} det \begin{pmatrix} a_{ii} & a_{ij} \\ a_{ji} & a_{jj} \end{pmatrix} \\
&= \sum_{1 \leq i \leq j \leq n} a_{ii}a_{jj} - a_{ij}a_{ji} \\
&= \sum_{1 \leq i \leq j \leq n} a_{ii}a_{jj} - \sum_{1 \leq i \leq j \leq n} a_{ij}^2 \\
&= -\sum_{v_i \sim v_j} a_{ij}^2 \\
&= -\sum_{v_i \sim v_j} [e(v_i) + e(v_j)]^2 .
\end{aligned}$$

Similarly, the expression for c_3 can be obtained. $\square$

Proposition 2. *Let $\mu_1, \mu_2, \ldots, \mu_n$ be the eigenvalues of the eccentricity sum matrix A_{ε^c}. Then $\sum_{i=1}^{n} \mu_i^2 = 2P$, where $P = \sum_{v_i \sim v_j} [e(v_i) + e(v_j)]^2$.*

Proposition 3. *For any connected graph G of order $n \geq 2$, the ε^c-spectrum of G contains at least two distinct eigenvalues.*

Proof. Suppose all the eigenvalues of the matrix A_{ε^c} are identical, that is, $\mu_1 = \mu_2 = \cdots = \mu_n$, then since $\sum_{i=1}^{n} \mu_i = 0$, it is clear that each of μ_i is zero. This implies $A_{\varepsilon^c} = 0$, which is impossible as the eccentricity of every vertex in G is nonzero. $\square$

Proposition 4. *Let G be an eccentricity regular graph in which the eccentricity of every vertex is r. Suppose A is the adjacency matrix of G, then $tr(A_{\varepsilon^c}^2) = 4r^2 tr(A^2)$.*

Proof. Let $\lambda_i (1 \leq i \leq n)$ and $\mu_i (1 \leq i \leq n)$ be the eigenvalues of A and A_{ε^c}, respectively. From the definition of A and A_{ε^c}, $\sum_{i=1}^{n} \lambda_i = \sum_{i=1}^{n} \mu_i = 0$. Also, $tr(A^2) = \sum_{i=1}^{n} \lambda_i^2 = 2m$, where m is the number of edges in G. From Proposition 2,

$$tr(A_{\varepsilon^c}^2) = \sum_{i=1}^{n} \mu_i^2 = 2 \sum_{v_i \sim v_j} (e(v_i) + e(v_j))^2 = 2 \sum_{v_i \sim v_j} (2r)^2 = 8r^2 m = 4r^2 tr(A^2),$$

as desired $\square$

The following two propositions give results similar to classical adjacency matrices. The proofs are direct and are omitted.

Proposition 5. *Let G be a connected graph with diameter D. Suppose A_{ε^c} has k distinct eigenvalues, then $k > D$.*

Proposition 6. *Let G be a connected bipartite graph. If μ is an eigenvalues of A_{ε^c}, then $-\mu$ is also an eigenvalues of A_{ε^c}.*

3. Bounds for ε^c- energy and ε^c- spectral radius

In this section, some bounds for ε^c- energy and ε^c- spectral radius are derived in terms of few graph parameters.

Theorem 1. *Let G be a connected graph with diameter D and μ_1, λ_1 are the ε^c-spectral radius and A-spectral radius of G respectively. Then,*

$$\mu_1 \leq 2D\lambda_1.$$

Equality holds if and only if G is self-centered.

Proof. For any two adjacent vertices v_i and v_j $(1 \leq i, j \leq n)$, we have $e(v_i)+e(v_j) \leq 2D$. Suppose $A(G)$ is the adjacency matrix of G, then $A_{\varepsilon^c}(G) \leq 2DA(G)$. Thus, $\mu_1 \leq 2D\lambda_1$.

□

Theorem 2. *Let G be a connected graph and μ_1 be the ε^c-spectral radius of G. Then*

$$EA_{\varepsilon^c} \geq 2\mu_1.$$

Proof. The ε^c-energy is given by $EA_{\varepsilon^c} = \sum\limits_{i=1}^{n} |\mu_i| = |\mu_1| + \sum\limits_{i=2}^{n} |\mu_i| \geq |\mu_1| + |\sum\limits_{i=2}^{n} \mu_i|$. Since $\sum\limits_{i=2}^{n} \mu_i = 0, \mu_1 = -\sum\limits_{i=2}^{n} \mu_i$ and $|\mu_1| = |\sum\limits_{i=2}^{n} \mu_i|$. This implies, $EA_{\varepsilon^c} \geq |\mu_1| + |\mu_1| = 2|\mu_1|$. □

Theorem 3. *Let G be a graph of order n and $P = \sum\limits_{v_i \sim v_j} [e(v_i) + e(v_j)]^2$. Then*

$$EA_{\varepsilon^c} \leq \sqrt{2nP}.$$

Proof. This follows from the Cauchy-Schwarz inequality given in Lemma 1. □

Theorem 4. *For any connected graph G of order n,*

$$\sqrt{tr(A_{\varepsilon^c})^2} \leq EA_{\varepsilon^c} \leq \sqrt{n \times tr(A_{\varepsilon^c})^2}.$$

Proof. The variance of the number $|\mu_i|$, $i = 1, 2, \ldots, n$ is given by $\frac{1}{n}\sum\limits_{i=1}^{n} |\mu_i|^2 - \left(\frac{1}{n}\sum\limits_{i=1}^{n} |\mu_i|\right)^2 \geq 0$. Since $\sum\limits_{i=1}^{n} \mu_i^2 = tr(A_{\varepsilon^c})^2$, we have $\sum\limits_{i=1}^{n} \mu_i \leq \sqrt{n \times tr(A_{\varepsilon^c})^2}$.
By Radon's inequality 2,

$$\sum_{i=1}^{n} |\mu_i| = \sum_{i=1}^{n} \frac{|\mu_i|^2}{|\mu_i|} \geq \frac{\sum\limits_{i=1}^{n} |\mu_i|^2}{\sum\limits_{i=1}^{n} |\mu_i|}$$

$$\Longrightarrow \sum_{i=1}^{n} |\mu_i| \geq \sqrt{\sum_{i=1}^{n} |\mu_i|^2} = \sqrt{tr(A_{\varepsilon^c})^2}.$$

□

Corollary 1. *Let G be an eccentricity regular graph in which the eccentricity of every vertex is r. Then*

$$2r\sqrt{2m} \leq EA_{\varepsilon^c} \leq 2r\sqrt{2mn}$$

where n, m are the order and size of G, respectively.

Proof. By Theorem 4, we have

$$EA_{\varepsilon^c} \leq \sqrt{n \times tr(A_{\varepsilon^c})^2} = \sqrt{2n\sum_{i=1}^{n}[e(v_i)+e(v_j)]^2} = \sqrt{2n\sum_{i=1}^{n}(r+r)^2} = 2r\sqrt{2mn}.$$

Similarly, $EA_{\varepsilon^c} \geq \sqrt{tr(A_{\varepsilon^c})^2} = \sqrt{2\sum_{i=1}^{n}[e(v_i)+e(v_j)]^2} = \sqrt{2\sum_{i=1}^{n}(r+r)^2} = 2r\sqrt{2m}.$

□

Theorem 5. *Let $F(G) = \sum_{v_i \sim v_j} e(v_i)^2 + e(v_j)^2$ and $M_1(G) = \sum_{v_i \sim v_j} e(v_i)e(v_j)$ be the forgotten index and the first Zagreb index associated with a graph G. Then*

$$EA_{\varepsilon^c} \geq \sqrt{2F(G) + 4M_1(G)}.$$

Proof. By Theorem 4, we have

$$\begin{aligned}
EA_{\varepsilon^c} &\geq \sqrt{tr(A_{\varepsilon^c})^2} \\
&= \sqrt{2\sum_{v_i \sim v_j}\Big[e(v_i)+e(v_j)\Big]^2} \\
&= \sqrt{2\sum_{v_i \sim v_j}\Big[e(v_i)^2+e(v_j)^2+2e(v_i)e(v_j)\Big]} \\
&= \sqrt{2\sum_{v_i \sim v_j} e(v_i)^2+e(v_j)^2 + 4\sum_{v_i \sim v_j} e(v_i)e(v_j)} \\
&= \sqrt{2F(G)+4M_1(G)}.
\end{aligned}$$

□

The sharpness of the above bounds can be observed by comparing them with the actual values for the following graph.

Example 1. Consider the graph G given in Figure 1.

The various bounds computed in this section and the actual values of $\mu_1(G), EA_{\varepsilon^c}(G)$ are listed in the following table.

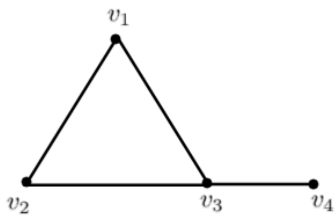

Figure 1. **The graph G with $\mu_1(G) = 7.0920, \lambda_1(G) = 2.1705$ and $EA_{\varepsilon^c}(G) = 16.5570$**

Hypothesis	Bound as per the hypothesis	Comparison with the actual value
Theorem 1	$\mu_1 \le 8.68$	$7.0920 \ge 8.68$
Theorem 2	$EA_{\varepsilon^c} \ge 14.8184$	$16.5570 \ge 14.8184$
Theorem 3	$EA_{\varepsilon^c} \le 18.5472$	$16.5570 \le 18.5472$
Theorem 4	$9.2736 \le EA_{\varepsilon^c} \le 18.547$	$9.2736 \le 16.5570 \le 18.547$
Theorem 5	$EA_{\varepsilon^c} \ge 9.2736$	$16.5570 \ge 9.2736$

Clearly, all of the above bounds are well-defined and notably precise.

4. ε^c- spectrum of some classes of graphs and their complements

This section gives the expression ε^c- energy of some classes of graphs like wheel graphs, bi-star graphs, cocktail party graphs, etc, and their complements if they are connected.

Theorem 6. *For a wheel graph $W_{1,n}$ on $n+1$ vertices,*

$$\varepsilon^c\text{-}spec(W_{1,n}) = \begin{pmatrix} 4+\sqrt{16+9n} & 4-\sqrt{16+9n} & \begin{matrix} 8cos\left(\frac{2\pi i}{n}\right) \\ i=1,\dots,(n-1) \end{matrix} \\ 1 & 1 & 1 \end{pmatrix}.$$

Proof. The eccentricity sum matrix of $W_{1,n}$ is a block matrix given by

$$A_{\varepsilon^c}(W_{1,n}) = \begin{pmatrix} 0_{1\times 1} & \mathbf{3}_{1\times n} \\ \mathbf{3}_{n\times 1} & 4A(C_n)_{n\times n} \end{pmatrix},$$

where $A(C_n)$ is the adjacency matrix of a cycle graph on n vertices. The block $\mathbf{3}$ above represents a matrix in which every entry is 3. Let $V = \begin{pmatrix} X_{1\times 1} & Y_{n\times 1} \end{pmatrix}^T$ be the nonzero eigenvector of $A_{\varepsilon^c}(W_{1,n})$ corresponding to the eigenvalue μ. That is,

$$A_{\varepsilon^c} V = \mu V \tag{4.1}$$

We obtain a pair of a nonzero vector V and a scalar μ satisfying Equation 4.1. Suppose $X = 0$ and $Y = \begin{pmatrix} y_1 & y_2 & \dots & y_n \end{pmatrix}^T$, then from Equation 4.1, we get

$$3(y_1 + y_2 + \cdots + y_n) = 0 \tag{4.2}$$

$$4A(C_n)Y = \mu Y \tag{4.3}$$

From Equation 4.2, the components in Y sum up to 0. Further, from Equation 4.3, suppose σ is an eigenvalue of $A(C_n)$ with an eigenvector Y whose components sum up to zero, then $\mu = 4\sigma$ is an eigenvalue of A_{ε^c}. We know that for all eigenvalues σ of $A(C_n)$, the components in the eigenvectors sum up to 0, except for $\sigma = 2$ (whose eigenvector is a unity). That is, if $\sigma \neq 2$ is an eigenvalue of $A(C_n)$ with the eigenvector Y, then $\mu = 4\sigma$ is an eigenvalue of A_{ε^c} with eigenvector $V = \begin{pmatrix} 0_{1\times 1} & Y_{n\times 1} \end{pmatrix}^T$. Since $A(C_n)$ have eigenvalues $2cos\left(\frac{2\pi i}{n}\right); j = 0, 1, \dots, (n-1)$, the $(n-1)$ eigenvalues of $A_{\varepsilon^c}(W_{1,n})$ are $8cos\left(\frac{2\pi i}{n}\right); j = 1, \dots, (n-1)$. The remaining two eigenvalues can be obtained from the quotient matrix. The quotient matrix Q of $A_{\varepsilon^c}(W_{1,n})$ is given by $Q = \begin{pmatrix} 0 & 3n \\ 3 & 8 \end{pmatrix}$, the eigenvalues of which are $4 \pm \sqrt{16 + 9n}$. By Lemma 5, $4 \pm \sqrt{16 + 9n}$ are eigenvalues of A_{ε^c}. □

Corollary 2. *Let $W_{1,n}$ be a wheel graph. Then $0 \in \varepsilon^c$-spec$(W_{1,n})$ if and only if $\frac{n}{2}$ is even.*

Proof. By Theorem 6, for each of the integers $i = 1, 2, \dots, (n-1)$, $8cos\left(\frac{2\pi i}{n}\right)$ are the eigenvalues of $W_{1,n}$. When $\frac{n}{2}$ is even, $\frac{n}{4}$ is an integer such that $\frac{n}{4} < (n-1)$. Clearly, when $i = \frac{n}{4}$, we get the eigenvalues 0. Conversely, suppose $0 \in \varepsilon^c$-spec$(W_{1,n})$, then it follows directly that, it is contributed from $8cos\left(\frac{2\pi i}{n}\right)$ for some integer $i \leq (n-1)$ and $\frac{n}{2}$ is even. This implies $\frac{n}{2}$ is even. □

A friendship graph is a graph in which every two distinct vertices have exactly one common adjacent vertex. A friendship graph, often denoted by F_n has $2n+1$ vertices, $2n$ of them being of degree two and the remaining one being of degree $2n$.

Theorem 7. *For a friendship graph F_n on $2n+1$ vertices, $EA_{\varepsilon^c}(F_n) = 8n$.*

Proof. The eccentricity sum matrix $A_{\varepsilon^c}(F_n)$ can be represented as a block matrix

$$A_{\varepsilon^c}(F_n) = \begin{pmatrix} 0_{1\times 1} & \mathbf{3}_{1\times 2n} \\ \mathbf{3}_{2n\times 1} & B_{2n\times 2n} \end{pmatrix},$$

where B is a block diagonal matrix $B = diag\left[4(J-I), 4(J-I), \dots, 4(J-I)\right]$, with $J - I = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Let the rows and columns of $A_{\varepsilon^c}(F_n)$ (indexed by the vertices

$v_1, v_2, \ldots, v_{2n+1}$) be named as $R_1, R_2, \ldots, R_{2n+1}$ and $C_1, C_2, \ldots, C_{2n+1}$ as shown below.

$$A_{\varepsilon^c}(F_n) = \begin{array}{c} \\ R_1 \\ R_2 \\ R_3 \\ R_4 \\ \vdots \\ R_{2n} \\ R_{2n+1} \end{array} \begin{array}{c} \begin{array}{cccccccc} C_1 & C_2 & C_3 & C_4 & C_5 & \ldots & C_{2n} & C_{2n+1} \end{array} \\ \begin{pmatrix} 0 & 3 & 3 & 3 & 3 & \ldots & 3 & 3 \\ 3 & 0 & 4 & 0 & 0 & \ldots & 0 & 0 \\ 3 & 4 & 0 & 0 & 0 & \ldots & 0 & 0 \\ 3 & 0 & 0 & 0 & 4 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 3 & 0 & 0 & 0 & 0 & \ldots & 0 & 4 \\ 3 & 0 & 0 & 0 & 0 & \ldots & 4 & 0 \end{pmatrix} \end{array}$$

Consider the characteristic polynomial, $det\,(A_{\varepsilon^c}(F_n) - \mu I)$.
We use the technique of elementary row and column operations. Let $R_j = R_j \pm R_k$ represent the operation, in which the entries in R_j are obtained by adding to /subtracting from each entry of R_j the corresponding entries from the row R_k. similarly, $C_j = C_j \pm C_k$ results in a new column C_j whose entries are obtained by adding/subtracting the corresponding entries from the column C_k.
For each $1 \leq i \leq n$, on performing the row operations $R_{2i} = R_{2i} - R_{2i+1}$, we get

$$A_{\varepsilon^c}(F_n) = \begin{array}{c} \\ R_1 \\ R_2 \\ R_3 \\ R_4 \\ \vdots \\ R_{2n} \\ R_{2n+1} \end{array} \begin{array}{c} \begin{array}{cccccccc} C_1 & C_2 & C_3 & C_4 & C_5 & \ldots & C_{2n} & C_{2n+1} \end{array} \\ \begin{pmatrix} -\mu & 3 & 3 & 0 & 3 & \ldots & 3 & 3 \\ 0 & \mu-4 & 4+\mu & 0 & 0 & \ldots & 0 & 0 \\ 0 & 0 & -\mu & 0 & 0 & \ldots & 0 & 0 \\ 0 & 0 & 0 & -\mu-4 & 4+\mu & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 3 & 0 & 0 & 0 & 0 & \ldots & 0 & 4 \\ 3 & 0 & 0 & -4+\mu & 0 & \ldots & 4 & -\mu \end{pmatrix} \end{array}$$

This can be written as $A_{\varepsilon^c}(F_n) - \mu I = (4+\mu)^n det(B)$, where B is a matrix of the form

$$B = \begin{array}{c} \\ R_1 \\ R_2 \\ R_3 \\ R_4 \\ \vdots \\ R_{2n} \\ R_{2n+1} \end{array} \begin{array}{c} \begin{array}{cccccccc} C_1 & C_2 & C_3 & C_4 & C_5 & \dots & C_{2n} & C_{2n+1} \end{array} \\ \begin{pmatrix} -\mu & 3 & 3 & 3 & 3 & \dots & 3 & 3 \\ 0 & -1 & 1 & 0 & 0 & \dots & 0 & 0 \\ 3 & 4 & -\mu & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \dots & -1 & 1 \\ 3 & 0 & 0 & 0 & 0 & \dots & 4 & -\mu \end{pmatrix} \end{array}$$

Now, on performing the column operation $C_{2j} = C_{2j} + C_{2j+1}$ for each $1 \leq j \leq n-1$ and then $C_{2j} = C_{2j} - C_{2j+2}$ sequentially, we get

$$A_{\varepsilon^c}(F_n) - \mu I = (4+\mu)^n(\mu-4)^{n-1}det(C),$$

where C is the reduced resultant matrix given by

$$C = \begin{array}{c} \\ R_1 \\ R_2 \\ R_3 \\ R_4 \\ \vdots \\ R_{2n} \\ R_{2n+1} \end{array} \begin{array}{c} \begin{array}{cccccccc} C_1 & C_2 & C_3 & C_4 & C_5 & \dots & C_{2n} & C_{2n+1} \end{array} \\ \begin{pmatrix} -\mu & 0 & 3 & 0 & 3 & \dots & 3 & 3 \\ 0 & 0 & 1 & 0 & 0 & \dots & 0 & 3 \\ 3 & 1 & -\mu & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \dots & -1 & 1 \\ 3 & 0 & 0 & -1 & 0 & \dots & 4 & -\mu \end{pmatrix} \end{array}$$

From this, one can note that ± 4 are the eigenvalues with respective multiplicities $(n-1)$ and n. For the remaining two eigenvalues, we make use of the quotient matrix $Q = \begin{pmatrix} 0 & 6n \\ 3 & 4 \end{pmatrix}$, eigenvalues of which can be easily computed as $2 \pm \sqrt{4+18n}$. By Lemma 5,$2 \pm \sqrt{4+18n} \in \varepsilon^c$- spectrum. Thus, $EA_{\varepsilon^c}(F_n) = 8n$. □

A crown graph $H_{n,n}$ is a graph obtained from the complete bipartite graph $K_{n,n}$ by removing a perfect matching.

Theorem 8. *For crown graph $H_{n,n}$ on $2n$ vertices, $EA_{\varepsilon^c}(H_{n,n}) = 24(n-1)$.*

Proof. The eccentricity sum matrix of the crown graph is $A_{\varepsilon^c}(H_{n,n}) = 6A(H_{n,n})$, where $A(H_{n,n})$ is the adjacency matrix $H(n,n)$ given by

$$A(H_{n,n}) = \begin{pmatrix} 0_{n\times n} & (J-I)_{n\times n} \\ (J-I)_{n\times n} & 0_{n\times n} \end{pmatrix}.$$

By Lemma 3, A-spec$(H_{n,n}) = spec\{(J-I), -(J-I)\}$. Since the eigenvalues of $J-I$ of order n are $n-1, -1$ with respective multiplicities 1 and $2n-1$, the A-spectrum contains $\pm(n-1)$ and ± 1 with respective multiplicities 1 and $(n-1)$. Thus ε^c- spec$(H_{n,n})$ contains $\pm 6(n-1)$ and ± 6 with repective multiplicities $1, 1, (n-1)$ and $(n-1)$ and $EA_{\varepsilon^c}(H_{n,n}) = 24(n-1)$. □

Theorem 9. *Let $\overline{H_{n,n}}$ be the complement of crown graph. Then $EA_{\varepsilon^c}(\overline{H_{n,n}}) = 16(n-1)$.*

Proof. The extended adjacency matrix of the complement of the crown graph is

$$A_{\varepsilon^c}(\overline{H_{n,n}}) = \begin{pmatrix} 4(J-I)_{n\times n} & 4I_{n\times n} \\ 4I_{n\times n} & 4(J-I)_{n\times n} \end{pmatrix}.$$

The eigenvalues of $4(J-I) + 4I = 4J$ are $4n, 0$ with respective multiplicities 1 and $n-1$. Similarly, $4(J-I) - 4I = 4J - 8I$ has the eigenvalues $4n-8, -8$ with respective multiplicities 1 and $(n-1)$. Thus by using Lemma 3, $EA_{\varepsilon^c}(\overline{H_{n,n}}) = 16(n-1)$. □

A bi-star graph $B(p,q)$ is a tree on $p+q$ vertices obtained by joining two star graphs $K_{1,p-1}$ and $K_{1,q-1}$ by an edge.

Theorem 10. *For a bi-star graph $B_{n,n}$ on $2n$ vertices, $EA_{\varepsilon^c}(B(n,n)) = 4\sqrt{25n-21}$.*

Proof. On labeling the vertices of $B(n,n)$ appropriately (where the rows in each of the blocks P, Q corresponds to the vertices inducing the stars $K_{1,n-1}$, the first row P and Q respectively corresponding to the central vertices of star graphs) can be written as,

$$A_{\varepsilon^c}(B_{n,n}) = \begin{pmatrix} P_{n\times n} & Q_{n\times n} \\ Q_{n\times n} & P_{n\times n} \end{pmatrix},$$

where $P = \begin{pmatrix} 0 & 5 & 5 & \dots & 5 \\ 5 & 0 & 0 & \dots & 0 \\ 5 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 5 & 0 & 0 & \dots & 0 \end{pmatrix}$ and $Q = \begin{pmatrix} 4 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix}$.

Using Lemma 3, the eigenvalues of A_{ε^c} consists of the eigenvalues of $P+Q$ and $P-Q$.

Consider $P + Q = \begin{pmatrix} 4 & 5 & 5 & \dots & 5 \\ 5 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 5 & 0 & 0 & \dots & 0 \end{pmatrix}$

Let $X = \left(x_1, x_2, \dots, x_n\right)^T$ be the nonzero eigenvector of $P + Q$ corresponding to the eigenvalue μ. Since $(P + Q)X = \mu X$, we have

$$4x_1 + 5\sum_{i=2}^{n} x_i = \mu x_1$$
$$5x_1 = \mu x_j \ for \ j = 2, 3, \dots, n$$

On solving this system of equations, for each of $j = 2, 3, \dots, n$, we get

$$x_j = \frac{5x_1}{\mu}$$
$$\left[\mu^2 - 4\mu + 25(n-1)\right] x_1 = 0$$

Since $x_1 \neq 0$, it follows that $\mu^2 - 4\mu + 25(n-1) = 0$, which results in $\mu = 2 \pm \sqrt{25n - 21}$. The other eigenvalues of $P + Q$ are 0, as it has the rank 2.

Similarly, $P - Q = \begin{pmatrix} -4 & 5 & 5 & \dots & 5 \\ 5 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 5 & 0 & 0 & \dots & 0 \end{pmatrix}$.

One can show that $P - Q$ has the eigenvalues $-2 \pm \sqrt{25n - 21}$ (each with multiplicity 1) and 0 (with multiplicity $n - 2$). Since ε^c-spec$(B(n, n))$ contains the eigenvalues of $P + Q$ and $P - Q$, $EA_{\varepsilon^c}(B_{n,n}) = 4\sqrt{25n - 21}$.

□

Theorem 11. *Let $\overline{B_{n,n}}$ be the complement of the bi-star graph. Then*

$$\varepsilon^c - spec(\overline{B_{n,n}}) = \begin{pmatrix} -4 & (4n-6) \pm \sqrt{k_1} & -2 \pm 2\sqrt{k_2} \\ 2n-4 & 1 & 1 \end{pmatrix},$$

where $k_1 = 16n^2 - 23n + 11$ and $k_2 = 25n - 21$.

Proof. The eccentricity sum matrix $A_{\varepsilon^c}(\overline{B_{n,n}})$ can be represented as a block matrix

$$A_{\varepsilon^c}(\overline{B_{n,n}}) = \begin{pmatrix} P_{n\times n} & Q_{n\times n} \\ Q_{n\times n} & P_{n\times n} \end{pmatrix},$$

where $P = \begin{pmatrix} 0_{1\times 1} & 5_{1\times n-1} \\ 5_{n-1\times 1} & 4(J-I)_{n-1\times n-1} \end{pmatrix}$ and $Q = \begin{pmatrix} 0_{1\times 1} & 0_{1\times n-1} \\ 0_{n-1\times 1} & 4J_{n-1\times n-1} \end{pmatrix}$.
Note that $P+Q = \begin{pmatrix} 0 & 5 \\ 5 & 8J-4I \end{pmatrix}$ and $P-Q = \begin{pmatrix} 0 & 5 \\ 5 & -4I \end{pmatrix}$.
Let the rows and columns of $P+Q$ (indexed by the vertices $v_1, v_2, \ldots, v_{2n}$) be named as $R_1, R_2, \ldots, R_{2n}$ and $C_1, C_2, \ldots, C_{2n}$, respectively.
Consider the characteristic polynomial of $P+Q$, $det\,((P+Q)-\mu I)$.
For each $2 \le i \le 2n-1$, on performing the row operations $R_i = R_i - R_{i+1}$, we get

$$(P+Q)-\mu I = (\mu+4)^{n-2} det(B),$$

where B is a matrix of the form

$$B = \begin{array}{c} \\ R_1 \\ R_2 \\ R_3 \\ R_4 \\ \vdots \\ R_{2n-1} \\ R_{2n} \end{array} \begin{array}{c} \begin{array}{cccccccc} C_1 & C_2 & C_3 & C_4 & C_5 & \ldots & C_{2n-1} & C_{2n} \end{array} \\ \begin{pmatrix} -\mu & 5 & 5 & 5 & 5 & \ldots & 5 & 5 \\ 0 & 1 & -1 & 0 & 0 & \ldots & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & \ldots & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & \ldots & 0 & 0 \\ \vdots & & & & & & & \\ 0 & 0 & 0 & 0 & 0 & \ldots & -1 & 1 \\ 5 & 8 & 8 & 8 & 8 & \ldots & 8 & 4-\mu \end{pmatrix} \end{array}$$

From this, -4 is an eigenvalue of $(P+Q)$ with multiplicity $n-2$. Similarly, one can show that -4 is an eigenvalue of $(P-Q)$ with the same multiplicity. The respective quotient matrices Q_1 and Q_2 of $P+Q$ and $P-Q$ are given by is given by $Q_1 = \begin{pmatrix} 0 & 5n-5 \\ 5 & 8n-12 \end{pmatrix}$ and $Q_2 = \begin{pmatrix} 0 & 5n-5 \\ 5 & -4 \end{pmatrix}$. The eiegenvalues of Q_1 and Q_2 are respectively, $(4n-6) \pm \sqrt{k_1}$ and $-2 \pm 2\sqrt{k_2}$, where $k_1 = 16n^2 - 23n + 11$ and $k_2 = 25n - 21$. Thus, by Lemma 3 and Lemma 5, the spectrum follows. □

Two edges in a graph are said to be independent if they do not share any vertex in common. A set of edges is said to be independent if every pair of edges in it are independent. The cocktail party graph of order $2n$, also called the hyperoctahedral graph is the graph obtained from a complete graph K_{2n} by removing n independent edges.

Theorem 12. *For a cocktail party graph $K_{n\times 2}$ on $2n$ vertices, $EA_{\varepsilon^c}(K_{n\times 2}) = 16(n-1)$.*

Proof. The extended adjacency matrix $A_{\varepsilon^c}(K_{n\times n})$ can be written as

$$A_{\varepsilon^c}(K_{n\times 2}) = (J - I)_{n\times n} \otimes B,$$

where $B = \begin{pmatrix} 4 & 4 \\ 4 & 4 \end{pmatrix}$.

Using Lemma 4, the eigenvalues of $A_{\varepsilon^c}(K_{n\times 2})$ are $\lambda_i\omega_j$ for each eigenvalues λ_i of $J - I$, and ω_j of B. One can easily compute the eigenvalues of $(J - I)$ as $(n - 1), -1$ (with respective multiplicities $1, (n - 1)$) and that of B as $8, 0$ (each with multiplicity 1). Thus $EA_{\varepsilon^c}(K_{n\times 2}) = 16(n - 1)$. □

The eccentricity sum matrix of the complement of the first two classes of graphs discussed above is not undertaken as they are disconnected.

5. Conclusion

The eccentricity sum matrix is a type of weighted adjacency matrix associated with graphs, where each weight is in terms of vertex eccentricities. This matrix a type of eccentricity based extended adjacency matrix. The scope of the eigenvalues and energy associated with extended adjacency matrices has been broadened in recent years, yielding intriguing insights in the field of molecular chemistry. That is, the energy and spectral radius of the extended adjacency matrix of molecular compounds have been discovered to have strong correlations with a number of biological activities and physicochemical characteristics.

This article effectively investigates the spectral properties of one of the extended adjacency matrices defined based on vertex eccentricities. New bounds for spectral radius and energy are derived. We obtained the spectrum and energy of certain graph classes and their complements wherever they are connected. The eccentricity sum matrix of the complement of the first two classes of graphs discussed above is not undertaken as they are disconnected.

Conflict of Interest: The authors declare that they have no conflict of interest.

Data Availability: Data sharing is not applicable to this article as no data sets were generated or analyzed during the current study.

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