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Research Article

Total double Roman domination stability in graphs

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Abstract: Let G be a graph with vertex set $V(G)$. A total double Roman dominating function (TDRD-function) on a graph G with no isolated vertices is a function $f : V(G) \to \{0, 1, 2, 3\}$ satisfying the conditions: (i) if $f(v) = 0$, then the vertex v must be adjacent to at least two vertices assigned 2 or one vertex assigned 3 under f, and if $f(v) = 1$, then the vertex v must be adjacent to at least one vertex assigned 2 or 3 and (ii) the subgraph of G induced by the set $\{v \in V(G) \mid f(v) \neq 0\}$ has no isolated vertices. The weight of a TDRD-function f is the sum of its function values over all vertices, and the minimum weight of a TDRD-function on G is the total double Roman domination number, $\gamma_{tdR}(G)$. The γ_{tdR}-stability (γ^-_{tdR}-stability, γ^+_{tdR}-stability) of G, denoted by $\mathrm{st}_{\gamma_{tdR}}(G)$ (resp. $\mathrm{st}^-_{\gamma_{tdR}}(G)$, $\mathrm{st}^+_{\gamma_{tdR}}(G)$), is defined as the minimum size of a set of vertices whose removal changes (resp. decreases, increases) the total double Roman domination number. In this paper, we first determine the exact values of the γ_{tdR}-stability of some special classes of graphs, and then we present some bounds on $\mathrm{st}_{\gamma_{tdR}}(G)$, $\mathrm{st}^-_{\gamma_{tdR}}(G)$ and $\mathrm{st}^+_{\gamma_{tdR}}(G)$). In particular, for a graph G with maximum degree $\Delta \geq 3$, we show that $\mathrm{st}^-_{\gamma_{tdR}}(G) \leq \Delta - 1$.

Keywords: total double Roman domination, total double Roman domination stability.

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1. Introduction

In this paper, G is a simple graph with vertex set $V(G)$ and edge set $E(G)$. For $v \in V(G)$, the *open neighborhood* of v is the set $N(v) = N_G(v) = \{u \in V(G) : uv \in E(G)\}$ and its *closed neighborhood* is the set $N[v] = N_G[v] = N(v) \cup \{v\}$. We denote the *degree* of a vertex v in G by $d(v) = d_G(v) = |N(v)|$. The minimum and maximum degrees among all vertices of G are denoted by $\delta = \delta(G)$ and $\Delta = \Delta(G)$, respectively. For $u, v \in V(G)$, the length of a shortest (u, v)-path in G is the *distance* $d(u, v)$ between u and v. The *diameter* $\text{diam}(G)$ of G is the maximum distance among all pairs of vertices. A shortest path whose length equals $\text{diam}(G)$ is called a *diametral path* of G. Let S be a set of vertices, and let $v \in S$. The *private neighbor set of* v, with respect to S, is defined by $\text{pn}[v, S] = N[v] \setminus N[S \setminus \{v\}]$.
A vertex of degree one is called a *leaf* and a vertex adjacent to (exactly) one leaf is called a (*weak*) *support vertex*. For $r, t \geq 1$, a *double star* $S_{r,t}$ is a tree with exactly two adjacent vertices that are not leaves, one of which is adjacent to r leaves and the other colorred one is adjacent to t leaves. As usual, the path, cycle and complete graph with n vertices are denoted by P_n, C_n and K_n, respectively. We denote by $K_{r,s}$ the complete bipartite graph having partite sets of cardinality r and s.
Inspired by the strategies for defending the Roman Empire presented in ReVelle and Rosing [14] and Stewart [16], Cockayne et al. [9] introduced in 2004 the concept of Roman domination. But since its introduction, Roman domination has been intensively studied which led to the emergence of several variants. There are currently over 250 papers published on topics related to this concept. For more details we refer the reader to the book chapters [4, 8] and surveys [5–7].
In 2020, Hao, Volkmann and Mojdeh [11] introduced a new variant which they called total double Roman domination defined as follows. A total double Roman dominating function (TDRD-function) on a graph G with no isolated vertices is a function $f : V(G) \to \{0, 1, 2, 3\}$ satisfying the conditions: (i) if $f(v) = 0$, then the vertex v must be adjacent to at least two vertices assigned 2 or one vertex assigned 3 under f, and if $f(v) = 1$, then the vertex v must be adjacent to at least one vertex assigned 2 or 3 and (ii) the subgraph of G induced by the set $\{v \in V(G) \mid f(v) \neq 0\}$ has no isolated vertices. The weight of a TDRD-function f is the sum of its function values over all vertices, and the minimum weight of a TDRD-function on G is the total double Roman domination number (abbreviated TDRD-number), denoted by $\gamma_{tdR}(G)$. For any TDRD-function f of G, let $V_i = \{v \in V \mid f(v_i) = i\}$, where $i \in \{0, 1, 2, 3\}$. Since these four sets determine f, we can write $f = (V_0, V_1, V_2, V_3)$. Also, a $\gamma_{tdR}(G)$-function is a TDRD-function of G with weight $\gamma_{tdR}(G)$. For more details on this parameter see [2, 12, 17].
In this paper, we are interested in studying the behavior of the TDRD-number with respect to the deletion of a set of vertices. We therefore define the total double Roman domination stability (TDRD-stability, or just γ_{tdR}-stability) of a graph G as being the minimum cardinality of a set of vertices whose removal changes the TDRD-number of G. On the basis of this definition, we can also define the γ^-_{tdR}-stability (resp. $\gamma^+_{tdR}(G)$-stability) to be the minimum cardinality of a set of vertices whose removal decreases

(resp. increases) the TDRD-number of G. By following the standard notations, let $\mathrm{st}_{\gamma_{tdR}}(G)$, $\mathrm{st}^-_{\gamma_{tdR}}(G)$ and $\mathrm{st}^+_{\gamma_{tdR}}(G)$ denote the γ_{tdR}-stability, γ^-_{tdR}-stability and γ^+_{tdR}-stability, respectively. Clearly, $\mathrm{st}_{\gamma_{tdR}}(G) = \min\{\mathrm{st}^-_{\gamma_{tdR}}(G), \mathrm{st}^+_{\gamma_{tdR}}(G)\}$ holds for every graph G. Furthermore, it is worth noting that it is possible that the removal of any set of vertices from a graph G does not increase $\gamma_{tdR}(G)$, and for such cases, we consider that $\mathrm{st}^+_{\gamma_{tdR}}(G) = \infty$. The concept of stability was first studied in 1983 by Bauer et al. [3] for the domination number, and was subsequently considered for other domination parameters, including the domination number [13], the Roman domination number [10], outer independent double Roman domination number [15] and very recently the restrained domination number [1].

In this paper, we first determine the exact values of the γ_{tdR}-stability of some special classes of graphs, and then we present some bounds on $\mathrm{st}_{\gamma_{tdR}}(G)$, $\mathrm{st}^-_{\gamma_{tdR}}(G)$ and $\mathrm{st}^+_{\gamma_{tdR}}(G)$). In particular, for a graph G with maximum degree $\Delta \geq 3$, we show that $\mathrm{st}^-_{\gamma_{tdR}}(G) \leq \Delta - 1$.

2. Exact values

In this section, we determine the TDRD-stability for some classes of graphs and present various bounds for this parameters. The following two results established in [11] will be useful.

Proposition 1 ([11]). *For $n \geq 2$,* $\gamma_{tdR}(P_n) = \begin{cases} 6 & \text{if } n = 4, \\ \lceil \frac{6n}{5} \rceil & \text{otherwise.} \end{cases}$

Proposition 2 ([11]). *For $n \geq 3$,* $\gamma_{tdR}(C_n) = \lceil \frac{6n}{5} \rceil$.

We first determine the γ^-_{tdR}-stability for paths.

Proposition 3. *For $n \geq 3$,* $\mathrm{st}^-_{\gamma_{tdR}}(P_n) = \begin{cases} 2 & \text{if } n = 5 \\ 1 & \text{otherwise.} \end{cases}$

Proof. Let $P_n = v_1 v_2 \ldots v_n$ be a path on n vertices. If $n \geq 6$, then it follows from Proposition 1 that $\gamma_{tdR}(P_n) = \lceil \frac{6n}{5} \rceil$, and moreover, $\gamma_{tdR}(P_n - v_n) = \gamma_{tdR}(P_{n-1}) = \lceil \frac{6(n-1)}{5} \rceil < \lceil \frac{6n}{5} \rceil$. Hence, $\mathrm{st}^-_{\gamma_{tdR}}(P_n) = 1$. Assume next that $n = 5$. We show that $\mathrm{st}^-_{\gamma_{tdR}}(P_5) \geq 2$. Let x be a vertex of P_5. If $P_5 - x = P_4$, then from Proposition 1 we have $\gamma_{tdR}(P_4) = 6 = \gamma_{tdR}(P_5)$. Henceforth, by definition we assume that $P_5 - x$ consists of two disjoint paths P_2. It follows from Proposition 1 that $2\gamma_{tdR}(P_2) = 6 = \gamma_{tdR}(P_5)$. It means that $st^-_{\gamma_{tdR}}(P_n) \geq 2$. On the other hand, by Proposition 1 we have $\gamma_{tdR}(P_5 - \{v_1, v_2\}) = 4 < 6 = \gamma_{tdR}(P_5)$, which yields $st^-_{\gamma_{tdR}}(P_5) \leq 2$. Therefore, $st^-_{\gamma_{tdR}}(P_5) = 2$. Finally Proposition 1 easily implies $\mathrm{st}^-_{\gamma_{tdR}}(P_n) = 1$ for $n = 3, 4$. □

Proposition 4. *For* $n \geq 3$, $\mathrm{st}^{+}_{\gamma_{tdR}}(P_n) = \begin{cases} k & \text{if } n = 5k+4 \\ \infty & \text{if } n \neq 5k+4. \end{cases}$

Proof. Let $P_n = v_1 v_2 \dots v_n$ and let $S = \{v_{i_1}, v_{i_2}, \dots, v_{i_s}\}$, where $i_1 < i_2 < \cdots < i_s$, be the smallest set of vertices whose removal from P_n increases the TDRD-number. Suppose that $P_{n_1}, P_{n_2}, \dots, P_{n_t}$ are the components of $P_n - S$. By definition, each component must have at least two vertices. Let t_1 be the number of non-P_4 components and let t_2 be the number of P_4 components of $P_n - S$. We have that

$$\begin{aligned}\sum_{i=1}^{t} \gamma_{tdR}(P_{n_i}) &\leq \sum_{n_i \neq 4} \frac{6n_i+4}{5} + \sum_{n_i=4} \frac{6n_i+6}{5} \\ &= \frac{6(\sum_{i=1}^{t} n_i) + 4t_1 + 6t_2}{5} \\ &= \frac{6(n-s) + 4t_1 + 6t_2}{5}.\end{aligned}$$

Note that $s \geq t_1 + t_2 - 1$. First, assume that $s \geq t_1 + t_2$. Then

$$\sum_{i=1}^{t} \gamma_{tdR}(P_{n_i}) \leq \frac{6(n-s) + 6(t_1+t_2) - 2t_1}{5} \leq \frac{6n - 2t_1}{5} \leq \lceil \frac{6n}{5} \rceil,$$

which contradicts the choice of S. Hence, $s = t_1 + t_2 - 1$. Since the number of removing vertices is exactly one less than the number of created components, two removing vertices cannot be adjacent to each other. Without loss of generality, we may assume that $P_{n_1}, P_{n_2}, \dots, P_{n_{t_2}}$ are P_4 components and the remaining components are non-P_4 components. If $t_1 \geq 3$ then we obtain the contradiction

$$\sum_{i=1}^{t} \gamma_{tdR}(P_{n_i}) \leq \frac{6(n-s) + 6(s+1) - 2t_1}{5} \leq \lceil \frac{6n}{5} \rceil.$$

Next assume that $t_1 = 2$ and that $P_{n_{s-1}}$ and P_{n_s} are the non-P_4 components. We consider five cases. In all these cases we note that $n = 5t_2 + n_{s-1} + n_s + 1$.

Case 1. $n = 5k+1$.
Then we have $5k+1 = 5t_2 + n_{s-1} + n_s + 1$, and so $5(k - t_2) \equiv n_{s-1} + n_s \equiv 0 \pmod 5$. Then, without loss of generality, $n_{s-1} \equiv n_s \equiv 0 \pmod 5$ or $n_{s-1} \equiv 1 \pmod 5$ and $n_s \equiv 4 \pmod 5$ or $n_{s-1} \equiv 2 \pmod 5$ and $n_s \equiv 3 \pmod 5$. In all cases we observe that

$$\lceil \frac{6n_{s-1}}{5} \rceil + \lceil \frac{6n_s}{5} \rceil \leq \frac{6(n_{s-1}+n_s)+5}{5}.$$

Therefore,

$$\begin{aligned}\sum_{i=1}^{t}\gamma_{tdR}(P_{n_i}) &\leq \frac{6(n_{s-1}+n_s)+5+(6\times 4t_2)+6t_2}{5}\\ &= \frac{6(5t_2+n_{s-1}+n_s+1)+5-6}{5}\\ &= \frac{6n-1}{5}\\ &< \lceil\frac{6n}{5}\rceil,\end{aligned}$$

a contradiction again.

Case 2. $n = 5k+2$.
In this case we have $n_{s-1}+n_s \equiv 1 \pmod 5$. Then, without loss of generality, $n_{s-1} \equiv n_s \equiv 3 \pmod 5$ or $n_{s-1} \equiv 4 \pmod 5$ and $n_s \equiv 2 \pmod 5$ or $n_{s-1} \equiv 0 \pmod 5$ and $n_s \equiv 1 \pmod 5$. In in all cases we have

$$\lceil\frac{6n_{s-1}}{5}\rceil + \lceil\frac{6n_s}{5}\rceil = \frac{6(n_{s-1}+n_s)+4}{5}.$$

Hence

$$\begin{aligned}\sum_{i=1}^{t}\gamma_{tdR}(P_{n_i}) &\leq \frac{6(n_{s-1}+n_s)+4+(6\times 4t_2)+6t_2}{5}\\ &= \frac{6(5t_2+n_{s-1}+n_s+1)+4-6}{5}\\ &= \frac{6n-2}{5}\\ &< \lceil\frac{6n}{5}\rceil,\end{aligned}$$

a contradiction.

Case 3. $n = 5k+3$.
As before we have $n_{s-1}+n_s \equiv 2 \pmod 5$. Then, without loss of generality, $n_{s-1} \equiv n_s \equiv 1 \pmod 5$ or $n_{s-1} \equiv 2 \pmod 5$ and $n_s \equiv 0 \pmod 5$ or $n_{s-1} \equiv 3 \pmod 5$ and $n_s \equiv 4 \pmod 5$. This yields to

$$\lceil\frac{6n_{s-1}}{5}\rceil + \lceil\frac{6n_s}{5}\rceil \leq \frac{6(n_{s-1}+n_s)+8}{5}.$$

Thus

$$\begin{aligned}\sum_{i=1}^{t}\gamma_{tdR}(P_{n_i}) &\le \frac{6(n_{s-1}+n_s)+8+(6\times 4t_2)+6t_2}{5}\\ &= \frac{6(5t_2+n_{s-1}+n_s+1)+2}{5}\\ &= \frac{6n+2}{5}\\ &= \lceil\frac{6n}{5}\rceil.\end{aligned}$$

which contradicts the choice of S.

Case 4. $n = 5k + 4$.
In this case we have $n_{s-1}+n_s \equiv 3 \pmod 5$. Then, without loss of generality, $n_{s-1} \equiv n_s \equiv 4 \pmod 5$ or $n_{s-1} \equiv 1 \pmod 5$ and $n_s \equiv 2 \pmod 5$ or $n_{s-1} \equiv 0 \pmod 5$ and $n_s \equiv 3 \pmod 5$. It follows that

$$\lceil\frac{6n_{s-1}}{5}\rceil+\lceil\frac{6n_s}{5}\rceil \le \frac{6(n_{s-1}+n_s)+7}{5}.$$

Hence

$$\begin{aligned}\sum_{i=1}^{t}\gamma_{tdR}(P_{n_i}) &\le \frac{6(n_{s-1}+n_s+1)+1+(6\times 4t_2)+6t_2}{5}\\ &= \frac{6(5t_2+n_{s-1}+n_s+1)+1}{5}\\ &= \frac{6n+1}{5}\\ &= \lceil\frac{6n}{5}\rceil,\end{aligned}$$

a contradiction.

Case 5. $n = 5k$.
Then we have $5k = 5t_2+n_{s-1}+n_s+1$, and so $n_{s-1}+n_s \equiv 4 \pmod 5$. Then, without loss of generality, $n_{s-1} \equiv n_s \equiv 2 \pmod 5$ or $n_{s-1} \equiv 0 \pmod 5$ and $n_s \equiv 4 \pmod 5$ or $n_{s-1} \equiv 1 \pmod 5$ and $n_s \equiv 3 \pmod 3$. This implies in all cases

$$\lceil\frac{6n_{s-1}}{5}\rceil+\lceil\frac{6n_s}{5}\rceil \le \frac{6(n_{s-1}+n_s)+6}{5}.$$

Therefore,

$$\begin{aligned}\sum_{i=1}^{t}\gamma_{tdR}(P_{n_i}) &\leq \frac{6(n_{s-1}+n_s)+6+(6\times 4t_2)+6t_2}{5}\\ &= \frac{6(5t_2+n_{s-1}+n_s+1)}{5}\\ &= \lceil\frac{6n}{5}\rceil,\end{aligned}$$

a contradiction.

Second, let $t_1 = 1$ and suppose P_{n_s} is not the P_4 component. Then $n = 5t_2 + n_s$ and $n \equiv n_s \pmod 5$. Now let $n = 5k + \ell$ where $\ell \in \{0, 1, 2, 3, 4\}$. Then $n_s \equiv \ell \pmod 5$. We have $\gamma_{tdR}(P_{n_s}) = \frac{6n_s+m}{5}$, where $m = 5 - \ell$ if $\ell \in \{1, 2, 3, 4\}$ and $m = 0$ if $\ell = 0$. So

$$\begin{aligned}\sum_{i=1}^{t}\gamma_{tdR}(P_{n_i}) &\leq \frac{6n_s+m+(6\times 4t_2)+6t_2}{5}\\ &= \frac{6(5t_2+n_s)+m}{5}\\ &= \frac{6n+m}{5}\\ &\leq \lceil\frac{6n}{5}\rceil.\end{aligned}$$

Finally, Let $t_1 = 0$ then $t = t_2$ and $n = 5(t-1)+4$. It follows that $\sum_{i=1}^{t}\gamma_{tdR}(P_{n_i}) = 6t > \lceil\frac{6n}{5}\rceil$, and thus $st^-_{\gamma_{tdR}}(P_{5(t-1)+4}) = |S| = t-1$ and the proof is complete. □

The next result is an immediate consequence of Propositions 3 and 4.

Corollary 1. *For* $n \geq 3$, $\mathrm{st}_{\gamma_{tdR}}(P_n) = \begin{cases} 2 & \text{if } n = 5 \\ 1 & \text{otherwise.}\end{cases}$

Proposition 5. *For* $n \geq 3$, $\mathrm{st}^-_{\gamma_{tdR}}(C_n) = \begin{cases} 2 & \text{if } n = 5 \\ 1 & \text{otherwise.}\end{cases}$

Proof. Let $C_n = v_1v_2\ldots v_nv_1$ be a cycle on n vertices. By Proposition 2 $\gamma_{tdR}(C_n) = \lceil\frac{6n}{5}\rceil$. First, suppose that $n \neq 5$. Note that $C_n - v_n = P_{n-1}$ and Proposition 1 leads to $\gamma_{tdR}(C_n - v_n) = \gamma_{tdR}(P_{n-1}) < \lceil\frac{6n}{5}\rceil$. Hence, $st^-_{\gamma_{tdR}}(C_n) = 1$. Now we consider the case of $n = 5$. by Proposition 2, $\gamma_{tdR}(C_5) = 6$. Note that it follows from Proposition 1 that $\gamma_{tdR}(C_5 - v_i) = \gamma_{tdR}(P_4) = 6 = \gamma_{tdR}(C_5)$. It means that $st^-_{\gamma_{tdR}}(C_5) \geq 2$. Next, let $S = \{v_1, v_5\}$ and note that $C_5 - S = P_3$. From Proposition 1, we obtain $\gamma_{tdR}(C_5 - S) = 4 < \lceil\frac{6n}{5}\rceil$, which yields $st^-_{\gamma_{tdR}}(C_5) \leq 2$. Therefore $st^-_{\gamma_{tdR}}(C_5) = 2$. □

Proposition 6. *For $n \geq 3$, $\mathrm{st}^+_{\gamma_{tdR}}(C_n) = \infty$*

Proof. Let $C_n = v_1v_2 \ldots v_n$ and let $S = \{v_{i_1}, v_{i_2}, \ldots, v_{i_s}\}$, where $i_1 < i_2 < \cdots < i_s$, be the smallest set of vertices whose removal from C_n increases the total double Roman domination number. Suppose that $P_{n_1}, P_{n_2}, \ldots, P_{n_s}$ are the components of $C_n - S$. By definition, each component must have at least two vertices. Let t_1 be the number of non-P_4 components and let t_2 be the number of P_4 components. As in the proof of Proposition 4, we see that

$$\sum_{i=1}^{t} \gamma_{tdR}(P_{n_i}) \leq \frac{6(n-s)+4t_1+6t_2}{5}.$$

Note that $s \geq t_1 + t_2$. Thus the last inequality leads to the contradiction

$$\sum_{i=1}^{t} \gamma_{tdR}(P_{n_i}) \leq \frac{6(n-s)+6(t_1+t_2)-2t_1}{5} \leq \frac{6n-2t_1}{5} \leq \lceil \frac{6n}{5} \rceil.$$

Hence $\mathrm{st}^+_{\gamma_{tdR}}(C_{13}) = \infty$. □

As a consequence of Propositions 5 and 6 we obtain the next result.

Corollary 2. *For $n \geq 3$, $\mathrm{st}_{\gamma_{tdR}}(C_n) = \begin{cases} 2 & \text{if } n = 5 \\ 1 & \text{otherwise.} \end{cases}$*

One can observe that for $n \geq 3$, $\gamma_{tdR}(K_n) = \gamma_{tdR}(K_{1,n-1}) = 4$, for $1 \leq r \leq t$, $\gamma_{tdR}(S_{r,t}) = 6$ and for $n \geq m \geq 2$,

$$\gamma_{tdR}(K_{m,n}) = \begin{cases} 5 & \text{if } m = 2 \\ 6 & \text{if } m \geq 3 \end{cases}$$

The above results, easily lead to the next corollaries.

Corollary 3. *For $n \geq 3$, $\mathrm{st}_{\gamma_{tdR}}(K_n) = \mathrm{st}^-_{\gamma_{tdR}}(K_n) = n - 2$ and $\mathrm{st}^+_{\gamma_{tdR}}(K_n) = \infty$.*

Corollary 4. *For $n \geq 3$, $\mathrm{st}_{\gamma_{tdR}}(K_{1,n-1}) = \mathrm{st}^-_{\gamma_{tdR}}(K_{1,n-1}) = n-2$ and $\mathrm{st}^+_{\gamma_{tdR}}(K_{1,n-1}) = \infty$.*

Corollary 5. *For $1 \leq r \leq t$, $\mathrm{st}_{\gamma_{tdR}}(S_{r,t}) = \mathrm{st}^-_{\gamma_{tdR}}(S_{r,t}) = r$ and $\mathrm{st}^+_{\gamma_{tdR}}(S_{r,t}) = \infty$.*

Corollary 6. *For integers $n \geq m \geq 2$, $\mathrm{st}^+_{\gamma_{tdR}}(K_{m,n}) = \infty$ and*

$$\mathrm{st}_{\gamma_{tdR}}(K_{m,n}) = \mathrm{st}^-_{\gamma_{tdR}}(K_{m,n}) = \begin{cases} 1 & \text{if } m = 2 \\ m-2 & \text{if } m \geq 3. \end{cases}$$

3. Bounds

In the sequel we present several simple bounds for the TDRD-stability of a graph. Since for any graph G of order at least 3, $\gamma_{tdR}(G) \geq 4$ with equality if and only if $\Delta(G) = n-1$ (see [11]), the proof of the first observation is trivial.

Observation 1. If G is a graph of order $n \geq 3$, then $\mathrm{st}^{-}_{\gamma_{tdR}}(G) \leq n-2$ with equality if and only if $\Delta(G) = n-1$, or equivalently $\gamma_{tdR}(G) = 4$.

Proposition 7. *Let G be a connected graph having a $\gamma_{tdR}(G)$-function $f = (V_0, V_1, V_2, V_3)$ with $V_3 \neq \emptyset$. Then*

$$\mathrm{st}^{-}_{\gamma_{tdR}}(G) \leq \min\{\deg(v) - 1 \mid v \in V_3\}.$$

Proof. Let v be an arbitrary vertex with $f(v) = 3$ and let $S = pn(v, V_2 \cup V_3) \cap V_0$. Since f is a TDRD-function, we have $|S| \leq \Delta - 1$ and the function g defined on $G - S$ by $g(v) = 2$ and $g(x) = f(x)$ for the remaining vertices, is a TDRD-function on $G - S$ of weight less than $\gamma_{tdR}(G)$. Hence $\mathrm{st}^{-}_{\gamma_{tdR}}(G) \leq \deg(v) - 1$ and the result follows. □

Proposition 8. *Let G be a connected graph of order $n \geq 3$ with $\Delta(G) \geq 3$ and $\gamma_{tdR}(G) \geq 5$. Then*

$$\mathrm{st}_{\gamma_{tdR}}(G) \leq \Delta - 1.$$

This bound is sharp double stars $S_{\Delta,\Delta}$.

Proof. Let $f = (V_0, V_1, V_2, V_3)$ be a $\gamma_{tdR}(G)$-function. If $V_3 \neq \emptyset$, then the result follows from Proposition 7. Thus we assume that $V_3 = \emptyset$. Then $V_2 \neq \emptyset$. If $|V_2| = 1$ and $V_2 = \{v\}$, then $V - \{v\} = V_1$ and any vertex in V_1 must be adjacent to v. Now for any vertex $w \in V_1$, the function f restricted to $G - w$ is a TDRD-function of weight $\omega(f) - 1$ so $\mathrm{st}_{\gamma_{tdR}}(G) = 1$. Henceforth, we assume that $|V_2| \geq 2$. Let $G_1, G_2, \ldots, G_k$ be the components of $G[V_1 \cup V_2]$ and let $n(G_1) = \max\{n(G_i) \mid 1 \leq i \leq k\}$. By definition the induced subgraph $G[V_1 \cup V_2]$ is an isolated free graph and so $n(G_i) \geq 2$. We distinguish two cases.

Case 1. Let $n(G_1) = 2$ with $V(G_1) = \{u, v\}$.
Without loss of generality, we may assume that $g(v) = 2$. Let $S_v = \{y \in N(v) \cap V_0 : |N(y) \cap V_2| = 2\}$. Consider two situations.

Subcase 1.1. Let $g(u) = 1$.
If $|S_v| = 0$, then the function f restricted to $G - \{u, v\}$ is a TDRD-function of weight $\gamma_{tdR}(G) - 3$ and so $\mathrm{st}_{\gamma_{tdR}}(G) \leq 2$. If $|S_v| = 1$ and $S_v = \{w\}$, then define the function g on $G - \{u, v\}$ by $g(w) = 1$ and $g(x) = f(x)$ for the remaining vertices. Obviously, g is a TDRD-function and $\gamma_{tdR}(G - \{u, v\}) < \gamma_{tdR}(G)$ and so $\mathrm{st}_{\gamma_{tdR}}(G) \leq 2 \leq \Delta - 1$. Finally, let $|S_v| \geq 2$ and $w_1, w_2 \in S$. Define g on $G - ((S_v - \{w_1, w_2\}) \cup \{u, v\})$ by $g(w_1) = g(w_2) = 1$ and $g(x) = f(x)$ otherwise. Obviously, g is a TDRD-function of $G - ((S_v - \{w_1, w_2\}) \cup \{u, v\})$ and so $\gamma_{tdR}(G - ((S_v - \{w_1, w_2\}) \cup \{u, v\})) < \gamma_{tdR}(G)$. Thus $\mathrm{st}_{\gamma_{tdR}}(G) \leq \Delta - 1$.

Subcase 1.2. Assume that $g(u) = 2$.
Let $S_u = \{y \in N(u) \cap V_0 : |N(y) \cap V_2| = 2\}$. Then the function g defined on $G - S_u$ by $g(u) = 1$ and $g(x) = f(x)$ for remaining vertices, is a TDRD-function of $G - S_u$ of weight less than $\omega(f)$ and so $\mathrm{st}_{\gamma_{tdR}}(G) \leq \Delta - 1$.

Case 2. Let $n(G_1) \geq 3$.
If G_1 has a spanning tree with a leaf z assigned 1 under f, then the restriction of f on $G - z$ is a TDRD-function of weight $\gamma_{tdR}(G) - 1$ leading to $\mathrm{st}_{\gamma_{tdR}}(G) = 1$. Henceforth we may assume that every leaf of a spanning tree of G_1 is assigned 2 under f. Let T be a spanning tree of G_1 and let $v_1 v_2 \dots v_k$ be a diametral path in T. Root T at v_k. Let $u_1 = v_1, u_2, \dots, u_t$ be the leaf neighbors of v_2 in T. By assumption $f(v_1) = f(u_1) = \cdots = f(u_t) = 2$. Note that Let $S_{u_i} = \{y \in N(u_i) \cap V_0 : |N_G(y) \cap V_2| = 2\}$ and $S'_{u_i} = \{y \in N_G(u_i) \cap V_1 : |N(y) \cap V_2| = 1\}$ for each $i \in \{1, \dots, t\}$. Note that

$$|N_G(u_i)| = |\big(N_G(u_i) \cap V_2\big)| + |S_{u_i}| + |\big(N_G(u_i) \cap V_0 \setminus S_{u_i}\big)| + |S'_{u_i}| + |\big(N(u_i) \cap V_1 \setminus S'_{u_i}\big)|.$$

If $f(v_2) = 2$, then reassigning v_1 the value 1, provides a TDRD-function of $G - (S_{u_1} \cup S'_{u_1})$ with weight less than $\omega(f)$ and so $\mathrm{st}_{\gamma_{tdR}}(G) \leq |S_{u_1} \cup S'_{u_1}| \leq |N(u_1) - \{v_2\}| \leq \Delta - 1$.
Let $f(v_2) = 1$. Assume first that $S'_{u_i} = \emptyset$ for some i. If $S_{u_i} = \emptyset$, then the function f restricted to $G - u_i$ is a TDRD-function of $G - u_i$ with weight less than $\omega(f)$ and so $\mathrm{st}_{\gamma_{tdR}}(G) = 1$. Assume that $S_{u_i} \neq \emptyset$ and let $w \in S_{u_i}$. Set $S = (S_{u_i} - \{w\}) \cup \{u_i\}$. Then clearly $|S| \leq \Delta - 1$ and the function g with $g(w) = 1$ and $g(x) = f(x)$ otherwise is a TDRD-function of $G - S$ with weight less than $\omega(f)$ and so $\mathrm{st}_{\gamma_{tdR}}(G) \leq \Delta - 1$. Now let $S'_{u_i} \neq \emptyset$ for every $i \in \{1 \dots, t\}$, and let $u'_i \in S'_{u_i}$. Then $T' = (T - \{v_3 v_2, v_2 u_i \mid 2 \leq i \leq t\}) + \{u_i u'_i \mid 1 \leq i \leq t\}$ is a spanning tree of G_1 with a leaf assigned 1 under f contradiction our earlier assumption. This completes the proof. □

The path P_5 and cycle C_5 show that the condition $\Delta(G) \geq 3$ in Proposition 8 is necessary.

Proposition 9. *Let G be a connected graph of order n with $\gamma_{tdR}(G) \geq 5$, then $\mathrm{st}^-_{\gamma_{tdR}}(G) \leq n - \Delta(G) - 1$.*

Proof. Let u be a vertex of G with $\deg(u) = \Delta(G)$. Note that $\gamma_{tdR}(G[N[u]]) = 4 < \gamma_{tdR}(G)$. Thus, we have that $st^-_{\gamma_{tdR}}(G) \leq |V(G) - N[u]| = n - \Delta(G) - 1$. □

The next corollaries are immediate consequence of Propositions 8 and 9.

Corollary 7. *Let G be a connected graph of order $n \geq 3$ with $\gamma_{tdR}(G) \geq 5$. Then $\mathrm{st}_{\gamma_{tdR}}(G) \leq \min\{\Delta, n - \Delta(G) - 1\}$.*

Corollary 8. *Let G be a connected graph of order $n \geq 3$ with $\gamma_{tdR}(G) \geq 5$ and $\Delta(G) \geq 3$. Then $\mathrm{st}_{\gamma_{tdR}}(G) \leq \frac{n-2}{2}$.*

Proof. If $\Delta \leq \frac{n}{2}$, then the result follows from Proposition 8. Assume that $\Delta > \frac{n}{2}$. It follows from Proposition 9 that $\mathrm{st}^{-}_{\gamma_{tdR}}(G) \leq n - \frac{n+1}{2} - 1 \leq \frac{n-2}{2}$ as desired. □

The double stars $S_{\Delta,\Delta}$ shows that Corollary 8 is sharp. The complete graph shows that the condition $\gamma_{tdR}(G) \geq 5$ in Corollary 8 is necessary. Using the above corollaries we can characterize all connected graph with large TDRD-stability.

Proposition 10. *Let G be a connected graph of order $n \geq 4$ with $\gamma_{tdR}(G) \geq 5$. Then $\mathrm{st}_{\gamma_{tdR}}(G) = n - 3$ if and only if $G \in \{P_4, P_5, C_4, C_5\}$.*

Proof. If $G \in \{P_4, P_5, C_4, C_5\}$, then by Corollaries 1 and 2, we have $\mathrm{st}_{\gamma_{tdR}}(G) = n - 3$. To prove the necessity, let G be a connected graph with $\mathrm{st}_{\gamma_{tdR}}(G) = n - 3$. Note that $\Delta(G) \geq 2$, since G is connected having at least four vertices. It follows from Proposition 9 that $\Delta(G) \leq n - \mathrm{st}_{\gamma_{tdR}}(G) - 1 = 2$ and thus $\Delta(G) = 2$. Therefore G is a path or cycle and we deduce from Corollaries 1 and 2 that $G \in \{P_4, P_5, C_4, C_5\}$. □

We conclude this section with a problem.

Problem. Characterize all graphs G with maximum degree $\Delta \geq 3$ and $\mathrm{st}^{-}_{\gamma_{tdR}}(G) = \Delta - 1$.

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Data Availability: Data sharing is not applicable to this article as no data sets were generated or analyzed during the current study.

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