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Research Article

A note on the maximum A_α-spectral radius of some classes of graphs

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Abstract: According to Nikiforov [V. Nikiforov, Merging the A- and Q-spectral theories, Appl. Anal. Discrete Math. 11 (2017), no. 1, 81–107], the A_α-matrix of a graph G is defined as $A_\alpha(G) = \alpha D(G) + (1-\alpha)A(G)$, where $\alpha \in [0,1]$, $D(G)$ is the diagonal matrix with the degrees of the vertices of G as the diagonal entries, and $A(G)$ is the adjacency matrix. The A_α-spectral radius of the A_α-matrix is its largest eigenvalue. In this study, we characterize the graph that maximizes the A_α-spectral radius within three specific classes of graphs: (i) graphs of order n, with vertex connectivity $\kappa(G) \leq k$ and minimum degree $\delta(G) \geq k$; (ii) bipartite graphs of order n with vertex connectivity k; and (iii) graphs of order n, connectivity k, and independence number r. Furthermore, we determine the location of the A_α-spectral radius for the class $\nu_{k,\delta,n}$.

Keywords: A_α-matrix, A_α-spectral radius, vertex connectivity, minimum degree, bipartite graphs, independence number.

AMS Subject classification: 05C12, 05C50, 15A18

1. Introduction

We consider a simple finite connected graph G with vertex set $V(G)$ and edge set $E(G)$. The *order* of a graph refers to the number of vertices, while its *size* refers to

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the number of edges. Denote by $N(v) = \{u \in V(G) \mid vu \in E(G)\}$ the *neighborhood* of a vertex $v \in V(G)$, and by $d_G(v) = |N(v)|$ the *degree* of v.

Assume that G has two nonadjacent vertices, x and y. By adding the edge xy to G, we obtain the graph $G+xy$. For a subset $S \subseteq V(G)$, the induced subgraph is denoted by $G[S]$.

Consider two graphs, $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$. The union $G_1 \cup G_2$ is defined as $G_1 \cup G_2 = (V_1 \cup V_2, E_1 \cup E_2)$. From $G_1 \cup G_2$, the join $G_1 + G_2$ is obtained by adding all the edges linking each vertex of G_1 to each vertex of G_2.

Let $\delta(G)$ denote the minimum degree of a graph G. The smallest number of vertices that must be removed from G to either disconnect the graph or reduce it to a trivial graph is referred to as its *vertex connectivity*.

In a graph, a set of vertices is called an *independent set* if no two vertices within the set are adjacent. The *independence number*, $\gamma(G)$, represents the cardinality of the largest independent set in G.

The signless Laplacian matrix of a graph G is defined as $D(G) + A(G)$, where $A(G)$ is the adjacency matrix and $D(G)$ is the diagonal matrix of vertex degrees. Both adjacency matrices and signless Laplacian matrices have been extensively studied, revealing many similarities between their properties. In this context, Nikiforov proposed the A_α-matrix, defined as $A_\alpha(G) = \alpha D(G) + (1-\alpha)A(G)$, where $\alpha \in [0,1]$. This formulation generalizes both matrices: the adjacency matrix is $A_0(G)$, half of the signless Laplacian matrix is $A_{\frac{1}{2}}(G)$, and $A_1(G) = D(G)$.

For $\alpha \in [0,1)$, the analysis of $A_\alpha(G)$ suffices. For any real $\alpha \in [0,1)$, $A_\alpha(G)$ is a nonnegative symmetric matrix, ensuring that all its eigenvalues are real. Furthermore, since $A_\alpha(G)$ is an irreducible matrix, its largest eigenvalue, known as the *A_α-spectral radius*, is positive and simple, with a corresponding positive eigenvector.

In recent years, research on the A_α-spectral radius has garnered significant interest. In [10], Nikiforov et al. identified the maximum and minimum A_α-spectral radii for every tree of order n. Additionally, in [12], C. Wang et al. determined the graph with the highest A_α-spectral radius given fixed vertex or edge connectivity. For further details, refer to [6], [2], [4], [7], [11], and [5]. The family of all connected graphs with order n, vertex connectivity k, and minimum degree $\delta(G) \geq k$ is denoted by $\nu_{k,\delta,n}$. Let $\xi_{k,r,n}$ represent the set of graphs of order n with connectivity k and independence number r, and let $\zeta^B_{n,k}$ denote the class of all connected bipartite graphs of order n with vertex connectivity k.

In [8], the authors determined the graphs with the highest adjacency spectral radius within the families $\nu_{k,\delta,n}$, $\zeta^B_{n,k}$, and $\xi_{k,r,n}$. In this study, we identify the graphs with the largest A_α-spectral radius among these three families. Additionally, we provide the location of the A_α-spectral radius for the class $\nu_{k,\delta,n}$.

2. Preliminaries

In this section we present several key ideas and lemmas that form the foundation of the primary proofs discussed in this article.

Let G be a graph with the vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ and edge set $E(G)$. The eigenvalues of $A_\alpha(G)$ are denoted as $\lambda_1(A_\alpha(G)) \geq \lambda_2(A_\alpha(G)) \geq \ldots \geq \lambda_n(A_\alpha(G))$. We denote the A_α-spectral radius of G by $\rho_\alpha := \lambda_1(A_\alpha(G))$.
Assume that $V(G)$ is partitioned into $\pi_\alpha = (V_1, V_2, \ldots, V_r)$. If a constant b_{ij}, independent of a vertex u, determines the number of neighbors u has in V_j, then this partition is called equitable. The quotient matrix Q [8] for $A(G)$ with respect to the partition π_α is given by $Q = (b_{ij})_{r\times r}$. In this context, we define the quotient matrix for $A_\alpha(G)$ as follows:
$$Q_\alpha(G) = \begin{cases} (1-\alpha)b_{ij}, & \text{if } i \neq j; \\ \alpha(b_{i1} + \ldots + b_{i(i-1)} + b_{i(i+1)} + \ldots + b_{ir}) + b_{ii}, & \text{otherwise.} \end{cases}$$
The following lemmas are derived from the definition of the A_α-spectral radius.

Lemma 1. [10, 12, 13] *Suppose $A_\alpha(G)$ be the A_α-matrix of a connected graph G, where $\alpha \in [0, 1)$. Let $v, w \in V(G)$, and $u \in T \subset V(G)$ such that $T \subset N(v) \setminus (N(w) \cup \{w\})$. Let G^* be the graph with vertex set $V(G)$ and edge set $E(G) \setminus \{uv, u \in T\} \cup \{uw, u \in T\}$, and X a unit eigenvector to $\rho_\alpha(A_\alpha(G))$. If $x_w > x_v$ and $|T| \neq \emptyset$, then $\rho_\alpha(G^*) > \rho_\alpha(G)$.*

Lemma 2. [1, 9, 12] *For $\alpha \in [0, 1)$,*

1. *if G^* is any proper subgraph of connected graph G, and ρ_α is the A_α-spectral radius, then $\rho_\alpha(G^*) < \rho_\alpha(G)$.*
2. *if X is a positive vector and r is a positive number such that $A_\alpha(G)X < rX$, then $\rho_\alpha(G) < r$.*

Let $\lambda_1(Q_\alpha)$ be the largest eigenvalue of Q_α. Then the following lemma holds.

Lemma 3. [3] *Let G be a graph and $\pi_\alpha = (V_1, V_2, \ldots, V_r)$ be a partition of $V(G)$ with quotient matrix Q_α. Then $\lambda_1(A_\alpha(G)) \geq \lambda_1(Q_\alpha)$ with equality if the partition is equitable.*

3. Graph with Maximal A_α- Spectral Radius in $\nu_{k,\delta,n}$

It is evident from the definition that $\nu_{k,\delta+1,n} \subseteq \nu_{k,\delta,n}$. The graph with the maximal A_α-spectral radius in $\nu_{k,\delta,n}$ can be identified by applying the following theorem.

Theorem 1. *The graph $K_k + (K_{\delta-k+1} \cup K_{n-\delta-1})$ uniquely maximizes the A_α-spectral radius among all the graphs in $\nu_{k,\delta,n}$.*

Proof. There is nothing to prove if $n = k + 1$, since K_{k+1} is the only k-connected graph of order n. For $n \geq k + 2$, let $G^* \in \nu_{k,\delta,n}$ be the graph with the greatest A_α-spectral radius. Given that G^* is not a complete graph, there exists a k-vertex cut, say $S = \{v_1, v_2, \ldots, v_k\}$. We now present the following claims:
Claim 1: There are precisely two components in $G^* - S$.
Assume, for the sake of contradiction, that $G^* - S$ consists of three components: G_1, G_2, and G_3. By connecting every possible edge between G_1 and G_3, we obtain a new

graph G' which still has S as a cut set and remains disconnected. However, according to Lemma 2, $G' \in \nu_{k,\delta,n}$ and has a higher A_α-spectral radius than G^*, which leads to a contradiction. Therefore, $G^* - S$ must consist of only two components: G_1 and G_2.
Claim 2: For each $i = 1, 2$, every subgraph of G^* induced by vertices $V(G_i) \cup S$ is complete.
Otherwise, there exists at least one pair of non-adjacent vertices (with one vertex in $V(G_i)$ and the other in S, or both vertices in $V(G_i)$ or both in S). By connecting these two vertices, we obtain a new graph in $\nu_{k,\delta,n}$ with a higher A_α-spectral radius than G^*, according to Lemma 2, leading to a contradiction.
As G_1 and G_2 are complete subgraphs according to Claim 2, let $G_i \cong K_{n_i}$, for $i = 1, 2$. Moreover, for $i = 1, 2$, $n_i \geq \delta - k + 1 > 0$, i.e., $n_i - 1 + k \geq \delta(G) \geq k$.
Claim 3: Either n_1 or n_2 is equal to $\delta - k + 1$.
If not, $n_1 > \delta - k + 1$ and $n_2 > \delta - k + 1$, respectively. When $\pi_\alpha = \{S, V(K_{n_1}), V(K_{n_2})\}$, the partition of the vertex set of G^* is considered, it is evident that this partition of G^* is equitable with respect to the quotient matrix

$$Q_\alpha = \begin{pmatrix} \alpha(n_1 + n_2) + (k-1) & (1-\alpha)n_1 & (1-\alpha)n_2 \\ (1-\alpha)k & \alpha k + (n_1 - 1) & 0 \\ (1-\alpha)k & 0 & \alpha k + (n_2 - 1) \end{pmatrix}.$$

According to the Perron-Frobenius theorem for the A_α-matrix, Q_α has a Perron vector, denoted as $a = (a_1, a_2, a_3)$, and its corresponding Perron value is $\rho_\alpha(Q_\alpha)$. From this, we will show that $a_3 > a_2$ if $n_2 > n_1$. The following two eigen-equations are derived from the final two rows of Q_α:

$$(1-\alpha)ka_1 + [\alpha k + (n_1 - 1)]a_2 = \rho_\alpha a_2, \tag{3.1}$$
$$(1-\alpha)ka_1 + [\alpha k + (n_2 - 1)]a_3 = \rho_\alpha a_3. \tag{3.2}$$

Now, (3.2)-(3.1) gives

$$\begin{aligned} (\alpha k + (n_2 - 1))a_3 - (\alpha k + (n_1 - 1))a_2 &= \rho_\alpha(a_3 - a_2) \\ \Rightarrow \alpha k(a_3 - a_2) + (n_2 - 1)(a_3 - a_2) + (n_2 - n_1)a_2 &= \rho_\alpha(a_3 - a_2) \\ \Rightarrow (\rho_\alpha - n_2 + 1)(a_3 - a_2) - \alpha k(a_3 - a_2) &= (n_2 - n_1)a_2 > 0 \\ \Rightarrow [\rho_\alpha - (n_2 + \alpha k - 1)](a_3 - a_2) &> 0. \end{aligned} \tag{3.3}$$

Note that the Perron-value $\rho_\alpha(Q_\alpha)$ is also the Perron-value of G^*. Since, $K_k + K_{n_2}$ is a subgraph of G^*, we have

$$(n_2 + \alpha k - 1) < (n_2 + k - 1) = \rho_\alpha(K_k + K_{n_2}) < \rho_\alpha(Q_\alpha).$$

Thus, $a_3 > a_2$ from (3.3), and we have an eigenvector $b = (\underbrace{a_1, \ldots, a_1}_{k}, \underbrace{a_2, \ldots, a_2}_{n_1}, \underbrace{a_3, \ldots, a_3}_{n_2})$ of G^* with $n_2 a_3 > n_1 a_2$. Let $x = \frac{1}{\sqrt{ka_1^2 + n_1 a_2^2 + n_2 a_3^2}} b$, such that

$xx^T = 1$, and x is a Perron-vector of G^*. For a vertex $u \in V(K_{n_1})$, Lemma 1 states that if $H = G^* - \{uu_i | u_i \in V(K_{n_1})\} + \{uv_i | v_i \in V(K_{n_2})\}$, then $H \in \nu_{k,\delta,n}$, and hence $\rho_\alpha(G^*) < \rho_\alpha(H)$, which is contradictory. As a result, let $n_1 = \delta - k + 1$. Consequently, $n_2 = n - \delta - 1$ will be implied and $G^* \cong K_k + (K_{\delta-k+1} \cup K_{n-\delta-1})$. □

Corollary 1. *Among all the connected graphs of order n, vertex connectivity k, and minimum degree exactly δ, the maximum A_α-spectral radius is attained by $K_k + (K_{\delta-k+1} \cup K_{n-\delta-1})$.*

Proof. If the minimum degree is fixed as exactly δ, then the required class of graphs is $\vartheta = \nu_{k,\delta,n} - \nu_{k,\delta+1,n}$. Since the graph $K_k + (K_{\delta-k+1} \cup K_{n-\delta-1}) \in \vartheta$, the result follows. □

The graph with the maximum A_α-spectral radius among all the graphs in $\nu_{k,\delta,n}$ is $K_k + (K_{\delta-k+1} \cup K_{n-\delta-1})$, according to Theorem 1.1 In the following result, we locate $\rho_\alpha(K_k + (K_{\delta-k+1} \cup K_{n-\delta-1}))$.

Theorem 2. *The A_α-spectral radius of $K_k + (K_{\delta-k+1} \cup K_{n-\delta-1})$ is the largest root of the equation:*
$x^3 - (\alpha k + \alpha n + n - 3)x^2 + [(\alpha n - \alpha k + k - 1)(2\alpha k + n - k - 2) + (\alpha k + \delta - k)(\alpha k + n - \delta - 2) - k(1-\alpha)^2(\delta - k + 1) - k(1-\alpha)^2(n - \delta - 1)]x + [k(1-\alpha)^2(\alpha k + n - \delta - 2)(\delta - k + 1) + k(1-\alpha)^2(n - \delta - 1)(\alpha k + \delta - k) - (\alpha k + \delta - k)(\alpha k + n - \delta - 2)(\alpha n - \alpha k + k - 1)] = 0.$

Proof. For the graph $G = K_k + (K_{\delta-k+1} \cup K_{n-\delta-1})$, let $\pi_\alpha = \{V(K_k), V(K_{\delta-k+1}), V(K_{n-\delta-1})\}$ be a partition of the vertex set. Since this partition is equitable, we have the quotient matrix for $A_\alpha(G)$ as follows:

$$Q_\alpha = \begin{pmatrix} \alpha(n-k) + (k-1) & (1-\alpha)(\delta-k+1) & (1-\alpha)(n-\delta-1) \\ (1-\alpha)k & \alpha k + (\delta - k) & 0 \\ (1-\alpha)k & 0 & \alpha k + (n-\delta-2) \end{pmatrix}.$$

Then we have the following equation:

$det(xI_3 - Q_\alpha)$

$$= det \begin{pmatrix} x - \alpha(n-k) - (k-1) & -(1-\alpha)(\delta-k+1) & -(1-\alpha)(n-\delta-1) \\ -(1-\alpha)k & x - \alpha k - (\delta - k) & 0 \\ -(1-\alpha)k & 0 & x - \alpha k - (n-\delta-2) \end{pmatrix}$$

$= x^3 - (\alpha k + \alpha n + n - 3)x^2 + [(\alpha n - \alpha k + k - 1)(2\alpha k + n - k - 2) + (\alpha k + \delta - k)(\alpha k + n - \delta - 2) - k(1-\alpha)^2(\delta - k + 1) - k(1-\alpha)^2(n - \delta - 1)]x + [k(1-\alpha)^2(\alpha k + n - \delta - 2)(\delta - k + 1) + k(1-\alpha)^2(n - \delta - 1)(\alpha k + \delta - k) - (\alpha k + \delta - k)(\alpha k + n - \delta - 2)(\alpha n - \alpha k + k - 1)].$

Lemma 3 now indicates that $\lambda_1(A_\alpha(G)) \geq \lambda_1(Q_\alpha)$. Equality holds because the partition is equitable, and the desired outcome is achieved. □

4. Graph with Maximal A_α- Spectral Radius in $\zeta^B_{n,k}$

Define a graph $H(n,r)$, constructed from $K_{r,n-r-1}$ by joining a new vertex w to k vertices of degree r of $K_{r,n-r-1}$. The graph with maximal A_α-spectral radius in $\zeta^B_{n,k}$ is found using the following theorem.

Theorem 3. *Let G^* be the graph having the maximal A_α-spectral radius in $\zeta^B_{n,k}$. Then G^* is of the form $H(n,r)$ where $k \le r \le n-k-1$.*

Proof. Let, A, B be the bipartition of the vertex set of G^* with $V(G^*) = \{v_1, v_2, \dots, v_n\}$. Since G^* has connectivity k, therefore $|A| \ge k$ and $|B| \ge k$, and $k \le \lfloor \frac{n}{2} \rfloor$. If n is odd and $k = \frac{n-1}{2}$ (resp. n is even and $k = \frac{n}{2}$), then $\zeta^B_{n,k} = \{H(n, \frac{n-1}{2}) = K_{\frac{n-1}{2},\frac{n+1}{2}}\}$ (resp. $\zeta^B_{n,k} = \{H(n, \frac{n}{2}-1) = K_{\frac{n}{2},\frac{n}{2}}\}$). Hence, the result follows in this case. Therefore, let us consider $k \le \lfloor \frac{n}{2} \rfloor - 1$, and $x = (x_{v_1}, x_{v_2}, \dots, x_{v_n})$ be the Perron-vector of $A_\alpha(G^*)$.
Let $S = \{v_1, v_2, \dots, v_k\}$ be a k-vertex cut of G^*, $S_A = S \cap A$, $S_B = S \cap B$, $T_A = A - S_A$, and $T_B = B - S_B$. We now have the following claims:
Claim 1: With partitions (S_A, S_B), $G^*[S_A \cup S_B]$ is a complete bipartite graph. Similarly, for each component C of $G^* - S$, $G^*[V(C) \cup S]$ is a complete bipartite graph.
If $G^*[S_A \cup S_B]$ is not a complete bipartite graph, then there exist $i_1, i_2 \in \{1, 2, \dots, k\}$ such that $v_{i_1} \in S_A$, $v_{i_2} \in S_B$ and $v_{i_1}v_{i_2} \notin E(G^*)$. Then by Lemma 2, $\rho_\alpha(G^* + v_{i_1}v_{i_2}) > \rho_\alpha(G^*)$, a contradiction as $G^* + v_{i_1}v_{i_2} \in \zeta^B_{n,k}$. Similarly, for any component C of $G^* - S$, we can show that $G^*[V(C) \cup S]$ is a complete bipartite graph.
Claim 2: If $S_A \neq \emptyset$ and $S_B \neq \emptyset$, then $G^* - S$ consists of two complete bipartite components.
Let $G^* - S$ have three related components, C_1, C_2, C_3. Now, if $T_A = \emptyset$ (or $T_B = \emptyset$), then $|S_A| < k$, which goes against the idea that G^* is k-connected, and S_A (resp. S_B) is a cut set of G^*. Consequently, $T_A \neq \emptyset$ and $T_B \neq \emptyset$. Without loss of generality, let us consider $V(C_1) \cap A \neq \emptyset$ and $V(C_2) \cap B \neq \emptyset$. Let $x \in V(C_1) \cap A$ and $y \in V(C_2) \cap B$. If there exists $z \in V(C_3) \cap B$ (or $z \in V(C_3) \cap A$), then $G^* + xz \in \zeta^B_{n,k}$ (resp. $G^* + yz \in \zeta^B_{n,k}$) and $\rho_\alpha(G^* + xz) > \rho_\alpha(G^*)$ (resp. $\rho_\alpha(G^* + yz) > \rho_\alpha(G^*)$), a contradiction. Therefore, $G^* - S$ has only two components; by Claim 1, both components are complete bipartite.
Claim 3: Either $S \subseteq A$ or $S \subseteq B$ and $G^* - S$ contains only two components one of which is a complete bipartite and another is an isolated vertex.
Let $S_A \neq \emptyset$ and $S_B \neq \emptyset$ initially. Then, according to Claim 3, $G^* - S$ is made up of C_1 and C_2, two complete bipartite components. Since G^* is k-connected, $C_i \cap A \neq \emptyset$ and $C_i \cap B \neq \emptyset$, for $i = 1, 2$; additionally, $|A| > k > |S_A|$ and $|B| > k > |S_B|$. Let $v \in V(G^*) \setminus S$ be such an element whose component x_v is the smallest among all the components in the Perron vector X corresponding to the elements in $V(G^*) \setminus S$. Without loss of generality, let us assume $v \in V(C_2) \cap B$. Let $G = G^* - \{zv | z \in V(C_2) \cap A\} + \{zu | z \in V(C_2) \cap A, u \in V(C_1) \cap B\}$. Lemma 1 thus indicates that

$\rho_\alpha(G^*) < \rho_\alpha(G)$. At this point, $d_G(v) \leq |S_A| = k - r < k$, with $|S_B| = r$ (say). Upon joining all r edges from any r vertices of A to v, we have a graph $F \in \zeta^B_{n,k}$, along with a contradiction $\rho_\alpha(G^*) < \rho_\alpha(G) < \rho_\alpha(F)$. Consequently, A or B contains S. Say $S \subseteq A$. Upon adding all the edges that are still not in F from every vertex in A to every vertex in B, excluding vertex v, we obtain $H(n, |B - v|) \in \zeta^B_{n,k}$, again leading to a contradiction $\rho_\alpha(G^*) < \rho_\alpha(G) < \rho_\alpha(F) < \rho_\alpha(H(n, |B - v|))$. Now, from the above discussion, $G^* = H(n, |B - v|)$. According to the structure of $H(n, |B - v|)$, $G^* - S$ consists of two parts: an isolated point and a complete bipartite subgraph.
Given that $k \leq |B|$, we examine the case where $|B| = k$, which leads to $|B - v| = k - 1$. In this scenario, the graph $H(n, |B - v|)$ is not a member of the class $\zeta^B_{n,k}$. Thus, it must be that $k \leq |B - v|$. Additionally, it is clear that $|B - v| \leq n - k - 1$. Hence, the desired conclusion follows. □

To identify the graph with the maximum A_α-spectral radius within $\zeta^B_{n,k}$, we conducted computations for various values of k and n, examining how α behaves across the corresponding values of r. Our observations consistently suggest that the range of r described in Theorem 3, from k to $n - k - 1$, in practice spans from $\frac{n-3}{2}$ to $n - k - 1$. However, we have not yet been able to formally prove this, and thus we propose the following conjecture.

Conjecture 1. Let G^* be the graph having the maximal A_α-spectral radius in $\zeta^B_{n,k}$. Then G^* is of the form $H(n, r)$ with $\frac{n-3}{2} \leq r \leq n - k - 1$.

5. Graph with Maximal A_α- Spectral Radius in $\xi_{k,r,n}$

The following theorem determines the graph with maximal A_α- spectral radius in $\xi_{k,r,n}$.

Theorem 4. *$K_k + (K_1 \cup (K_{n-k-r} + (r - 1)K_1))$ uniquely maximizes the A_α-spectral radius among all graphs in $\xi_{k,r,n}$.*

Proof. Let G^* be the graph with the maximum A_α-spectral radius in $\xi_{k,r,n}$, and $V(G^*) = \{v_1, v_2, \ldots, v_n\}$. Let $x = (x_{v_1}, x_{v_2}, \ldots, x_{v_n})$ be the Perron vector of $A_\alpha(G^*)$, and $\rho_\alpha(G^*)$ be the A_α-spectral radius. Since G^* is k-connected and independent number is r, so $n \geq k + r$. Let $S = \{v_{i_1}, v_{i_2}, \ldots, v_{i_r}\}$ and $T = \{v_{j_1}, v_{j_2}, \ldots, v_{j_k}\}$ be the independent set and cut set of G^*, respectively. Let C_1 be a component of $G^* - T$ such that $a_1 = |V(C_1) \cap S|$, $a_2 = |V(C_1) - S|$, $b_1 = |T \cap S|$, and $b_2 = |T - S|$.
Claim 1: $C_1 = K_{a_2} + a_1K_1$, $G^*[T] = K_{b_2} + b_1K_1$ and $G^*[V(C_1) \cup T] = K_{a_2+b_2} + (a_1 + b_1)K_1$.
Assume $w \in |V(C_1) \cap S|$ and $u, v \in |V(C_1) - S|$. Let uv not be in $E(G^*)$. Then $\rho_\alpha(G^* + uv) > \rho_\alpha(G^*)$, a contradiction, because S is still an independent set and T is a cut set of $G^* + uv$. Similarly, we can show that $uw \in G^*$. Therefore, $C_1 = K_{a_2} + a_1K_1$. The proofs of the remaining parts of the claim are similar and hence avoided.

If $n = k + r$, then $G - S$ is a cut set. With similar proof to Claim 1, we have $G^* = K_k + rK_1$. So, we can take $n \geq k + r + 1$.

Claim 2: $G^* - T$ contains only two components, one of which is an isolated vertex. Otherwise, let C_1, C_2, C_3 be three components of $G^* - T$. If possible suppose $V(C_2) \geq 2$. Then, $V(C_2) - S \neq \emptyset$, and let $u \in V(C_2) - S,\ v \in V(C_3)$. Then, $G^* + uv \in \xi_{k,r,n}$ and we get a contradiction as $\rho_\alpha(G^* + uv) \geq \rho_\alpha(G^*)$. Hence, $G^* - T$ contains exactly two components. Also, it can be seen that $C_2 = K_{h_2} + h_1 K_1$, where, $h_1 = |V(C_2) \cap S|$ and $h_2 = |V(C_2) - S|$. Consider $w \in V(C_1) - S$. Now, if $x_u > x_w$, and if $G_1 = G^* - \{xu | x \in N_{c_2}(u)\} + \{xw | x \in N_{C_2}(u)\}$, then $\rho_\alpha(G_1) > \rho_\alpha(G^*)$. But $G_1 \in \xi_{k,r,n}$, giving a contradiction. Else if $x_w > x_u$, with a similar discussion, we can obtain another contradiction. Hence, $V(C_2)$ is a singleton set say $\{v_1\}$.

Claim 3: $v_1 \in S$.

Otherwise, suppose v_1 is not in S. Since G^* is k-connected, $N_{G^*}(v_1) = T$. Therefore $N_{G^*}(v_1) \cap S \neq \emptyset$ i.e., $S \cap T \neq \emptyset$. Let, $u_1 \in S \cap T$. Suppose, $|S \cap T| \geq 2$. As we have $n \geq k + r + 1$, $V(C_1) - S \neq \emptyset$. Let $u_2 \in V(C_1) - S$. If $x_{u_1} \leq x_{u_2}$, then let, $G' = G^* - u_1 v_1 + u_2 v_1$. It is easy to see that $(T - u_1) \cup \{u_2\}$ is a cut set and S is still a maximum independent set of G'. Therefore, $G' \in \xi_{k,r,n}$ and $\rho_\alpha(G') > \rho_\alpha(G^*)$ which is a contradiction. Else, if $x_{u_2} \leq x_{u_1}$, let $G'' = G^* - \{xu_2 | x \in S - u_1\} + \{xu_1 | x \in S - u_1\}$. Then, T is still a cut set and $(S - u_1) \cup \{u_2\}$ is still a maximum independent set of G''. Hence, $G'' \in \xi_{k,r,n}$. So we get a contradiction $\rho_\alpha(G'') > \rho_\alpha(G^*)$. Therefore, $|S \cap T|$ contains only one element say $S \cap T = \{u_1\}$, and if we take $G_2 = G^* + \{xu_1 | x \in V(G^*) - N_{G^*}(u_1)\}$. Clearly, T is still a cut set of G_2 and $(S - u_1) \cup v_1$ is a maximum independent set of G_2, and also $\rho_\alpha(G_2) > \rho_\alpha(G^*)$, contradicting to the choice of G^*. Thus the claim.

Now, by Claim 3, $S \cap T = \emptyset$. Also, by Claim 1 and Claim 2, G^* can be obtained as a union of $K_k + (K_{n-k-r} + (r-1)K_1)$ and $K_k + K_1$. Hence, $G^* \cong K_k + [K_1 \cup (K_{n-k-r} + (r-1)K_1)]$. □

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