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Research Article

A full-NT step interior-point method for weighted linear complementarity problem over symmetric cones

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Abstract: A full Nesterov-Todd step interior point method is designed and analyzed in this paper to solve the weighted linear complementarity problem in Euclidean Jordan algebra. Under appropriate conditions, it is proven that the full Nesterov-Todd step is strictly feasible and the algorithm has a quadratic convergence rate to the target point on the central path in the framework of Euclidean Jordan algebras. The obtained iteration bound for the algorithm matches the best known current iteration bound for this problem. To the best of our knowledge, this is the first full-step interior point algorithm for the weighted complementarity problem in the space of Euclidean Jordan algebras.

Keywords: weighted linear complementarity problem, Euclidean Jordan algebra, interior-point method, polynomial complexity

AMS Subject classification: 90C51, 90C33

1. Introduction

The Weighted Linear Complementarity Problem (WLCP) proposed by Potra [16] seeks to find a pair of vectors (x, s) belonging to the intersection of a manifold and a cone such that their product in a given algebra, $x \circ s$, is equal to a non-negative weight vector w. The WLCP becomes a classical Linear Complementarity Problem (LCP) when $w = 0$. The WLCP in $\mathbf{R}^n$ algebra is discussed in [16] and it is shown that the

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Fisher market equilibrium problem [22] and the linear programming and weighted centering problem [3] can be written in the form of a WLCP. In the following, two interior point methods (IMPs) for WLCP are proposed, both of which follow the central path. In subsequent work, Potra [17] introduced the notion of sufficiency for WLCP and extended the properties of the sufficient LCP introduced by Cottle et al. [7] to the sufficient WLCP. Zhang [12] proposed a smoothing Newton algorithm for solving monotone WLCP, which requires only one linear system of equations and performs only one line search per iteration. Alzalg and Gafour [2] analyzed the iteration complexity for a weighted logarithmic barrier decomposition algorithm for two-stage stochastic convex quadratic semidefinite optimization with discrete support. Alzalg [1] developed a primal-dual central path interior-point algorithm for symmetric cone optimization (SCO) that uniquely equips the central path algorithm with different choices selections of the displacement step while solving SCO.

Full-Newton step IPMs were first proposed by Roos et al. [19] for linear optimization (LO), and the advantage of these methods is that they do not require line-searches during the solution process to update the iterates. These methods have been extended to different formulations of LCPs in the literature. Darvay's key idea [8] for finding search directions is to consider the algebraic equivalent transformation (AET) of the central path system. He applied a continuously differentiable, invertible, and monotone increasing function to both sides of the centering equation of the central path. Furthermore, used Newton's method to determine the search directions presenting a full-Newton step IPM for LO. An infeasible version of this algorithm was introduced by Kheirfam [14] for SCO.

Asadi et al. [4] proposed a full-Newton step IPM for monotone WLCP and proved that the proposed algorithm has a quadratic rate of convergence to the target point on the central path. Recently, infeasible IPMs as well as their computational complexities were proposed in [5, 6] for solving the special WLCP. Based on the AET technique, Kheirfam [13] proposed a full-Newton step IPM for the monotone WLCP and proved that the iteration bound is the same as the one obtained for this problem.

Motivated by results as mentioned above, in this paper, we consider a full-Nesterov-Todd (NT) step IPM for WLCP in the setting of Euclidean Jordan algebras (EJA). With a nonzero weight vector, the theory of WLCP becomes more complicated than the theory of LCP. However, we apply the Newton's method to the system defining the weighted central path for WLCP to get search directions and take full steps along these directions. We prove the quadratic rate of convergence to the target points on the weighted central path. By choosing appropriate values for the parameters, we derive an iteration bound for the WLCP.

The paper is organized as follows. In Sect. 2, we briefly describe the WLCP and recall the notion of the weighted central path and the NT scaling scheme. In Sect. 3, we present our full-NT step IPM. Sect. 4 is devoted to the analysis of the algorithm. The iteration bound is derived in Sect. 5. Concluding remarks are given in Sect. 6.

2. WLCP and its central path

In this paper, we consider that the reader is well acquainted with EJA and symmetric cones. Hence, we will mention well-known primary results and lemmas about EJA in the appropriate place required. To learn more about Jordan algebras, we refer the reader to the book of Faraut and Korányi [9].
Throughout this paper, let $(\mathcal{J}, \circ)$ denote an EJA with $\operatorname{rank}(\mathcal{J}) = r$ and Jordan product $x \circ s$. For any element $x \in \mathcal{J}$, the Lyapunov transformation $L : \mathcal{J} \to \mathcal{J}$ is given by

$$L(x)y := x \circ y, \forall y \in \mathcal{J}.$$

Furthermore, the quadratic representation of x in $\mathcal{J}$ is defined by

$$P(x) := 2L(x)^2 - L(x^2),$$

where $L(x)^2 = L(x)L(x)$.
Let $\mathcal{K} = \{x^2 = x \circ x : x \in \mathcal{J}\}$ denote the corresponding symmetric cone (formed by the squares of its elements). Given two linear transformations A and B in $\mathcal{J}$, a weight vector $w \in \mathcal{K}$ and $q \in \mathcal{J}$, the WLCP follows $(x, s) \in \mathcal{K} \times \mathcal{K}$ such that

$$\begin{aligned} Ax + Bs &= q, \\ x \circ s &= w. \end{aligned} \tag{2.1}$$

Let $\mathcal{F} = \{(x, s) \in \mathcal{K} \times \mathcal{K} : Ax + Bs = q\}$ denote the set of all feasible points of (2.1). Define the solution set of (2.1) as $\mathcal{F}^* = \{(x, s) \in \mathcal{F} : \ x \circ s = w\}$, and the set of all strictly feasible points of (2.1) as $\mathcal{F}^0 = \{(x, s) \in \mathcal{F} : \ (x, s) \in \operatorname{int}(\mathcal{K}) \times \operatorname{int}(\mathcal{K})\}$, where $\operatorname{int}(\mathcal{K})$ denotes the interior of $\mathcal{K}$. Given an initial point $(x^0, s^0) \in \mathcal{F}^0$. Define

$$t^0 = \tfrac{\operatorname{tr}(x^0 \circ s^0)}{n}, \ c = x^0 \circ s^0, \ \gamma = \tfrac{\lambda_{\min}(c)}{t^0}, \ w(t) = (1 - \tfrac{t}{t^0})w + \tfrac{t}{t^0}c, \tag{2.2}$$

where $t \in]0, t^0]$ and $\lambda_{\min}(c)$ is the smallest eigenvalue of c. The central path of WLCP (2.1) is the set of all points $(x, s; t)$, with $t \in [0, t^0]$, satisfying

$$\begin{aligned} Ax + Bs &= q, \\ x \circ s &= w(t), \\ x > 0, &\ \ s > 0, \end{aligned} \tag{2.3}$$

where $x, s > 0$ means that $x, s \in \operatorname{int}(\mathcal{K})$. A common way to define the search direction is to use Newton's method and linearize the second equation in (2.3). However, the resulting Newton system has no unique solution for the search directions. The reason is that x and s are not commutative operators in general. To overcome this problem, a scaling scheme is applied to system (2.3) to guarantee that the scaled operators x and s are commutative. Scaling is done based on the following lemma.

Lemma 1. ([20, Lemma 28]) *Let x, s, and z be elements of an EJA $\mathcal{J}$. Additionally, let $x, s \in \operatorname{int}(\mathcal{K})$ and let z be an invertible element. Then $x \circ s = \alpha e$ iff $P(z)x \circ P(z)^{-1}s = \alpha e$ for some scalar α, where e is an identity element of $\mathcal{J}$.*

Now we replace the second equation of system (2.3) with $P(z)x \circ P(z)^{-1}s = w(t)$ which leads to the following system:

$$\begin{aligned} Ax + Bs &= q, \\ P(z)x \circ P(z)^{-1}s &= w(t), \\ (P(z)x, P(z)^{-1}s) &\in \operatorname{int}(\mathcal{K}) \times \operatorname{int}(\mathcal{K}). \end{aligned} \tag{2.4}$$

Applying Newton's method to the system (2.4), we have

$$\begin{aligned} A(x + \Delta x) + B(s + \Delta s) &= q, \\ P(z)(x + \Delta x) \circ P(z)^{-1}(s + \Delta s) &= w(t). \end{aligned}$$

In needs, for any feasible $x > 0$ and $s > 0$ and neglecting the term $P(z)\Delta x \circ P(z)^{-1}\Delta s$, we want to find direction $(\Delta x, \Delta s)$ such that

$$\begin{aligned} A\Delta x + B\Delta s &= 0, \\ P(z)x \circ P(z)^{-1}\Delta s + P(z)^{-1}s \circ P(z)\Delta x &= w(t) - P(z)x \circ P(z)^{-1}s. \end{aligned} \tag{2.5}$$

In this paper, we consider the NT-scaling scheme, the resulting direction is called NT search direction. This scaling scheme was first proposed by Nesterov and Todd [15] for self-scaled cones and then updated by Faybusovich [10] for symmetric cones.

Lemma 2. ([10, Lemma 3.2]) *Let $x, s \in \operatorname{int}(\mathcal{K})$. Then there exists a unique $u \in \operatorname{int}(\mathcal{K})$ such that*

$$x = P(u)s.$$

Moreover,

$$u = P(x^{\frac{1}{2}})\big(P(x^{\frac{1}{2}})s\big)^{-\frac{1}{2}} \Big[= P(s^{-\frac{1}{2}})\big(P(s^{\frac{1}{2}})x\big)^{\frac{1}{2}}\Big].$$

The point u is called the scaling point of x and s. Let $z = u^{-\frac{1}{2}}$, where u is the NT-scaling point of x and s. Define

$$v = \frac{P(u)^{-1/2}x}{\sqrt{t}} \left[= \frac{P(u)^{1/2}s}{\sqrt{t}}\right], \tag{2.6}$$

and

$$d_x = \frac{P(u)^{-1/2}\Delta x}{\sqrt{t}}, \qquad d_s = \frac{P(u)^{1/2}\Delta s}{\sqrt{t}}. \tag{2.7}$$

With these notations, system (2.5) can be rewritten as

$$\begin{aligned} \tilde{A}d_x + \tilde{B}d_s &= 0, \\ d_x + d_s &= v^{-1} \circ \left(\tfrac{w(t)}{t} - v^2 \right), \end{aligned} \tag{2.8}$$

where $\tilde{A} = \sqrt{t}P(u)^{1/2}A$, $\tilde{B} = \sqrt{t}P(u)^{-1/2}B$. The search directions d_x and d_s are obtained by solving (2.8) so that Δx and Δs are computed via (2.7). By taking a full-NT step along these directions, a new iterate is obtained as follows:

$$x^+ = x + \Delta x, \qquad s^+ = s + \Delta s. \tag{2.9}$$

To measure the proximity of each approximation (x, s) of $(x(t), s(t))$ as a solution to (2.3), we introduce the proximity of a point $(x, s; t)$ to the central path (2.4) by the function

$$\delta(v) := \delta(x, s; t) = \left\| \frac{w(t)}{t} - v^2 \right\|_F,$$

where $\|x\|_F^2 = tr(x^2)$ denotes the Frobenius norm of $x \in \mathcal{J}$. It is worth noting that for any $(x, s) \in \mathcal{F}$, we have

$$x \circ s = w(t) \Leftrightarrow v^2 = \frac{w(t)}{t} \Leftrightarrow \delta(x, s; t) = 0.$$

We now outline the method.

Algorithm : Full – NT step IPM for WHLCP

Input

An accuracy parameter $\varepsilon > 0$;

A threshold parameter $0 < \tau < 1$;

An update parameter $0 < \theta < 1$;

Let $(x^0, s^0) \in \mathcal{F}^0$ with $\delta(x^0, s^0; t^0) \leq \tau$, where $t^0 = \frac{tr(x^0 \circ s^0)}{n}$;

Set $k = 0$;

while $\|w - x^k \circ s^k\| > \varepsilon$ **do**;

Set $t^{k+1} = (1 - \theta)t^k$;

Obtain the search direction $(\Delta x^k, \Delta s^k)$ by solving (2.8) and using (2.7);

Set $(x^{k+1}, s^{k+1}) = (x^k, s^k) + (\Delta x^k, \Delta s^k)$;

Set $k := k + 1$;

end while.

3. Analysis of the algorithm

Let $x, s \in \operatorname{int}(\mathcal{K})$ and $t > 0$. Using (2.6), (2.7) and (2.9) we have

$$x^+ = \sqrt{t}P(u)^{1/2}(v + d_x), \qquad s^+ = \sqrt{t}P(u)^{-1/2}(v + d_s). \tag{3.1}$$

Since $P(u)^{1/2}$ and $P(u)^{-1/2}$ are automorphisms of $\operatorname{int}(\mathcal{K})$ [9, Theorem III.2.1 and Proposition III.2.2], x^+ and s^+ will belong to $\operatorname{int}(\mathcal{K})$ if and only if $v + d_x$ and $v + d_s$ belong to $\operatorname{int}(\mathcal{K})$. The following lemma provides a necessary and sufficient condition for the strict feasibility of the iterates after a full-NT step.

Lemma 3. *The full-NT step is strictly feasible iff* $\frac{w(t)}{t} + d_x \circ d_s > 0$.

Proof. We have

$$\begin{aligned}(v + d_x) \circ (v + d_s) &= v^2 + v \circ (d_x + d_s) + d_x \circ d_s \\ &= v^2 + v \circ \big(v^{-1} \circ (\frac{w(t)}{t} - v^2)\big) + d_x \circ d_s \\ &= \frac{w(t)}{t} + d_x \circ d_s, \end{aligned} \tag{3.2}$$

where the second equality is due to the second equation of (2.8). Therefore, $v + d_x > 0$ and $v + d_s > 0$ implies that $\frac{w(t)}{t} + d_x \circ d_s > 0$. For the converse, we define

$$v_x(\alpha) = v + \alpha d_x, \qquad v_s(\alpha) = v + \alpha d_s,$$

where $\alpha \in [0, 1]$. It follows from the second equation of (2.8) that

$$\begin{aligned} v_x(\alpha) \circ v_s(\alpha) &= (v + \alpha d_x) \circ (v + \alpha d_s) \\ &= v^2 + \alpha v \circ (d_x + d_s) + \alpha^2 d_x \circ d_s \\ &= v^2 + \alpha\big(\frac{w(t)}{t} - v^2\big) + \alpha^2 d_x \circ d_s \\ &= (1 - \alpha)v^2 + \alpha\big(\frac{w(t)}{t} + \alpha d_x \circ d_s\big). \end{aligned} \tag{3.3}$$

Since $\frac{w(t)}{t} + d_x \circ d_s > 0$ implies that $d_x \circ d_s > -\frac{w(t)}{t}$. By substituting in (3.3), we get

$$v_x(\alpha) \circ v_s(\alpha) > (1 - \alpha)\big(v^2 + \alpha\frac{w(t)}{t}\big).$$

Since $v^2 \in \operatorname{int}(\mathcal{K})$, we have $v^2 + \alpha\frac{w(t)}{t} \in \operatorname{int}(\mathcal{K})$. Hence,

$$v_x(\alpha) \circ v_s(\alpha) > (1 - \alpha)\big(v^2 + \alpha\frac{w(t)}{t}\big) \geq 0, \text{ for all } \alpha \in [0, 1],$$

this means that by [21, Lemma 2.15] $\det(v_x(\alpha))$ and $\det(v_s(\alpha))$ do not vanish for $\alpha \in [0,1]$. Since $\det(v_x(0)) = \det(v_s(0)) = \det(v) > 0$, by continuity, $\det(v_x(\alpha))$ and $\det(v_s(\alpha))$ remain positive for all $\alpha \in [0,1]$. Furthermore, by [9, Theorem III.1.2], it follows that all eigenvalues of $v_x(\alpha)$ and $v_s(\alpha)$ remain positive for all $\alpha \in [0,1]$. So we find that all eigenvalues $v_x(1) = v + d_x$ and $v_s(1) = v + d_s$ are non-negative. The proof is complete. $\square$

Corollary 1. *The full-NT step is strictly feasible if* $\|d_x \circ d_s\|_F < \gamma$.

Proof. For each i, we have

$$\begin{aligned}\lambda_i\Big(\frac{w(t)}{t} + d_x \circ d_s\Big) &\geq \lambda_{\min}\Big(\frac{w(t)}{t} + d_x \circ d_s\Big)\\ &\geq \frac{\lambda_{\min}(w(t))}{t} - \|d_x \circ d_s\|_F\\ &\geq \frac{\lambda_{\min}(c)}{t^0} - \|d_x \circ d_s\|_F = \gamma - \|d_x \circ d_s\|_F,\end{aligned}$$

where the second inequality is due to Lemma 14 in [20] and the third inequality follows from the fact that $w(t)/t \geq c/t_0$. By Lemma 3, the full-NT step is strictly feasible if $\frac{w(t)}{t} + d_x \circ d_s > 0$. This certainly holds if $\|d_x \circ d_s\|_F < \gamma$. Therefore, the proof is completed. $\square$

Lemma 4. *The full-NT step is strictly feasible if* $\delta(v) < \frac{2\gamma}{1+\sqrt{1+\sqrt{2}}}$.

Proof. Since $\mathrm{tr}(d_x \circ d_s) \geq 0$, by [11, Lemma 2.13 (ii)], we have

$$\begin{aligned}\|d_x \circ d_s\|_F \leq \frac{1}{2\sqrt{2}}\|d_x + d_s\|_F^2 &= \frac{1}{2\sqrt{2}}\Big\|v^{-1} \circ \big(\frac{w(t)}{t} - v^2\big)\Big\|_F^2\\ &\leq \frac{1}{2\sqrt{2}\lambda_{\min}^2(v)}\Big\|\frac{w(t)}{t} - v^2\Big\|_F^2\\ &= \frac{\delta^2(v)}{2\sqrt{2}\lambda_{\min}^2(v)} \leq \frac{\delta^2(v)}{2\sqrt{2}(\gamma - \delta(v))},\end{aligned} \tag{3.4}$$

where the first equality is due to the second equation of (2.8), the second inequality follows from [18, Lemma 2.9], the second equality holds by the definition of $\delta(v)$, and the last inequality follows from the following

$$\begin{aligned}\delta(v) = \Big\|\frac{w(t)}{t} - v^2\Big\|_F = \sqrt{\sum_{i=1}^{r}\lambda_i^2\big(\frac{w(t)}{t} - v^2\big)} &\geq \Big|\lambda_{\min}\big(\frac{w(t)}{t} - v^2\big)\Big|\\ &\geq \lambda_{\min}\big(\frac{w(t)}{t} - v^2\big) \geq \lambda_{\min}\big(\frac{w(t)}{t}\big) - \lambda_{\min}^2(v)\\ &\geq \lambda_{\min}\big(\frac{c}{t^0}\big) - \lambda_{\min}^2(v) = \gamma - \lambda_{\min}^2(v).\end{aligned}$$

The full-NT step is strictly feasible, by Corollary 1 and (3.4), if

$$\frac{\delta^2(v)}{2\sqrt{2}(\gamma-\delta(v))} < \gamma,$$

or equivalently

$$\delta^2(v) + 2\sqrt{2}\gamma\delta(v) - 2\sqrt{2}\gamma^2 < 0,$$

and it can readily be verified that this inequality holds if

$$\delta(v) < \frac{2\gamma}{1+\sqrt{1+\sqrt{2}}}.$$

The proof is complete. □

Lemma 5. *Let $x, s \in \operatorname{int}(\mathcal{K})$ and $t > 0$, then*

$$\operatorname{tr}(x^+ \circ s^+) \le \operatorname{tr}(w(t)\circ e) + \frac{t\sqrt{r}\delta^2(v)}{2\sqrt{2}(\gamma-\delta(v))}.$$

Proof. According to (3.1) and (3.2) it follows that

$$\begin{aligned}
\operatorname{tr}(x^+ \circ s^+) &= \operatorname{tr}\Big(\sqrt{t}P(u)^{1/2}(v+d_x)\circ\sqrt{t}P(u)^{-1/2}(v+d_s)\Big)\\
&= t\operatorname{tr}\big((v+d_x)\circ(v+d_s)\big) = \operatorname{tr}(e\circ w(t)) + t\operatorname{tr}(d_x\circ d_s)\\
&\le \operatorname{tr}(e\circ w(t)) + t\sqrt{r}\|d_x\circ d_s\|_F\\
&\le \operatorname{tr}(e\circ w(t)) + \frac{t\sqrt{r}\delta^2(v)}{2\sqrt{2}(\gamma-\delta(v))},
\end{aligned}$$

where the last inequality is deduced from (3.4). The proof is complete. □

The quadratic rate of convergence to the target point $(x(t), s(t))$ is fixed in the next lemma when considering full-NT steps. According to (2.6), the v-vector after the step given by

$$v^+ := \frac{P(u^+)^{-1/2}x^+}{\sqrt{t}}\left[= \frac{P(u^+)^{1/2}s^+}{\sqrt{t}}\right], \tag{3.5}$$

where u^+ is the scaling point of x^+ and s^+.

Lemma 6. *Let $\delta(v) < \frac{2\gamma}{1+\sqrt{1+\sqrt{2}}}$, then the full-NT step is strictly feasible and*

$$\delta(v^+) \le \frac{\delta^2(v)}{2\sqrt{2}(\gamma-\delta(v))}.$$

Proof. Since $\delta(v) < \frac{2\gamma}{1+\sqrt{1+\sqrt{2}}}$, from Lemma 4, it follows that $v + d_x$ and $v + d_s$ belong to the int$(\mathcal{K})$. We have

$$\begin{aligned}\delta(v^+) = \left\|\frac{w(t)}{t} - (v^+)^2\right\|_F &= \left\|\frac{w(t)}{t} - P(v+d_x)^{1/2}(v+d_s)\right\|_F \\ &= \left\|\frac{w(t)}{t} - (v+d_x)\circ(v+d_s)\right\|_F \\ &= \left\|\frac{w(t)}{t} - \frac{w(t)}{t} - d_x\circ d_s\right\|_F \\ &= \left\|d_x\circ d_s\right\|_F \leq \frac{\delta^2(v)}{2\sqrt{2}(\gamma-\delta(v))},\end{aligned}$$

where the second equality is due to [21, Proposition 5.9.3], the third equality follows from [20, Lemma 30], the fourth equality is obtained from (3.2) and the inequality is deduced from (3.4). The proof is complete. □

Corollary 2. *If $\delta(v) \leq \left(1 - \frac{1}{2\sqrt{2}}\right)\gamma$, then the full-NT step is strictly feasible and*

$$\delta(v^+) \leq \left(\frac{\delta}{\sqrt{\gamma}}\right)^2.$$

Lemma 7. *Let $t^+ = (1-\theta)t$, where $0 < \theta < 1$. Then,*

$$\delta(x, s; t^+) \leq \frac{\delta(v) + \frac{\theta}{t^0}\|w-c\|_F}{1-\theta}.$$

Proof. Due to (2.2), we have

$$w(t^+) = w(t) + \frac{t\theta}{t^0}(w-c).$$

Therefore, we have

$$\begin{aligned}\delta(x, s; t^+) = \left\|\frac{w(t^+)}{t^+} - \frac{v^2}{1-\theta}\right\|_F &= \left\|\frac{w(t) + \frac{t\theta}{t^0}(w-c)}{(1-\theta)t} - \frac{v^2}{1-\theta}\right\|_F \\ &= \frac{1}{1-\theta}\left\|\left(\frac{w(t)}{t} - v^2\right) + \frac{\theta}{t^0}(w-c)\right\|_F \\ &\leq \frac{1}{1-\theta}\left(\left\|\frac{w(t)}{t} - v^2\right\|_F + \frac{\theta}{t^0}\|w-c\|_F\right) \\ &= \frac{1}{1-\theta}\left(\delta(v) + \frac{\theta}{t^0}\|w-c\|_F\right).\end{aligned}$$

The proof is complete. □

4. Iteration bound

In this section, we determine the values of the barrier parameter θ and the threshold parameter τ to ensure that the proposed algorithm is well-defined; that is, if $\delta(v) \leq \tau$, then $\delta(x^+, s^+; t^+) \leq \tau$. Then, we obtain an upper bound on the number of iterations of the algorithm.
From Lemmas 7 and 6, we deduce that

$$\begin{aligned}\delta(x^+, s^+; t^+) &\leq \frac{1}{1-\theta}\Big(\delta(v^+) + \frac{\theta}{t^0}\|w-c\|_F\Big)\\ &\leq \frac{1}{1-\theta}\Big(\frac{\delta^2(v)}{2\sqrt{2}(\gamma-\delta(v))} + \frac{\theta}{t^0}\|w-c\|_F\Big),\end{aligned}$$

because $\delta(v) \leq \tau < \gamma$, we have

$$\delta(x^+, s^+; t^+) \leq \frac{1}{1-\theta}\Big(\frac{\tau^2}{2\sqrt{2}(\gamma-\tau)} + \frac{\theta}{t^0}\|w-c\|_F\Big).$$

Substituting $t^0 = \frac{\lambda_{\min}(c)}{\gamma}$, due to (2.2), in the last inequality, we get

$$\delta(x^+, s^+; t^+) \leq \frac{1}{1-\theta}\Big(\frac{\tau^2}{2\sqrt{2}(\gamma-\tau)} + \frac{\gamma\theta}{\lambda_{\min}(c)}\|w-c\|_F\Big),$$

if we take $\tau = \frac{\gamma}{2}$, we get

$$\delta(x^+, s^+; t^+) \leq \frac{1}{1-\theta}\Big(\frac{\tau}{2\sqrt{2}} + \frac{2\tau\theta}{\lambda_{\min}(c)}\|w-c\|_F\Big).$$

Therefore, the condition $\delta(x^+, s^+; t^+) \leq \tau$ holds if

$$\frac{1}{1-\theta}\Big(\frac{\tau}{2\sqrt{2}} + \frac{2\tau\theta}{\lambda_{\min}(c)}\|w-c\|_F\Big) \leq \tau$$

or equivalently

$$\frac{1}{1-\theta}\Big(\frac{1}{2\sqrt{2}} + \frac{2\theta}{\lambda_{\min}(c)}\|w-c\|_F\Big) \leq 1.$$

If we take

$$\theta = \frac{\lambda_{\min}(c)}{5(\lambda_{\min}(c) + \|w-c\|_F)}, \tag{4.1}$$

then we have

$$\frac{1}{1-\theta}\Big(\frac{1}{2\sqrt{2}} + \frac{2\theta}{\lambda_{\min}(c)}\|w-c\|_F\Big) \leq \frac{5}{4}\Big(\frac{1}{2\sqrt{2}} + \frac{2}{5}\Big) = 0.9419 < 1.$$

Theorem 1. *If $\tau = \frac{\gamma}{2}$ and θ is defined as in (4.1), then Algorithm finds an ε-approximate solution $(x, s) \in \mathcal{F}^0$ such that $\|w - x \circ s\|_F \leq \varepsilon$ in at most*

$$\left\lceil \frac{5\big(\lambda_{\min}(c) + \|c - w\|_F\big)}{\lambda_{\min}(c)} \log \frac{\frac{\lambda_{\min}(c)}{2} + \|w - c\|_F}{\varepsilon} \right\rceil$$

iterations.

Proof. We have

$$\begin{aligned} \|w - x \circ s\|_F &\leq \|w(t) - x \circ s\|_F + \|w(t) - w\|_F \\ &= t\left\|\frac{w(t)}{t} - v^2\right\|_F + \|w(t) - w\|_F \\ &= t\delta(v) + \|w(t) - w\|_F \\ &\leq \frac{t\gamma}{2} + \frac{t}{t^0}\|c - w\|_F \\ &= \frac{t}{t^0}\Big(\frac{\lambda_{\min}(c)}{2} + \|c - w\|_F\Big). \end{aligned}$$

Therefore, after k iterations, we deduce that $\|w - x \circ s\|_F \leq \varepsilon$ is satisfied if

$$\Big(\frac{\lambda_{\min}(c)}{2} + \|c - w\|_F\Big)(1 - \theta)^k \leq \varepsilon.$$

Taking logarithms on both sides, we obtain

$$k \log(1 - \theta) \leq -\log \frac{\frac{\lambda_{\min}(c)}{2} + \|c - w\|_F}{\varepsilon},$$

and using $\log(1 - \theta) \leq -\theta$ for $\theta \in (0, 1)$, we observe that the above inequality holds if

$$-k\theta \leq -\log \frac{\frac{\lambda_{\min}(c)}{2} + \|c - w\|_F}{\varepsilon}.$$

This gives

$$k \geq \frac{1}{\theta} \log \frac{\frac{\lambda_{\min}(c)}{2} + \|c - w\|_F}{\varepsilon},$$

which completes the proof. □

5. Concluding remarks

Based on Euclidean Jordan algebras, we extended the full-Newton step IPM to WLCP on the cone symmetric. It uses full steps, and the iterates always lie in the quadratic convergence neighborhood. Finally, the order of the iteration bound coincides with the currently best-known iteration bound for this type of problem.

Conflict of Interest: The authors declare that they have no conflict of interest.

Data Availability: Data sharing is not applicable to this article as no data sets were generated or analyzed during the current study.

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