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Research Article

On the essential graph of a poset

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Abstract: Let $(P, \leq)$ be an atomic partially ordered set (briefly, a poset) with a minimum element 0, and let $\mathcal{I}(P)$ be the set of all nontrivial ideals of P. The essential graph of P, denoted by $G_e(P)$, is an undirected, simple graph with the vertex set $\mathcal{I}(P)$ and two distinct vertices $I, J \in \mathcal{I}(P)$ are adjacent in $G_e(P)$ if and only if $I \cup J$ is an essential ideal of P. We study the connections between the graph-theoretic properties of this graph and the algebraic properties of a poset. We prove that $G_e(P)$ is connected with diameter at most three. Furthermore, all posets are characterized based on the diameters of their essential graphs. Also, all posets with planar $G_e(P)$ are classified. Among other results, the clique number and chromatic number of $G_e(P)$ are determined.

Keywords: poset, essential graph, diameter, planar, clique number.

AMS Subject classification: 06A07, 05C25

1. Introduction

Recently, a major part of research in algebraic combinatorics has been devoted to the application of graph theory and combinatorics in abstract algebra. There are a lot of papers which apply combinatorial methods to obtain algebraic results in poset theory (for example see [2–5] and [7]).

Let $(P, \leq)$ be a poset with a least element 0. The set of all ideals of P is denoted by $\mathcal{J}(P)$ and $\mathcal{I}(P) = \mathcal{J}(P) \setminus \{\{0\}, P\}$. For every element x of P, the ideal $(x] := \{y \in$

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$P|y \leq x\}$(the filter $[x) := \{y \in P|x \leq y\}$) is called *principal* whose generator is x. Let $x, y \in P$. We say that y *covers* x in P if $x < y$ and no element in P lies strictly between x and y. If y covers x, then we write $x \sqsubset y$. Any cover of 0 in P is called an *atom*. The set of all atoms of P is denoted by $Atom(P)$. The poset P is called *atomic* if, for every non-zero element $x \in P$, we have $(x] \cap Atom(P) \neq \emptyset$. A nonzero ideal I of P is called *essential* if I has a nonzero intersection with any nonzero ideal of P. The set of all principal ideals generated by $Atom(P)$ and $P \setminus Atom(P)$ are denoted by $\mathcal{A}(P)$ and $\mathcal{B}(P)$, respectively. For any undefined notation or terminology in poset theory, we refer the reader to [8, 9].

Let $G = (V, E)$ be a graph, where $V = V(G)$ is the set of vertices and $E = E(G)$ is the set of edges. The degree of a vertex v is denoted by $deg(v)$. The maximum degree and the minimum degree of G are denoted by $\Delta(G)$ and $\delta(G)$, respectively. The set of all adjacent vertices to $v \in V$ is denoted by $N(v)$. If u and v are two adjacent vertices of G, then we write $u - v$. We say that G is a *connected graph* if there is a path between each pair of distinct vertices of G. For two vertices x and y, let $d(x, y)$ denote their distance. The diameter of G is denoted by $diam(G)$. A complete graph of order n and a complete bipartite graph with par sizes m, n are denoted by K_n and $K_{m,n}$, respectively. A graph G is said to be *planar* if it can be drawn in the plane such that its edges intersect only at their endpoints. The *subdivision* of G is a graph obtained from G by subdividing some of the edges, that is, by replacing the edges by paths having at most their endvertices in common. A *clique* of G is a maximal complete subgraph of G and the number of vertices in the largest clique of G, denoted by $\omega(G)$, is called the *clique number* of G. For a graph G, let $\chi(G)$ denote the *vertex chromatic number* of G, i.e., the minimal number of colors which can be assigned to the vertices of G in such a way that every two adjacent vertices have different colors. Note that for every graph G, $\omega(G) \leq \chi(G)$. A graph G is said to be *weakly perfect* if $\omega(G) = \chi(G)$. The graph $H = (V_0, E_0)$ is a subgraph of G if $V_0 \subseteq V$ and $E_0 \subseteq E$. Moreover, H is called an *induced subgraph by* V_0, denoted by $G[V_0]$, if $V_0 \subseteq V$ and $E_0 = \{\{u, v\} \in E \,|\, u, v \in V_0\}$. Also, a set $S \subseteq V(G)$ is a *dominating set* if every vertex in $V(G)$ is either in S or is adjacent to a vertex in S. The *domination number* of G denoted by $\gamma(G)$ is the minimum cardinality among the dominating sets of G. For any undefined notation or terminology in graph theory, we refer the reader to [1, 10].

Throughout this paper P is a nontrivial atomic poset with a minimum element 0 and $Atom(P) = \{a_1, a_2, \ldots, a_n, \ldots\}$ (at most countably infinite). The *essential graph of* P, denoted by $G_e(P)$, is an undirected and simple graph with the vertex set $\mathcal{I}(P)$ and two distinct vertices $I, J \in \mathcal{I}(P)$ are adjacent in $G_e(P)$ if and only if $I \cup J$ is an essential ideal of P. The idea of essential graph of a ring was first introduced and studied in [6]. Motivated by [6], we define and study the essential graph of a poset. In this paper, we study some connections between the graph-theoretic properties of this graph and some algebraic properties of a poset. We prove that $G_e(P)$ is connected and $diam(G_e(P)) \leq 3$. Furthermore, all posets are characterized based on the diameters of their essential graphs. Also, planar essential graphs are characterized. Finally, some additional parameters of $G_e(P)$ are studied.

2. Connectivity and Diameter of $G_e(P)$

In this section, it is shown that $G_e(P)$ is always a connected graph and $diam(G_e(P)) \leq 3$. Moreover, all posets are characterized based on the diameters of their essential graphs. The following lemma plays a key role in this paper.

Lemma 1. *Let $(P, \leq)$ be a poset and $I, J \in \mathcal{I}(P)$. Then I and J are adjacent in $G_e(P)$ if and only if $Atom(P) \subseteq I \cup J$.*

Proof. First suppose that I and J are adjacent in $G_e(P)$. If $Atom(P) \nsubseteq I \cup J$, then there exists $a \in Atom(P)$ such that $a \notin I \cup J$. This implies that $\{0, a\} \nsubseteq I \cup J$ and so $I \cup J$ is not an essential ideal of P, a contradiction.
Conversely, suppose that $Atom(P) \subset I \cup J$, for some $I, J \in \mathcal{I}(P)$ with $I \neq J$ and J' is an arbitrary ideal of P. Since P is atomic, there exists $a \in Atom(P)$ such that $a \in J'$. This means that $\{0, a\} \subseteq (I \cup J) \cap J'$ and thus $I \cup J$ is an essential ideal of P. Hence I and J are adjacent in $G_e(P)$. □

We are now ready to prove that $G_e(P)$ is connected with a diameter of at most 3.

Theorem 1. *Let $(P, \leq)$ be a poset. Then $G_e(P)$ is connected and $diam(G_e(P)) \leq 3$.*

Proof. Let I and J be two distinct vertices of $G_e(P)$. We find a path of length at most 3 between I and J. To see this, consider the following cases:

Case 1. $P = Atom(P) \cup \{0\}$. If $I \cup J = P$, then by Lemma 1, I is adjacent to J. So let $I \cup J \neq P$. This implies that $Atom(P) \nsubseteq I \cup J$. If $I \subset J$, then put $K = (P \setminus I) \cup \{0\}$. It is not hard to see that K is a proper ideal of P and $K \cup I = K \cup J = P$. Hence by Lemma 1, K is adjacent to both of I and J. The situation $J \subset I$ is similar. Let $I \nsubseteq J$, $J \nsubseteq I$ and put $L_1 = ((P \setminus I) \cup (I \cap J))$ and $L_2 = ((P \setminus J) \cup (I \cap J))$. We claim that L_1, L_2 are proper ideals of P. Since $P = Atom(P) \cup \{0\}$, it is enough to show that $P \neq L_1$ and $P \neq L_2$. Clearly, $L_1 = ((P \setminus I) \cup (I \cap J)) = I^c \cup (I \cap J) = I^c \cup J$. Since $I \nsubseteq J$, there exists $a \in I$, such that $a \notin I^c$ and $a \notin J$. This implies that $a \notin I^c \cup J$ and thus $I^c \cup J \neq P$. Similarly, $P \neq L_2$ and so the claim is proved. Obviously, $L_1 \cup I = P$ and $L_2 \cup J = P$. By Lemma 1, I (J) is adjacent to L_1 (L_2). Moreover, $L_1 \cup L_2 = (I^c \cup (I \cap J)) \cup (J^c \cup (I \cap J)) = (I^c \cup J) \cup (J^c \cup I) = P$. Thus by Lemma 1, L_1 and L_2 are adjacent to each other. Therefore, $I - L_1 - L_2 - J$ is a path of length 3 between I and J.

Case 2. $P \neq Atom(P) \cup \{0\}$. If $I \cup J = P$, then by Lemma 1, there is nothing to prove. So let $I \cup J \neq P$ and put $K = Atom(P) \cup \{0\}$. Clearly, K is a proper ideal of P, $(I \cup K) \cap L \neq \{0\}$ and $(J \cup K) \cap L \neq \{0\}$, for every $L \in \mathcal{I}(P)$. Therefore, $I - K - J$ is a path between I and J. □

The following theorem gives a necessary and sufficient condition under which $diam(G_e(P)) = 1$.

Theorem 2. *Let $(P, \leq)$ be a poset. Then $G_e(P)$ is complete if and only if one of the following statements hold.*
(1) $|Atom(P)| = 1$.
(2) $|Atom(P)| = 2$ and either $P = Atom(P) \cup \{0\}$ or $Atom(P) \subset (x]$, for every $x \in P \setminus Atom(P)$.

Proof. Suppose that $G_e(P)$ is complete and $|Atom(P)| \neq 1$, we show that (2) is hold. Assume to the contrary, $\{a_1, a_2, a_3\}$ is a subset of $Atom(P)$. Then $\{0, a_1\}, \{0, a_1, a_2\} \in \mathcal{I}(P)$. This means that $Atom(P) \nsubseteq \{0, a_1\} \cup \{0, a_1, a_2\}$ and thus $\{0, a_1\}$ is not adjacent to $\{0, a_1, a_2\}$ in $G_e(P)$, a contradiction. So $|Atom(P)| = 2$. Suppose that $P \neq Atom(P) \cup \{0\}$ and $Atom(P) \nsubseteq (x]$, for some $x \in P \setminus Atom(P)$. Hence $a_1 \notin (x]$ or $a_2 \notin (x]$. With no loss of generality, assume that $a_2 \notin (x]$. By Lemma 1, $\{0, a_1\}$ is not adjacent to $(x]$, a contradiction.

Conversely, if $|Atom(P)| = 1$, then by Lemma 1, $G_e(P)$ is complete. If $|Atom(P)| = 2$ and $P = Atom(P) \cup \{0\}$, then $G_e(P) = K_2$. Finally, if $|Atom(P)| = 2$, $P \neq Atom(P) \cup \{0\}$ and $Atom(P) \subset (x]$, for every $x \in P \setminus Atom(P)$, then $Atom(P) \subset I \cup J$, for every two distinct vertices $I, J \in \mathcal{I}(P)$. Now, by Lemma 1, $G_e(P)$ is complete. □

Next, all posets whose essential graphs have diameter 3 are characterized.

Theorem 3. *Let $(P, \leq)$ be a poset. Then $diam(G_e(P)) = 3$ if and only if $P = Atom(P) \cup \{0\}$ and $|Atom(P)| \geq 3$.*

Proof. First suppose that $diam(G_e(P)) = 3$. If $P \neq Atom(P) \cup \{0\}$, then $K = Atom(P) \cup \{0\}$ is a proper ideal of P and so by Lemma 1, K is adjacent to every other vertex. This contradicts the hypothesis $diam(G_e(P)) = 3$. Hence $P = Atom(P) \cup \{0\}$. Also, this shows that $|Atom(P)| \geq 3$.

Conversely, suppose that $P = Atom(P) \cup \{0\}$ and $|Atom(P)| \geq 3$. By Theorem 2, $diam(G_e(P)) \geq 2$. Let $diam(G_e(P)) = 2$ and $I_1 = \{0, a_1\}$ and $I_2 = \{0, a_2\}$ be two vertices of $G_e(P)$. Then there exists $K \in \mathcal{I}(P)$ such that $Atom(P) \subset K \cup I_1$ and $Atom(P) \subset K \cup I_2$. Hence $Atom(P) \subset K$, a contradiction. Thus $d(I_1, I_2) \geq 3$, i.e., $diam(G_e(P)) \geq 3$. It follows from Theorem 1 that $diam(G_e(P)) = 3$. □

In light of Theorems 2 and 3, we state the last result of this section.

Theorem 4. *Let $(P, \leq)$ be a poset. Then $diam(G_e(P)) = 2$ if and only if one of the following statements holds.*
(1) $P \neq Atom(P) \cup \{0\}$, $|Atom(P)| = 2$ and $Atom(P) \nsubseteq (x]$, for some $x \in P \setminus Atom(P)$.
(2) $P \neq Atom(P) \cup \{0\}$ and $|Atom(P)| \geq 3$.

3. Planar Essential Graphs of Posets

One of the most of important invariant in graph theory is the planarity. Our focus in this section is on the planarity of essential graphs of posets. First, we need a celebrated theorem due to Kuratowski.

Theorem 5. [1, Theorem 10.30] *A graph is planar if and only if it contains no subdivision of either K_5 or $K_{3,3}$.*

In what follows, we first study the case $P = Atom(P) \cup \{0\}$.

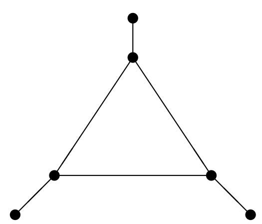
Theorem 6. *Let $(P, \leq)$ be a poset and $P = Atom(P) \cup \{0\}$. Then $G_e(P)$ is planar if and only if $|Atom(P)| \leq 3$.*

Proof. Suppose that $G_e(P)$ is planar. If $|Atom(P)| \geq 4$, then we consider the following cases.
Case 1. $|Atom(P)| \geq 5$. Let $S = \{I_i \in \mathcal{I}(P) | \ I_i = P \setminus \{a_i\}\}$. Lemma 1 implies that $G_e(P)[S]$ is complete. Hence $G_e(P)$ contains a subdivision of K_5, a contradiction.
Case 2. $|Atom(P)| = 4$. Put $I_1 = P \setminus \{a_2\}$, $I_2 = \{0, a_1, a_3\}$, $I_3 = \{0, a_1, a_4\}$, $J_1 = P \setminus \{a_1\}$, $J_2 = \{0, a_2, a_3\}$, $J_3 = \{0, a_2, a_4\}$, $L_1 = P \setminus \{a_3\}$ and $L_2 = P \setminus \{a_4\}$. Then by Lemma 1, it is not hard to check that the vertices of the set $\{I_1, I_2, I_3\}$, the vertices of the set $\{J_1, J_2, J_3\}$ together with the paths $I_2 - L_1 - J_2$ and $I_3 - L_2 - J_3$ form a subdivision of $K_{3,3}$, a contradiction.
Conversely, suppose that $|Atom(P)| \leq 3$. Indeed, we have the following cases.

Case 1. $|Atom(P)| = 1$. In this case $G_e(P) = K_1$ and thus it is planar.

Case 2. $|Atom(P)| = 2$. In this case $G_e(P) = K_2$ and thus it is planar.

Case 3. $|Atom(P)| = 3$. In this case the following figure shows that $G_e(P)$ is planar. □



$G_e(P)$ with $|Atom(P)| = 3$

Next we study the case $P \neq Atom(P) \cup \{0\}$. First, we show that if $|Atom(P)| \geq 4$, then $G_e(P)$ is not planar.

Theorem 7. *Let $(P, \leq)$ be a poset and $P \neq Atom(P) \cup \{0\}$. If $|Atom(P)| \geq 4$, then $G_e(P)$ is not planar.*

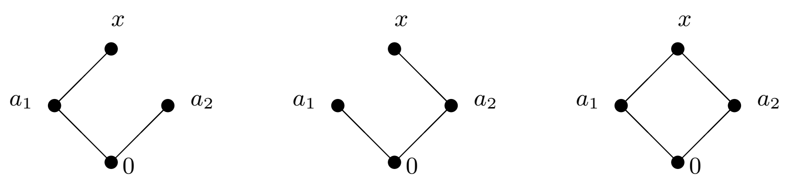
Proof. Put $I_i = (Atom(P) \setminus \{a_i\}) \cup \{0\}$, for every $1 \leq i \leq 4$ and $I_5 = Atom(P) \cup \{0\}$. Clearly, I_i is a proper ideal of P, for every $1 \leq i \leq 5$. The subgraph induced by the set $S = \{I_1, I_2, I_3, I_4, I_5\}$ is complete, by Lemma 1. Thus $G_e(P)$ is not planar. □

Remaining results of this section are devoted to investigate the cases $|Atom(P)| = 1$, $|Atom(P)| = 2$ and $|Atom(P)| = 3$.

Theorem 8. *Let $(P, \leq)$ be a poset, $P \neq Atom(P) \cup \{0\}$ and $|Atom(P)| = 1$. Then $G_e(P)$ is planar if and only if $\mathcal{I}(P) \leq 4$.*

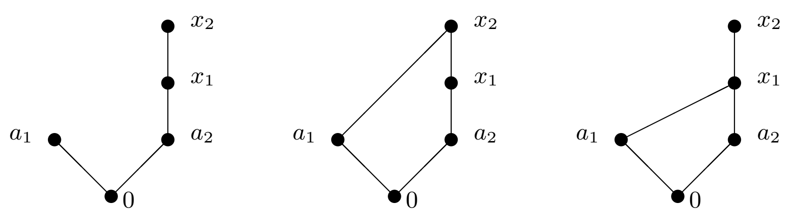
Proof. The result follows from Theorems 2 and 5. □

Remark 1. Let $(P, \leq)$ be a poset with $P \neq Atom(P) \cup \{0\}$, $|Atom(P)| = 2$ and $M = \{0 \neq x \in P \setminus Atom(P) |\ (x] \cup Atom(P) \neq P\}$. Then the following statements hold.
(1) If $|M| = 0$, then the van diagram of P is one of the following figures.

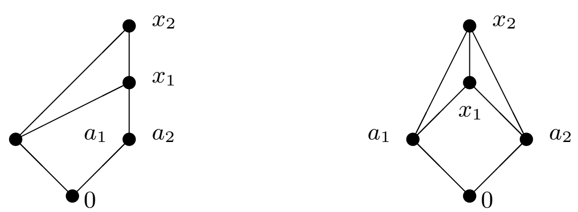


(i) (ii) (iii)

(2) If $|M| = 1$, then the van diagram of P is one of the following figures.



(i) (ii) (iii)



(iv) (v)

Theorem 9. *Let $P \neq Atom(P) \cup \{0\}$ be a poset with $|Atom(P)| = 2$ and let $M = \{0 \neq x \in P \setminus Atom(P) | \ (x] \cup Atom(P) \neq P\}$. Then the following statements hold.*
(1) *If $|M| = 0$, then $G_e(P)$ is planar.*
(2) *If $|M| = 1$, then $G_e(P)$ is not planar if and only if $(x] \neq P$, for some $x \in P \setminus (Atom(P) \cup M)$.*
(3) *If $|M| \geq 2$, then $G_e(P)$ is not planar.*

Proof. (1) Suppose that $|M| = 0$. Then clearly, $|P| = 4$ and thus by part (1) of Remark 1, one may easily see that $|V(G_e(P))| \leq 4$. Therefore $G_e(P)$ is planar.
(2) Suppose that $|M| = 1$. Then by part (2) of Remark 1, the van diagram of P is one of the figures $(i), (ii), (iii), (iv), (v)$. If the van diagram of P is one of $(ii), (iii), (iv), (v)$, then $|\mathcal{I}(P)| \leq 5$ and $G_e(P)$ is not complete. Thus $G_e(P)$ is planar, by Kuratowski's Theorem. If the van diagram of P is figure (i), then let
$I_1 = \{0, a_1\}$,
$I_2 = \{0, a_2\}$,
$I_3 = \{0, a_1, a_2\}$,
$I_4 = \{0, a_1, a_2, x_1\}$,
$I_5 = \{0, a_2, x_1\}$,
$I_6 = \{0, a_2, x_1, x_2\}$.
Clearly, $I_1, I_2, I_3, I_4, I_5, I_6$ are proper ideals of P. By Lemma 1, the vertices of the set $\{I_2, I_5, I_6\}$ and the vertices of the set $\{I_1, I_3, I_4\}$ form $K_{3,3}$ and thus $G_e(P)$ is not planar.
(3) Suppose that $|M| \geq 2$ and let
$I_1 = \{0, a_1\}$,
$I_2 = \{0, a_2\}$,
$I_3 = \{0, a_1, a_2\}$,
$I_4 = (x_1] \cup Atom(P)$,
$I_5 = (x_2] \cup Atom(P)$,
where $x_1, x_2 \in M$. By Lemma 1, $G_e(P)$ contains K_5 and thus $G_e(P)$ is not planar. □

The next result completes the study of planarity in $G_e(P)$.

Theorem 10. *Let $(P, \leq)$ be a poset with $|Atom(P)| = 3$ and let $M = \{0 \neq x \in P \setminus Atom(P) | \ (x] \cup Atom(P) \neq P\}$. If $P \neq Atom(P) \cup \{0\}$, then the following statements hold.*
(1) *$|M| = 0$ if and only if $P = \{0, a_1, a_2, a_3, x\}$, where $x \in P \setminus Atom(P)$.*
(2) *If $|M| = 0$, then $G_e(P)$ is planar if and only if $|(x] \cap Atom(P)| = 3$, where $x \in P \setminus (Atom(P))$.*
(3) *If $|M| \geq 1$, then $G_e(P)$ is not planar.*

Proof. (1) It is clear.
(2) Suppose that $G_e(P)$ is planar and $|(x] \cap Atom(P)| \neq 3$, for some $x \in P \setminus (Atom(P))$. Consider the following cases:

Case 1. $|(x] \cap Atom(P)| = 1$. With no loss of generality, suppose that $(x] \cap Atom(P) = \{a_1\}$. Let
$I_1 = \{0, a_1, a_3\}$,
$I_2 = \{0, a_2, a_3\}$,
$I_3 = \{0, a_1, a_2, a_3\}$,
$I_4 = \{0, a_1, a_2, x\}$,
$I_5 = \{0, a_1, a_3, x\}$,
$I_6 = \{0, a_1, a_2\}$.
Clearly, every I_i, $1 \leq i \leq 6$, is a proper ideal of P. It is not hard to see that

$$I_1, I_2, I_3, I_4, I_5$$

together with the path

$$I_1 - I_6 - I_5$$

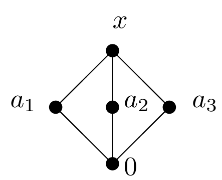
form a subdivision of K_5, a contradiction.
Case 2. $|(x] \cap Atom(P)| = 2$. With no loss of generality, suppose that $(x] \cap Atom(P) = \{a_1, a_2\}$. Let
$I_1 = \{0, a_1, a_2\}$
$I_2 = \{0, a_1, a_3\}$
$I_3 = \{0, a_2, a_3\}$
$I_4 = \{0, a_1, a_2, a_3\}$
$I_5 = \{0, a_1, a_2, x\}$
$I_6 = \{0, a_3\}$
Clearly, every I_i, $1 \leq i \leq 6$, is a proper ideal of P. Then the vertices

$$I_1, I_2, I_3, I_4, I_5$$

together with the path

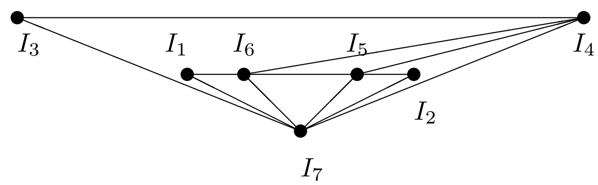
$$I_1 - I_6 - I_5$$

form a subdivision of K_5, a contradiction. Thus by Cases 1 and 2, $|(x] \cap Atom(P)| = 3$.
Conversely, suppose that $|(x] \cap Atom(P)| = 3$. Then the van diagram of P is as follows:



and $\mathcal{I}(P) = \{I_1, I_2, I_3, I_4, I_5, I_6, I_7\}$, where
$I_1 = \{0, a_1\}$,
$I_2 = \{0, a_2\}$,

$I_3 = \{0, a_3\}$,
$I_4 = \{0, a_1, a_2\}$,
$I_5 = \{0, a_1, a_3\}$,
$I_6 = \{0, a_2, a_3\}$,
$I_7 = \{0, a_1, a_2, a_3\}$.
Now, the following figure shows that $G_e(P)$ is planar.



(3) Suppose that $|M| \geq 1$ and let
$I_1 = \{0, a_1, a_2\}$,
$I_2 = \{0, a_1, a_3\}$,
$I_3 = \{0, a_2, a_3\}$,
$I_4 = \{0, a_1, a_2, a_3\}$,
$I_5 = (x] \cup Atom(P)$,
where $x \in M$. By Theorem 1, $G_e(P)$ contains K_5 and thus $G_e(P)$ is not planar. $\square$

4. Some Further Results on $G_e(P)$

In this section, we first study the coloring of $G_e(P)$. It is shown that $G_e(P)$ is weakly perfect and the clique number of $G_e(P)$ is given. Moreover, the domination number of $G_e(P)$ is studied. Finally, the maximum and minimum degree of $G_e(P)$ are determined.
First, the clique number of $G_e(P)$ is determined, if $P = Atom(P) \cup \{0\}$.

Theorem 11. *Let $(P, \leq)$ be a poset and $P = Atom(P) \cup \{0\}$. Then*

$$\omega(G_e(P)) = \chi(G_e(P)) = |Atom(P)|.$$

Proof. Let $S = \{I \in \mathcal{I}(P) |\ I = P \setminus \{a_i\}, for\, some\, a_i\}$ and $I_i, I_j \in S$ with $i \neq j$. Since $I_i \cup I_j = P$, we deduce from Lemma 1, I_i is adjacent to I_j. Hence $G_e(P)[S]$ is a complete graph and so

$$\chi(G_e(P)) \geq \omega(G_e(P)) \geq |Atom(P)|.$$

To complete the proof, we show that $\chi(G_e(P)) \leq |Atom(P)|$. To see this, define the coloring $f : V(G_e(P)) \to \{i |\ i \geq 1\}$ with $f(I) = min\{i |\ a_i \notin I\}$, where $I \in V(G_e(P))$. Now, we show that f is a proper coloring on $V(G_e(P))$. Suppose to the contrary, I and J are two adjacent vertices and $f(I) = f(J) = i$. Thus $a_i \notin I \cup J$, i.e., I and J are not adjacent, a contradiction. $\square$

It is worth mentioning that one may consider $G_e(P)$ as a complete multipartite graph based on the previous result.
Next, we study $\omega(G_e(P))$, in the case where $P \neq Atom(P) \cup \{0\}$.

Theorem 12. *Let $(P, \leq)$ be a poset and $P \neq Atom(P) \cup \{0\}$. If $M = \{I \in \mathcal{I}(P) | \ Atom(P) \subseteq Atom(P)\}$, then*

$$\omega(G_e(P)) = \chi(G_e(P)) = |Atom(P)| + |M|.$$

Proof. Let $S = \{I \in \mathcal{I}(P) | \ I = P \setminus \{a_i\}, for\, some\, a_i\}$ and $C = S \cup M$. By Lemma 1 and Theorem 11, C is a maximal clique of $G_e(P)$. Obviously, every vertex contained in C needs a different color. In a similar manner to the proof of Theorem 11, one can color vertices out of C. □

The domination number of $G_e(P)$ is studied in the next result.

Theorem 13. *Let $(P, \leq)$ be a poset. Then the following statements hold.*
(1) $\gamma(G_e(P)) = 1$ *if and only if one of the following statements hold.*
(i) $|Atom(P)| = 1$.
(ii) $P = \{0, a_1, a_2\}$.
(iii) $P \neq Atom(P) \cup \{0\}$.
(2) $\gamma(G_e(P)) = |Atom(P)|$ *if and only if* $P = Atom(P) \cup \{0\}$ *and* $|Atom(P)| \geq 3$ *or* $P \neq Atom(P) \cup \{0\}$ *and* $|Atom(P)| = 1$.

Proof. (1) Clearly, if (i) or (ii) holds, then $\gamma(G_e(P)) = 1$. Also, if (iii) holds, then the vertex $I = Atom(P) \cup \{0\}$ is adjacent to every other vertex. Hence $\gamma(G_e(P)) = 1$. Conversely, suppose that $\gamma(G_e(P)) = 1$. Thus $diam(G_e(P)) \leq 2$. Now, the proof follows from by Theorems 2 and 4.
(2) If $P \neq Atom(P) \cup \{0\}$ and $|Atom(P)| = 1$, then by Theorem 2, $\gamma(G_e(P)) = |Atom(P)| = 1$. So let $P = Atom(P) \cup \{0\}$ and $|Atom(P)| \geq 3$. Let $I_i = \{0, a_i\}$ and $J_i = P \setminus \{a_i\}$, for very $i \geq 1$. Clearly, $deg(I_i) = 1$ and I_i is adjacent to J_i. This implies that every minimum dominating set of $G_e(P)$ must contain the set $S = \{J_i \in \mathcal{I}(P) | \ 1 \leq i \leq n\}$. So

$$\gamma(G_e(P)) \geq |S| = |Atom(P)|.$$

Now, it is straightforward to show that $N[S] = V(G_e(P))$ and thus $\gamma(G_e(P)) = |Atom(P)|$.
The converse is easily obtained. □

Suppose that $P = Atom(P) \cup \{0\}$. To determine the maximum and minimum degrees in $G_e(P)$, we need the following result.

Theorem 14. *Let $(P, \leq)$ be a poset and $P = Atom(P) \cup \{0\}$. Then the following statements hold.*
(1) *If $|Atom(P)| = n$, for some positive integer n, then $|\mathcal{I}(P)| = 2^n - 2$.*
(2) *If $|I| = m$, for some positive integer m, then $deg(I) = 2^{m-1} - 1$.*

Proof. (1) Since $P = Atom(P) \cup \{0\}$, the number of ideals generated by r atoms is equal to $\binom{n}{r}$ and so

$$|\mathcal{I}(P)| = \Sigma_{r=1}^{n-1} \binom{n}{r} = 2^n - 2.$$

(2) If $|I| = m$, then the number of atoms contained in I is $m - 1$. By Lemma 1, $(P \setminus I) \subseteq J$, for every J adjacent to I. Thus the number of proper ideals containing $(P \setminus I)$ is equal to

$$deg(I) = \Sigma_{r=0}^{m-2} \binom{m-1}{r} = 2^{m-1} - 1.$$

□

We conclude this paper with the following corollary.

Corollary 1. *Let $(P, \leq)$ be a poset and $P = Atom(P) \cup \{0\}$. If $|Atom(P)| = n$ for some positive integer n, then the following statements hold.*
(1) $\Delta(G_e(P)) = 2^{n-1} - 1$.
(2) $\delta(G_e(P)) = 1$.

Proof. The proof follows from Theorems 2 and 14. □

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