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Research Article

Edge adding stability of graphs

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Abstract: For an arbitrary invariant $\rho(G)$ of a graph G the ρ-edge adding stability number $eas_\rho(G)$ is the minimum number of edges of the complement $\overline{G}$ whose addition to G results in a graph $H \supseteq G$ with $\rho(H) \neq \rho(G)$. If such an edge set does not exist, then we set $eas_\rho(G) = \infty$. In the first part of this paper we give some general results for $eas_\rho(G)$. We prove among others a Gallai's theorem type result for invariants that are based on the ρ-edge adding stability number.

Keywords: edge stability number, edge adding stability number, graph invariant.

AMS Subject classification: 05C15, 05C35, 05C70

1. Introduction

Let $G = (V(G), E(G))$ be a finite simple graph and $\mathcal{I}$ be the class of all these graphs. An empty graph is a graph with empty edge set. The graph $G - E'$ with $E' \subseteq E(G)$ is the graph $(V(G), E(G) \setminus E')$. The graph $G + E''$ with $E'' \subseteq \binom{V(G)}{2}$ is the graph $(V(G), E(G) \cup E'')$. We consider invariants of graphs and restrict ourselves to non-negative invariant values.

Definition 1. A (graph) invariant $\rho(G)$ is a function $\rho : \mathcal{I} \to \mathbb{R}_0^+ \cup \{\infty\}$ with $\rho(G_1) = \rho(G_2)$ if G_1 is isomorphic to G_2. An invariant is real-valued if its codomain is $\mathbb{R}_0^+$. An invariant $\rho(G)$ is monotone increasing if $H \subseteq G$ implies $\rho(H) \leq \rho(G)$, and monotone decreasing if $H \subseteq G$ implies $\rho(H) \geq \rho(G)$; $\rho(G)$ is monotone if it is monotone increasing or monotone decreasing. If the conditions hold for certain classes of subgraphs (for example, induced or spanning subgraphs), then we say that $\rho(G)$ is monotone (increasing or decreasing) with respect to the class.

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For example, the maximum degree $\Delta(G)$ of a graph G is monotone increasing. The minimum degree $\delta(G)$ is not monotone, but monotone increasing with respect to spanning subgraphs. The independence number $\alpha(G)$ is not monotone, but it is monotone increasing with respect to induced subgraphs and monotone decreasing with respect to spanning subgraphs.

It is an interesting topic to determine the stability of an arbitrary invariant $\rho(G)$ of a graph G with respect to specific graph operations, that is, to determine if (or when) the value of the invariant $\rho(H)$ differs from the value of $\rho(G)$ where H is obtained by performing (successively) the given operation on G. This topic was introduced in the 1980s (see e.g. [6, 9, 10, 17]) and again gained interest recently. For example, the stability with respect to removing vertices was studied e.g. in [2, 5, 15], and the stability with respect to subdividing edges in [12, 14].

The stability with respect to removing edges from G leads to the following invariant which was discussed in several papers, for example in [1, 3, 4, 6, 7, 11–17].

Definition 2. The ρ-edge (removing) stability number $es_\rho(G)$ of a graph G is the minimum number of edges of G whose removal results in a graph $H \subseteq G$ with $\rho(H) \neq \rho(G)$. If such an edge set does not exist, then we set $es_\rho(G) = \infty$.

If instead of removing edges we add edges between non-adjacent vertices, then we obtain the following invariant. The complement $\overline{G}$ of a graph $G = (V, E)$ is the graph $\overline{G} = \left(V, \binom{V}{2} \setminus E\right)$.

Definition 3. The ρ-edge adding stability number $eas_\rho(G)$ of G is the minimum number of edges of the complement $\overline{G}$ whose addition to G results in a graph $H \supseteq G$ with $\rho(H) \neq \rho(G)$. If such an edge set does not exist, then we set $eas_\rho(G) = \infty$.

We present general results for the edge adding stability number. If $\rho(G)+\rho(\overline{G}) = f(n)$ for every graph G where $\rho(G)$ is an arbitrary invariant and f a function of the order $n = n(G)$ of G, then it holds that $eas_\rho(G) = es_\rho(\overline{G})$. We extend this result to the sum of two invariants and to other operations. Moreover, we obtain exact results for the edge adding stability number of the join of disjoint graphs.

Gallai proved in 1959 [8] results on independence (edge independence) and vertex covering (edge covering) of graphs. We prove a corresponding result for invariants that are based on the edge adding stability number $eas_\rho(G)$.

2. General results for the edge adding stability number

In this section we provide some general results for the edge adding stability number.

Proposition 1. *Let H be a spanning supergraph of G obtained from G by adding k edges. Then $eas_\rho(G) \leq eas_\rho(H) + k$. Moreover, if $\rho(G) \neq \rho(H)$, then $eas_\rho(G) \leq k$.*

Proof. Let $H = G+E'$ where $E' \subseteq E(\overline{G})$ with $|E'| = k$. If $\rho(G) \neq \rho(H) = \rho(G+E')$, then $eas_\rho(G) \leq |E'| = k \leq eas_\rho(H) + k$.
Therefore, assume in the following that $\rho(G) = \rho(H)$. If $\rho(H)$ cannot be changed by edge additions then $eas_\rho(H) = \infty$, and the assertion is trivial. Otherwise, let E'' be a set of edges of $\overline{H}$ such that $|E''| = eas_\rho(H)$ and $\rho(H+E'') \neq \rho(H)$. Set $E''' = E' \cup E''$ with $|E'''| = |E'| + |E''| = k + eas_\rho(H)$. It follows that $\rho(G) = \rho(H) \neq \rho(H + E'') = \rho(G + E''')$ which implies $eas_\rho(G) \leq |E'''| = eas_\rho(H) + k$. □

We are interested in connections between the invariants $eas_\rho(G)$ and $es_\rho(G)$. Considering the complements of graphs, removing an edge of G corresponds to adding an edge to $\overline{G}$. This implies that $\overline{G - E'} = \overline{G} + E'$ for a set $E' \subseteq E(G)$. Therefore, it is possible to obtain results for the edge adding stability number from already known results on the edge stability number.

Example 1. For a complete graph K_n of order n it holds that $eas_\rho(K_n) = \infty$ by definition since there are no edges that can be added. For an empty graph $N_n \cong \overline{K_n}$ of order n it holds that $es_\rho(\overline{K_n}) = \infty$ by definition since there are no edges that can be removed. Hence, $eas_\rho(K_n) = es_\rho(\overline{K_n}) = \infty$ independently of the invariant ρ.

Example 2. The order $n(G)$ of a graph G does not change by edge deletions or additions, or in other words, $n(G)$ is constant for graphs with the same vertex set. By definitions we have $eas_n(G) = eas_n(\overline{G}) = es_n(G) = es_n(\overline{G}) = \infty$ for every graph G.

Example 3. The size $m(G)$ of a graph G does change by any possible edge addition or deletion. Therefore, $eas_m(G) = es_m(\overline{G}) = \begin{cases} 1 & \text{if } G \text{ not complete,} \\ \infty & \text{if } G \text{ complete.} \end{cases}$

Example 4. For the degree of a vertex v of a graph G it holds that $d_G(v) + d_{\overline{G}}(v) = n-1$. A vertex has minimum degree in G if and only if it has maximum degree in $\overline{G}$, and vice versa. Therefore, $\delta(G) + \Delta(\overline{G}) = \Delta(G) + \delta(\overline{G}) = n - 1$.
If $\Delta(G) = n - 1$ and thus $\delta(\overline{G}) = 0$, then $eas_\Delta(G) = es_\delta(\overline{G}) = \infty$ by definitions. If $\Delta(G) < n - 1$, then it suffices to add one edge incident to a vertex of maximum degree in G in order to increase $\Delta(G)$. On the other hand, $\delta(\overline{G}) > 0$ and it suffices to remove one edge incident to a vertex of minimum degree in $\overline{G}$ in order to reduce $\delta(\overline{G})$ which implies $eas_\Delta(G) = es_\delta(\overline{G}) = 1$.
Therefore, $eas_\Delta(G) = es_\delta(\overline{G})$ and $es_\delta(G) = eas_\Delta(\overline{G})$ by considering the complements.

It is possible to generalize the last example by considering two graph invariants with either constant sum or a sum that only depends on the order of the graph. Note that the order does not change by edge additions or edge deletions.

Theorem 1. *Let ρ and σ be two graph invariants and f be a function of the order $n = n(G)$ of a graph G.*

(1) *If $\rho(G) + \sigma(G) = f(n)$ for every graph G, then $es_\rho(G) = es_\sigma(G)$ and $eas_\rho(G) = eas_\sigma(G)$.*

(2) *If* $\rho(G) + \sigma(\overline{G}) = f(n)$ *for every graph* G*, then* $es_\rho(G) = eas_\sigma(\overline{G})$ *and* $eas_\rho(G) = es_\sigma(\overline{G})$.

Proof. Let G be a graph of order $n = n(G)$.

(1) If one of the invariants, say $\rho(G)$, does not change by edge deletions, then $\rho(G - E') = \rho(G)$ for any set $E' \subseteq E(G)$ which implies $\sigma(G - E') = f(n) - \rho(G - E') = f(n) - \rho(G) = \sigma(G)$, that is, also the other invariant does not change by edge deletions. By definition, $es_\rho(G) = es_\sigma(G) = \infty$.

Otherwise, there is a set $E' \subseteq E(G)$ of edges such that $|E'| = es_\rho(G)$ and $\rho(G - E') \neq \rho(G)$, hence $\sigma(G - E') = f(n) - \rho(G - E') \neq f(n) - \rho(G) = \sigma(G)$ which implies $es_\sigma(G) \leq |E'| = es_\rho(G)$. By exchanging ρ and σ we also obtain $es_\rho(G) \leq es_\sigma(G)$, that is, $es_\rho(G) = es_\sigma(G)$.

The proof of $eas_\rho(G) = eas_\sigma(G)$ runs analogously to the above proof.

(2) If $\rho(G)$ does not change by edge deletions, then $\sigma(\overline{G} + E') = \sigma(\overline{G - E'}) = f(n) - \rho(G - E') = f(n) - \rho(G) = \sigma(\overline{G})$ for any set $E' \subseteq E(G)$, that is, $\sigma(\overline{G})$ does not change by edge additions, and vice versa. In this case we have $es_\rho(G) = eas_\sigma(\overline{G}) = \infty$.

Otherwise, there is a set $E' \subseteq E(G)$ of edges such that $|E'| = es_\rho(G)$ and $\rho(G - E') \neq \rho(G)$, hence $\sigma(\overline{G} + E') = \sigma(\overline{G - E'}) = f(n) - \rho(G - E') \neq f(n) - \rho(G) = \sigma(\overline{G})$ which implies $eas_\sigma(\overline{G}) \leq |E'| = es_\rho(G)$. On the other hand, there is a set $E'' \subseteq E(G)$ of edges such that $|E''| = eas_\sigma(\overline{G})$ and $\sigma(\overline{G} + E'') \neq \sigma(\overline{G})$, hence $\rho(G - E'') = f(n) - \sigma(\overline{G - E''}) = f(n) - \sigma(\overline{G} + E'') \neq f(n) - \sigma(\overline{G}) = \rho(G)$. Therefore, $es_\rho(G) \leq |E''| = eas_\sigma(\overline{G})$, and equality follows.

By exchanging ρ and σ as well as G and $\overline{G}$ we obtain $eas_\rho(G) = es_\sigma(\overline{G})$. □

If we consider just one invariant, that is, $\rho(G) = \sigma(G)$, then the assertion of Theorem 1 (1) is obvious, while (2) implies that if $\rho(G) + \rho(\overline{G}) = f(n)$ for every graph G, then $es_\rho(G) = eas_\rho(\overline{G})$ and $eas_\rho(G) = es_\rho(\overline{G})$.

Example 5. Returning to above Example 4, it is possible to determine the invariant $eas_\delta(G)$ from the already known $es_\Delta(G)$: $es_\Delta(G) = \infty$ if $\Delta(G) = 0$ and $es_\Delta(G) = |V_\Delta| - \alpha'(G[V_\Delta])$ if $\Delta(G) > 0$, where V_Δ is the set of vertices of G of maximum degree $\Delta(G)$ and $\alpha'(G)$ is the edge independence number or matching number of G (see [11]). Therefore, by Theorem 1(2), $eas_\delta(G) = es_\Delta(\overline{G})$, that is, $eas_\delta(G) = \infty$ if G is complete and $eas_\delta(G) = |V_\delta| - \alpha'(\overline{G}[V_\delta])$ otherwise, where V_δ is the set of vertices of G of minimum degree $\delta(G)$.

Theorem 1 is stated for the sum of two invariants. Obviously, this can be replaced e.g. by the difference $\rho(G) - \sigma(G)$ and by other operations. Moreover, the conditions of equality (e.g. $\rho(G) + \sigma(G) = f(n)$ for every graph G) are too strong. The equality is only required in the proof for subgraphs $G - E'$ or supergraphs $G + E'$ of a given graph G, that is, for graphs with the same vertex set as G and hence of the same order as G. Therefore, we generalize the latter theorem as follows.

Theorem 2. *Let ρ and σ be two graph invariants, G be a graph of order $n = n(G)$, and $h_n : D \to R$ be a bijection with $D, R \subseteq \mathbb{R}_0^+ \cup \{\infty\}$.*

(1) *If $\sigma(H) = h_n(\rho(H))$ for every spanning subgraph $H = G - E'$ of G, $E' \subseteq E(G)$, then $es_\rho(G) = es_\sigma(G)$.*

(2) *If $\sigma(H) = h_n(\rho(H))$ for every spanning supergraph $H = G + E'$ of G, $E' \subseteq E(\overline{G})$, then $eas_\rho(G) = eas_\sigma(G)$.*

(3) *If $\sigma(\overline{H}) = h_n(\rho(H))$ for every spanning subgraph $H = G - E'$ of G, $E' \subseteq E(G)$, then $es_\rho(G) = eas_\sigma(\overline{G})$.*

(4) *If $\sigma(\overline{H}) = h_n(\rho(H))$ for every spanning supergraph $H = G + E'$ of G, $E' \subseteq E(\overline{G})$, then $eas_\rho(G) = es_\sigma(\overline{G})$.*

Proof. Let G be a graph of order $n = n(G)$.

(1) If $\rho(G)$ does not change by edge deletions, then $\rho(G - E') = \rho(G)$ for every set $E' \subseteq E(G)$ which implies $\sigma(G - E') = h_n(\rho(G - E')) = h_n(\rho(G)) = \sigma(G)$, that is, also $\sigma(G)$ does not change by edge deletions. We get the converse result by considering $\rho(H) = h_n^{-1}(\sigma(H))$ where h_n^{-1} is the inverse of h_n. Therefore, $es_\rho(G) = es_\sigma(G) = \infty$ by definition.

Otherwise, there is a set $E' \subseteq E(G)$ of edges such that $|E'| = es_\rho(G)$ and $\rho(G - E') \neq \rho(G)$, hence $\sigma(G - E') = h_n(\rho(G - E')) \neq h_n(\rho(G)) = \sigma(G)$ by the injectivity of h_n. This implies $es_\sigma(G) \leq |E'| = es_\rho(G)$.

On the other hand, there is a set $E'' \subseteq E(G)$ of edges such that $|E''| = es_\sigma(G)$ and $\sigma(G - E'') \neq \sigma(G)$, hence $\rho(G - E'') = h_n^{-1}(\sigma(G - E'')) \neq h_n^{-1}(\sigma(G)) = \rho(G)$ by the injectivity of h_n^{-1}. This implies $es_\rho(G) \leq |E''| = es_\sigma(G)$ and thus $es_\rho(G) = es_\sigma(G)$.

(2) The proof of $eas_\rho(G) = eas_\sigma(G)$ runs analogously to the above proof of (1).

(3) Let $E' \subseteq E(G)$ be an arbitrary set of edges. If $\rho(G)$ does not change by edge deletions, then $\sigma(\overline{G} + E') = \sigma(\overline{G - E'}) = h_n(\rho(G - E')) = h_n(\rho(G)) = \sigma(\overline{G})$, that is, $\sigma(\overline{G})$ does not change by edge additions. Conversely, if $\sigma(\overline{G})$ does not change by edge additions, then $\rho(G - E') = h_n^{-1}(\sigma(\overline{G - E'})) = h_n^{-1}(\sigma(\overline{G} + E')) = h_n^{-1}(\sigma(\overline{G})) = \rho(G)$, that is, $\rho(G)$ does not change by edge deletions. Therefore, $es_\rho(G) = eas_\sigma(\overline{G}) = \infty$ by definition.

Otherwise, there is a set $E' \subseteq E(G)$ of edges such that $|E'| = es_\rho(G)$ and $\rho(G - E') \neq \rho(G)$, hence $\sigma(\overline{G} + E') = \sigma(\overline{G - E'}) = h_n(\rho(G - E')) \neq h_n(\rho(G)) = \sigma(\overline{G})$ by the injectivity of h_n which implies $eas_\sigma(\overline{G}) \leq |E'| = es_\rho(G)$. On the other hand, there is a set $E'' \subseteq E(G)$ of edges such that $|E''| = eas_\sigma(\overline{G})$ and $\sigma(\overline{G} + E'') \neq \sigma(\overline{G})$, hence $\rho(G - E'') = h_n^{-1}(\sigma(\overline{G - E''})) = h_n^{-1}(\sigma(\overline{G} + E'')) \neq h_n^{-1}(\sigma(\overline{G})) = \rho(G)$ by the injectivity of h_n^{-1}. Therefore, $es_\rho(G) \leq |E''| = eas_\sigma(\overline{G})$, and equality follows.

(4) The proof of $eas_\rho(G) = es_\sigma(\overline{G})$ runs analogously to the above proof of (3) (it results from (3) by exchanging ρ and σ, h_n and h_n^{-1}, G and $\overline{G}$, H and $\overline{H}$). □

Note that Theorem 1 follows from Theorem 2 by using the bijection $h_n(x) = f(n) - x$. This implies that $\sigma(G) = h_n(\rho(G)) = f(n) - \rho(G)$, that is, $\rho(G) + \sigma(G) = f(n)$.
In Theorem 2 spanning subgraphs and supergraphs of a given graph are considered. If we have general conditions for all graphs (as in Theorem 1) or for every graph of a given order n, then we obtain general conclusions. We state this as follows, omitting the proof which runs analogously to the above one.

Theorem 3. *Let ρ and σ be two graph invariants and $h_n : D \to R$ be a bijection with $D, R \subseteq \mathbb{R}_0^+ \cup \{\infty\}$ and $n \in \mathbb{N}$.*

(1) *If $\sigma(G) = h_n(\rho(G))$ for every graph G of order n, then $es_\rho(G) = es_\sigma(G)$ and $eas_\rho(G) = eas_\sigma(G)$ for every graph G of order n.*

(2) *If $\sigma(\overline{G}) = h_n(\rho(G))$ for every graph G of order n, then $es_\rho(G) = eas_\sigma(\overline{G})$ and $eas_\rho(G) = es_\sigma(\overline{G})$ for every graph G of order n.*

In 1959 Gallai proved the following results which are nowadays known as Gallai's Theorem [8]. Let G be a graph of order $n(G)$ without isolated vertices, $\alpha(G)$ be the independence number, that is, the maximum number of mutually non-adjacent vertices of G, $\beta(G)$ the vertex covering number, that is, the minimum number of vertices of G such that every edge of G is incident to at least one of these vertices, $\alpha'(G)$ the edge independence number or matching number, that is, the maximum number of mutually non-adjacent edges of G, and $\beta'(G)$ the edge covering number, that is, the minimum number of edges of G such that every vertex of G is incident to at least one of these edges. Then (1) $\alpha(G) + \beta(G) = n(G)$ and (2) $\alpha'(G) + \beta'(G) = n(G)$.
By Theorem 1 or 3 and (1) it follows that $es_\alpha(G) = es_\beta(G)$ and $eas_\alpha(G) = eas_\beta(G)$. These theorems cannot be directly applied to (2), but it follows from Theorem 2(2) that $eas_{\alpha'}(G) = eas_{\beta'}(G)$ for every graph G without isolated vertices since adding edges to a graph without isolated vertices does not create any isolated vertices. Note that we cannot directly use Theorem 2(1) since removing edges may create subgraphs $H = G - E'$ with isolated vertices for which the equality $\alpha'(H) + \beta'(H) = n(H)$ does not hold anymore.

Example 6. The clique number $\omega(G)$ is the maximum number of pairwise adjacent vertices in G. Obviously, $\omega(G) = \alpha(\overline{G})$, thus the Theorem of Gallai implies that $\omega(G) + \beta(\overline{G}) = \alpha(\overline{G}) + \beta(\overline{G}) = n(G)$. By Theorem 1 and the above remarks it follows that $eas_\omega(G) = es_\beta(\overline{G}) = es_\alpha(\overline{G})$ and $es_\omega(G) = eas_\beta(\overline{G}) = eas_\alpha(\overline{G})$.

The join $H_1 \vee H_2$ of two disjoint graphs H_1 and H_2 is the graph composed by a copy of H_1 and a copy of H_2 in which each vertex of H_1 is connected to all vertices of H_2.

Theorem 4. *If $\rho(G)$ is a real-valued invariant with $\rho(H_1 \vee H_2) = \rho(H_1) + \rho(H_2)$ for the join of any two graphs H_1, H_2, then $eas_\rho(H_1 \vee H_2) = \min\{eas_\rho(H_1), eas_\rho(H_2)\}$.*

Proof. If H_1 and H_2 both are complete, then also $H_1 \vee H_2$ is complete and $eas_\rho(H_1 \vee H_2) = eas_\rho(H_1) = eas_\rho(H_2) = \infty$ follows by definition. More generally, if it is not possible to change $\rho(H_1)$ and $\rho(H_2)$ by edge additions, then it is also not possible to change $\rho(H_1 \vee H_2) = \rho(H_1) + \rho(H_2)$ by edge additions, which implies $eas_\rho(H_1 \vee H_2) = eas_\rho(H_1) = eas_\rho(H_2) = \infty$.
In all other cases it is possible to change at least one of the invariants $\rho(H_1), \rho(H_2)$ by edge additions, and therefore also $\rho(H_1 \vee H_2) = \rho(H_1) + \rho(H_2)$. Without loss of generality, assume that $eas_\rho(H_1) \leq eas_\rho(H_2)$ and $eas_\rho(H_1) < \infty$. Let $E' \subseteq E(\overline{H_1})$ such that $|E'| = eas_\rho(H_1)$ and $\rho(H_1 + E') \neq \rho(H_1)$. Then $\rho((H_1 \vee H_2) + E') = \rho((H_1 + E') \vee H_2) = \rho(H_1 + E') + \rho(H_2) \neq \rho(H_1) + \rho(H_2) = \rho(H_1 \vee H_2)$ which implies that $eas_\rho(H_1 \vee H_2) \leq |E'| = eas_\rho(H_1) = \min\{eas_\rho(H_1), eas_\rho(H_2)\}$.
Consider now $H_1 \vee H_2$ and let $E''' \subseteq E(\overline{H_1 \vee H_2})$ such that $|E'''| = eas_\rho(H_1 \vee H_2)$ and $\rho(H_1 \vee H_2 + E''') \neq \rho(H_1 \vee H_2)$. Since $\overline{H_1 \vee H_2} = \overline{H_1} \cup \overline{H_2}$, $E''' = E' \cup E''$ with $E' \subseteq E(\overline{H_1})$ and $E'' \subseteq E(\overline{H_2})$. Moreover, $\rho(H_1 \vee H_2 + E''') = \rho((H_1 + E') \vee (H_2 + E'')) = \rho(H_1 + E') + \rho(H_2 + E'') \neq \rho(H_1) + \rho(H_2) = \rho(H_1 \vee H_2)$. This implies $\rho(H_1 + E') \neq \rho(H_1)$ or $\rho(H_2 + E'') \neq \rho(H_2)$, therefore $|E'| \geq eas_\rho(H_1)$ or $|E''| \geq eas_\rho(H_2)$ which implies $eas_\rho(H_1 \vee H_2) = |E'''| = |E'| + |E''| \geq \min\{eas_\rho(H_1), eas_\rho(H_2)\}$. □

Example 7. It holds that $\omega(H_1 \vee H_2) = \omega(H_1) + \omega(H_2)$ for the clique number and $\chi(H_1 \vee H_2) = \chi(H_1) + \chi(H_2)$ for the chromatic number. Therefore, Theorem 4 implies $eas_\omega(H_1 \vee H_2) = \min\{eas_\omega(H_1), eas_\omega(H_2)\}$ and $eas_\chi(H_1 \vee H_2) = \min\{eas_\chi(H_1), eas_\chi(H_2)\}$.

In a certain sense, Theorem 4 corresponds to Theorem 11 of [11] on the edge stability number of union of graphs for additive invariants (invariants such that $\rho(H_1 \cup H_2) = \rho(H_1) + \rho(H_2)$ for disjoint graphs H_1, H_2) which states that $es_\rho(H_1 \cup H_2) = \min\{es_\rho(H_1), es_\rho(H_2)\}$. Note that it is not possible to remove edges between the graphs H_1 and H_2 of the union $H_1 \cup H_2$ and it is not possible to add edges between the graphs H_1 and H_2 of the join $H_1 \vee H_2$, so removing edges from $H_1 \cup H_2$ (adding edges to $H_1 \vee H_2$) means removing edges from (adding edges to) H_1 or H_2, and this keeps the structure of the graphs unchanged.
Iteratively applying Theorem 4 to the join $G = H_1 \vee \cdots \vee H_k$ of $k \geq 2$ graphs gives $eas_\rho(G) = \min\{eas_\rho(H_i) : 1 \leq i \leq k\}$.
It is possible to transfer Theorem 12 of [11] on the edge stability number of the union of graphs for maxing invariants (invariants such that $\rho(H_1 \cup H_2) = \max\{\rho(H_1), \rho(H_2)\}$ for disjoint graphs H_1, H_2) as follows.

Theorem 5. *Let $\rho(G)$ be a monotone decreasing invariant with respect to spanning subgraphs and with $\rho(H_1 \vee H_2) = \max\{\rho(H_1), \rho(H_2)\}$ for any two graphs H_1, H_2. Let $G = H_1 \vee \cdots \vee H_k$, $k \geq 2$, and $s \in \{1, \ldots, k\}$ such that $\rho(H_i) = \rho(G)$ if and only if $1 \leq i \leq s$. Then $eas_\rho(G) = \sum_{i=1}^{s} eas_\rho(H_i)$.*

Proof. If there is a graph H_j, $1 \leq j \leq s$, such that $\rho(H_j)$ cannot be changed by edge additions, then $\rho(G) = \rho(H_j) = \rho(G + E')$ for every $E' \subseteq E(\overline{G})$, since the invariant is

monotone decreasing with respect to spanning subgraphs (that is, adding edges does not increase the invariant). Therefore, $eas_\rho(G) = \infty$.
Otherwise, let $E' = E'_1 \cup \cdots \cup E'_s$ with $E'_i \subseteq E(\overline{H_i})$, $|E'_i| = eas_\rho(H_i)$, and $\rho(H_i + E'_i) \neq \rho(H_i) = \rho(G)$ for $i = 1, \ldots, s$. Because of the property of the invariant, $\rho(G + E') = \max\{\rho(H_i + E'_i) : 1 \leq i \leq s\} \cup \{\rho(H_i) : s+1 \leq i \leq k\} \neq \rho(G)$ which implies $eas_\rho(G) \leq |E'| = \sum_{i=1}^{s} eas_\rho(H_i)$. If an edge set E'' with less than $|E'|$ edges is added to G, then there is a subgraph H_j, $1 \leq j \leq s$, to which less than $eas_\rho(H_j)$ edges are added, which implies $\rho(H_j + E'') = \rho(H_j)$ and thus $\rho(G + E'') = \rho(H_j) = \rho(G)$. Therefore, $eas_\rho(G) = |E'| = \sum_{i=1}^{s} eas_\rho(H_i)$. □

Example 8. For the independence number it holds that $\alpha(H_1 \vee H_2) = \max\{\alpha(H_1), \alpha(H_2)\}$. Moreover, adding edges does not increase the independence number, that is, $\alpha(G)$ is monotone decreasing with respect to spanning subgraphs. Therefore, Theorem 5 implies $eas_\alpha(H_1 \vee H_2) = eas_\alpha(H_1) + eas_\alpha(H_2)$ if $\alpha(H_1) = \alpha(H_2)$ and $eas_\alpha(H_1 \vee H_2) = eas_\alpha(H_1)$ if $\alpha(H_1) > \alpha(H_2)$.

3. Theorem-of-Gallai type results

Let G be a graph of order $n(G)$ without isolated vertices. Gallai's Theorem [8] states that (1) $\alpha(G) + \beta(G) = n(G)$ and (2) $\alpha'(G) + \beta'(G) = n(G)$ (see above).
Analogous Theorem-of-Gallai type results were proved in [15] for invariants based on the vertex stability (stability with respect to removing vertices) and in [13] for invariants based on the edge stability number $es_\rho(G)$ of a graph G.
In [13] the invariants $\alpha'_\rho(G)$ and $\beta'_\rho(G)$ are defined as follows. If $\rho(G)$ is an invariant, then $\alpha'_\rho(G)$ is the maximum number of edges of a spanning subgraph H of G with $\rho(H) \neq \rho(G)$. If such a subgraph does not exist (that is, if $\rho(H)$ is constant for all spanning subgraphs H of G), then we set $\alpha'_\rho(G) = \infty$.
Let $\beta'_\rho(G)$ be the minimum number of edges of G that cover all nonempty spanning subgraphs H of G with $\rho(H) = \rho(G)$, that is, each such subgraph must contain at least one edge of the covering set.
If $es_\rho(G) < \infty$, then $es_\rho(G) = m(G) - \alpha'_\rho(G)$ where $m(G)$ is the size of G. Moreover, if $\rho(G)$ is monotone with respect to spanning subgraphs and $es_\rho(G) < \infty$, then $\alpha'_\rho(G) + \beta'_\rho(G) = m(G)$, and therefore $es_\rho(G) = \beta'_\rho(G)$ (see [13]).

We can transfer these results if we consider the operation of adding edges between non-adjacent vertices.

Definition 4. If $\rho(G)$ is an invariant, then $\alpha''_\rho(G)$ is defined to be the minimum number of edges of a spanning supergraph H of G with $\rho(H) \neq \rho(G)$. If such a supergraph does not exist (that is, if $\rho(H)$ is constant for all spanning supergraphs H of G), then we set $\alpha''_\rho(G) = \infty$.
Let $\beta''_\rho(G)$ be the minimum number of edges of a set $\overline{E} \subseteq E(\overline{G})$ such that for each non-complete spanning supergraph H of G with $\rho(H) = \rho(G)$ there is an edge in $\overline{E}$ which is not contained in H.

In the definitions of $\alpha'_\rho(G)$ and $\alpha''_\rho(G)$ we consider specific graphs "closest" to the graph G, therefore, the number of edges is maximal for spanning subgraphs while it is minimal for spanning supergraphs.
It holds that $0 \leq \beta''_\rho(G) \leq m(\overline{G})$. If $\rho(H)$ is constant for all spanning supergraphs H of G, then $eas_\rho(G) = \alpha''_\rho(G) = \infty$ by the definitions and $\beta''_\rho(G) = m(\overline{G})$ (including the case that G is complete) by considering the complete graphs without an edge $e \in E(\overline{G})$.
In the following we require that $\rho(H)$ is not constant for all spanning supergraphs H of G which is equivalent to $eas_\rho(G) < \infty$.

Lemma 1. *If $eas_\rho(G) < \infty$, then $eas_\rho(G) = \alpha''_\rho(G) - m(G)$.*

Proof. Since $\rho(G)$ can be changed by edge additions, there are sets $E' \subseteq E(\overline{G})$ with $\rho(G + E') \neq \rho(G)$. The size of $G + E'$ is minimal if and only if $|E'|$ is minimal, that is, $|E'| = eas_\rho(G)$.
This implies $\alpha''_\rho(G) = m(G+E') = m(G)+|E'| = m(G)+eas_\rho(G)$, that is, $eas_\rho(G) = \alpha''_\rho(G) - m(G)$. □

Theorem 6. *If $\rho(G)$ is monotone with respect to spanning supergraphs and $eas_\rho(G) < \infty$, then $\alpha''_\rho(G) - \beta''_\rho(G) = m(G)$.*

Proof. Note that $eas_\rho(G) < \infty$ implies $\alpha''_\rho(G) < \infty$ and that G is not complete.
Let $G' = (V(G), E')$ be a spanning supergraph of G with $E' \supsetneq E(G)$, $|E'| = \alpha''_\rho(G)$, and $\rho(G') \neq \rho(G)$. Then the difference $E'' = E' \setminus E(G)$ has the property that each spanning supergraph H of G with $\rho(H) = \rho(G)$ does not contain all edges of E'', that is, at least one edge is missing. Suppose not, then there is a spanning supergraph H of G with $\rho(H) = \rho(G)$ that contains all edges of E'', that is, $E(H) \supseteq E'$ and H is a spanning supergraph of G'. But $\rho(G)$ is monotone with respect to spanning supergraphs, so either $\rho(G) < \rho(G') \leq \rho(H)$, or $\rho(G) > \rho(G') \geq \rho(H)$, that is, $\rho(H) \neq \rho(G)$, a contradiction.
This implies $\beta''_\rho(G) \leq |E''| = \alpha''_\rho(G) - m(G)$ by the minimality of $\beta''_\rho(G)$, that is, $\alpha''_\rho(G) - \beta''_\rho(G) \geq m(G)$.
Conversely, let $E'' \subseteq E(\overline{G})$ be a set of $\beta''_\rho(G)$ edges such that each spanning supergraph H of G with $\rho(H) = \rho(G)$ does not contain an edge of E''. The graph $G' = G + E''$ is a spanning supergraph of G with all edges of the selected set E'' which implies $\rho(G') \neq \rho(G)$. By the minimality, $\alpha''_\rho(G) \leq m(G') = m(G) + \beta''_\rho(G)$, that is, $\alpha''_\rho(G) - \beta''_\rho(G) \leq m(G)$ and thus equality follows. □

Corollary 1. *If $\rho(G)$ is monotone with respect to spanning supergraphs and $eas_\rho(G) < \infty$, then $eas_\rho(G) = \beta''_\rho(G)$.*

Proof. By Lemma 1 and Theorem 6, $eas_\rho(G) = \alpha''_\rho(G) - m(G) = \beta''_\rho(G)$. □

These results imply that only one of the invariants $eas_\rho(G)$, $\alpha''_\rho(G)$, $\beta''_\rho(G)$ needs to be determined in order to know also the other two invariants.

Example 9. Consider the chromatic number $\chi(G)$ of a graph G which is a monotone increasing invariant. Lemma 1 and Corollary 1 state that $eas_\chi(G) = \alpha''_\chi(G) - m(G) = \beta''_\chi(G)$. If G is a bipartite graph with $\Delta(G) \geq 2$, then adding a single edge between two neighbors of a vertex results in a K_3 which gives a supergraph with chromatic number 3. Therefore, $eas_\chi(G) = \beta''_\chi(G) = 1$ and $\alpha''_\chi(G) = m(G) + 1$.

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