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Research Article

Skew-cyclic and skew-quasi-cyclic codes over a general infinite family of rings

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Abstract: We study structural properties of cyclic codes, and their generalization, over a general infinite family of rings, namely the ring $\mathcal{R}_k$ defined by $R[v_1, v_2, \ldots, v_k]$ with conditions $v_i^2 = v_i$, for $i \in [1, k]_{\mathbb{Z}}$, where R is any finite commutative Frobenius ring. We derived necessary and sufficient condition for the codes to be cyclic, quasi-cyclic, skew-cyclic as well as to be quasi-skew-cyclic. As an application, we constructed optimal linear codes over $\mathbb{Z}_4$ as a Gray images of our codes.

Keywords: commutative Frobenius ring, cyclic code, quasi-cyclic code, skew-cyclic code, skew-quasi-cyclic code, optimal codes over $\mathbb{Z}_4$.

AMS Subject classification: 94A05, 94B15

1. Introduction

In its early development, linear codes used finite fields as codes alphabet. Codes over finite rings were introduced later in 1970s by Blake [4, 5]. He [4] showed how to construct codes over $\mathbb{Z}_m$ from cyclic codes over $\mathbb{F}_p$, where p is a prime factor of m. He [5] then further observed the structure of codes over $\mathbb{Z}_{p^r}$. Spiegel [18, 19] generalized Blake's results to codes over $\mathbb{Z}_m$, where m is an arbitrary positive integer.

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Study of linear codes over finite rings attracted great interest after the work of Hammons, Kumar, Calderbank, Sloane and Solé in 1994 [7], where they showed that certain good nonlinear binary codes can be constructed from linear codes over $\mathbb{Z}_4$ via the Gray map. Recently, many people consider some special cases of linear codes over the ring of the form $R[v_1, v_2 \ldots, v_k]$, where $v_i^2 = v_i$ for all $i \in [1, k]_{\mathbb{Z}}$, and R is a certain finite commutative ring. The reasons why it attracts the attention of many researchers in coding theory is, among other thing, because codes over such kind of rings have a lot of nice structures. For example, in [1], [9], [11] and [12], they considered skew-cyclic codes over the ring $\mathbb{F}_2 + v\mathbb{F}_2, \mathbb{F}_p + v\mathbb{F}_p$, $\mathbb{F}_2[v_1, v_2 \ldots, v_k]$, and $\mathbb{F}_{p^r}[v_1, v_2 \ldots, v_k]$, respectively. Moreover, in [8], [15], [6], and [17], they studied the structural properties of linear codes over $\mathbb{F}_2[v_1, v_2 \ldots, v_k], \mathbb{Z}_4 + v\mathbb{Z}_4$, $\mathbb{Z}_4 + u\mathbb{Z}_4 + v\mathbb{Z}_4 + w\mathbb{Z}_4 + uv\mathbb{Z}_4 + uw\mathbb{Z}_4 + vw\mathbb{Z}_4 + uvw\mathbb{Z}_4$, and $\mathbb{Z}_{2^m} + v\mathbb{Z}_{2^m}$, respectively, including MacWilliams identity, self-dual codes, cyclic codes, constacyclic codes, *etc.* Also, we can find constructions of good and new $\mathbb{Z}_4$-linear codes in [15] and [6].

In our previous works [12, 13, 16], we studied structural properties of linear codes over an infinite family of ring B_k which is defined as $\mathbb{F}_{p^r}[v_1, v_2, \ldots, v_k]$, with condition $v_i^2 = v_i$ for all $i \in [1, k]_{\mathbb{Z}}$, where $\mathbb{F}_{p^r}$ is a finite field of order p^r. In this paper, we generalize some results obtained in [12] as well as [13] to a very general setting of ring, namely to the ring $\mathcal{R}_k$ which is defined by $R[v_1, v_2, \ldots, v_k]$, with condition $v_i^2 = v_i$ for all $i \in [1, k]_{\mathbb{Z}}$, where R is a finite commutative Frobenius ring. We investigate the structures of cyclic codes, quasi-cyclic codes, skew-cyclic codes, and quasi-skew-cyclic codes over the ring $\mathcal{R}_k$. As an application, we provide several new and optimal linear codes over $\mathbb{Z}_4$ obtained as a Gray map of linear codes. This is a sequel of our previous paper [14].

We follow many standard books in coding theory (see, for instance, [10]) for undefined terms.

2. Cyclic and quasi-cyclic codes

Let $n = md$, for some positive integers m and d. Let C be a linear code of length n over the ring $\mathcal{R}_k$. Also, let $\mathbf{c} \in \mathcal{R}_k^n$, with $\mathbf{c} = \left(\mathbf{c}^{(1)}|\mathbf{c}^{(2)}|\cdots|\mathbf{c}^{(d)}\right)$, where $\mathbf{c}^{(i)} \in \mathcal{R}_k^m$, for all $i \in [1, d]_{\mathbb{Z}}$. Let σ_d be a map from $\mathcal{R}_k^n$ to $\mathcal{R}_k^n$ such that

$$\sigma_d(\mathbf{c}) = \left(T\left(\mathbf{c}^{(1)}\right)|T\left(\mathbf{c}^{(2)}\right)|\cdots|T\left(\mathbf{c}^{(d)}\right)\right),$$

where T is a cyclic shift from $\mathcal{R}_k^m$ to $\mathcal{R}_k^m$. A code C of length n over ring $\mathcal{R}_k$ is said to be a *quasi-cyclic* code of index d if $\sigma_d(C) = C$. Note that, our definition of quasi-cyclic here is permutation-equivalent to the usual definition of quasi-cyclic codes. Also, a code C is said to be *cyclic* if it is quasi-cyclic of index $d = 1$. We have the following characterization for quasi-cyclic codes over the ring $\mathcal{R}_k$, by using the second Gray map $\overline{\Psi}$ from $\mathcal{R}_k^n$ to $R^{2^k \times n}$ as defined in [14], page 55.

Theorem 1. *A linear code C of length n over $\mathcal{R}_k$ is a quasi-cyclic code with index d if and only if $C = \overline{\Psi}^{-1}(C_1, C_2, \ldots, C_{2^k})$, where $C_1, C_2, \ldots, C_{2^k}$ are quasi-cyclic codes of length n with index d over R.*

Proof. ($\Longrightarrow$) For $i \in [1, 2^k]_{\mathbb{Z}}$, take $\mathbf{c} \in C_i$ and let $\mathbf{c} = (c_{1,1}, \ldots, c_{1,m}, \ldots, c_{d,1}, \ldots, c_{d,m})$. Since C is a quasi-cyclic code of index d, we have that

$$\overline{\Psi}^{-1}\begin{pmatrix} \mathbf{0} \\ \vdots \\ \mathbf{0} \\ \sigma_d(\mathbf{c}) \\ \mathbf{0} \\ \vdots \\ \mathbf{0} \end{pmatrix} = \overline{\Psi}^{-1}\begin{pmatrix} 0 & 0 & \ldots & 0 & \ldots & 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & 0 & \ldots & 0 & 0 & \ldots & 0 \\ c_{1,m} & c_{1,1} & \ldots & c_{1,m-1} & \ldots & c_{d,m} & c_{d,1} & \ldots & c_{d,m-1} \\ 0 & 0 & \ldots & 0 & \ldots & 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & 0 & \ldots & 0 & 0 & \ldots & 0 \end{pmatrix}$$

$$= \sum_{S \supseteq S_i} (-1)^{|S|-|S_i|} v_S \left(c_{1,m}, c_{1,1}, \ldots, c_{1,m-1}, \ldots, c_{d,m}, c_{d,1}, \ldots, c_{d,m-1}\right) \in C.$$

This gives $\sigma_d(\mathbf{c}) \in C_i$, for all $i \in [1, 2^k]_{\mathbb{Z}}$.
($\Longleftarrow$) For any $\mathbf{w}$ in C, there exist codewords $\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_{2^k}$, where $\mathbf{w}_i \in C_i$, for all $i \in [1, 2^k]_{\mathbb{Z}}$, such that $\mathbf{w} = \overline{\Psi}^{-1}\left((\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_{2^k})^T\right)$. Also, we have that

$$\sigma_d(\mathbf{w}) = \sigma_d\left(\overline{\Psi}^{-1}\left((\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_{2^k})^T\right)\right) = \overline{\Psi}^{-1}\left((\sigma_d(\mathbf{w}_1), \sigma_d(\mathbf{w}_2), \ldots, \sigma_d(\mathbf{w}_{2^k}))^T\right).$$

Since C_i is a quasi-cyclic code of index d, we have $\sigma_d(\mathbf{w}_i)$ is in C_i, for all $i \in [1, 2^k]_{\mathbb{Z}}$. Hence,

$$(\sigma_d(\mathbf{w}_1), \sigma_d(\mathbf{w}_2), \ldots, \sigma_d(\mathbf{w}_{2^k}))$$

is in $\overline{\Psi}(C)$. This means $\sigma_d(\mathbf{w})$ is in C. $\square$

As a consequence of the above theorem, we have the following property.

Corollary 1. *A linear code C of length n over $\mathcal{R}_k$ is cyclic if and only if $C = \overline{\Psi}^{-1}(C_1, C_2, \ldots, C_{2^k})$, where $C_1, C_2, \ldots, C_{2^k}$ are cyclic codes of length n over R.*

We also have the characterization of quasi-cyclic codes, by using the ingredient of the first Gray map $\overline{\varphi}_j$ from $\mathcal{R}_j^n$ to $\mathcal{R}_{j-1}^{nl_j}$ defined in [14], page 54, as given in the theorem below.

Theorem 2. *A linear code C of length n over $\mathcal{R}_j$ is a quasi-cyclic code with index d if and only if $\overline{\varphi}_j(C)$ is a quasi-cyclic code of length nl_j with index $l_j d$ over $\mathcal{R}_{j-1}$.*

Proof. For any $\mathbf{c}'$ in $\overline{\varphi}_j(C)$, there exists $\mathbf{c}$ in C such that $\overline{\varphi}_j(\mathbf{c}) = \mathbf{c}'$. Now, let

$$\mathbf{c} = \left(\alpha^{(1)} \mid \alpha^{(2)} \mid \cdots \mid \alpha^{(d)}\right),$$

where $\alpha^{(i)} = (\alpha_{i1} + \alpha'_{i1}v_j, \alpha_{i2} + \alpha'_{i2}v_j, \ldots, \alpha_{im} + \alpha'_{im}v_j)$, for all $i \in [1,d]_{\mathbb{Z}}$. Then we have

$$\begin{aligned}\mathbf{c}' &= \overline{\varphi}_j(\mathbf{c}) \\ &= \left(\beta_0^{(1)} \mid \beta_0^{(2)} \mid \cdots \mid \beta_0^{(d)} \mid \beta_1^{(1)} \mid \beta_1^{(2)} \mid \cdots \mid \beta_1^{(d)} \mid \cdots \mid \beta_{l_j-1}^{(1)} \mid \beta_{l_j-1}^{(2)} \mid \cdots \mid \beta_{l_j-1}^{(d)}\right),\end{aligned}$$

where $\beta_0^{(i)} = (\alpha_{i1}, \alpha_{i2}, \ldots, \alpha_{im})$, for all $i \in [1,d]_{\mathbb{Z}}$, and

$$\beta_r^{(i)} = (\beta_r\alpha_{i1} + \beta'_r\alpha'_{i1}, \beta_r\alpha_{i2} + \beta'_r\alpha'_{i2}, \ldots, \beta_r\alpha_{im} + \beta'_r\alpha'_{im}),$$

for all $r \in [1, l_j - 1]_{\mathbb{Z}}$ and $i \in [1,d]_{\mathbb{Z}}$. Consider also

$$\begin{aligned}\overline{\varphi}_j(\sigma_d(\mathbf{c})) = &\left(\sigma\left(\beta_0^{(1)}\right) \mid \sigma\left(\beta_0^{(2)}\right) \mid \cdots \mid \sigma\left(\beta_0^{(d)}\right) \mid \sigma\left(\beta_1^{(1)}\right) \mid \sigma\left(\beta_1^{(2)}\right) \mid \cdots\right. \\ &\left. \mid \sigma\left(\beta_1^{(d)}\right) \mid \cdots \mid \sigma\left(\beta_{l_j-1}^{(1)}\right) \mid \sigma\left(\beta_{l_j-1}^{(2)}\right) \mid \cdots \mid \sigma\left(\beta_{l_j-1}^{(d)}\right)\right) \\ = &\ \sigma_{l_j d}(\mathbf{c}').\end{aligned}$$

Therefore, $\sigma_d(\mathbf{c}) \in C$ if and only if $\sigma_{l_j d}(\mathbf{c}') \in \overline{\varphi}_j(C)$. □

The result below is a direct consequences of Theorem 2.

Corollary 2. *A linear code C of length n over $\mathcal{R}_j$ is a cyclic code if and only if $\overline{\varphi}_j(C)$ is a quasi-cyclic code of length nl_j with index l_j over $\mathcal{R}_{j-1}$.*

Again, by applying Theorem 2 repeatedly while considering the image of $\overline{\Phi}_k := \overline{\varphi}_1 \circ \overline{\varphi}_2 \circ \cdots \circ \overline{\varphi}_k$, we obtain the following theorem. Note that $\overline{\Phi}_k$ is the first Gray map defined in [14].

Theorem 3. *A linear code C of length n over $\mathcal{R}_k$ is a quasi-cyclic code with index d if and only if $\overline{\varphi}_1 \circ \overline{\varphi}_2 \circ \cdots \circ \overline{\varphi}_k(C)$ is a quasi-cyclic code of length $nl_1l_2\cdots l_k$ with index $d \cdot l_1l_2\cdots l_k$ over R.*

For the case of cyclic codes, namely quasi-cyclic codes of index $d = 1$, we obtain the following immediate consequence.

Corollary 3. *A linear code C of length n over $\mathcal{R}_k$ is a cyclic code if and only if $\overline{\varphi}_1 \circ \overline{\varphi}_2 \circ \cdots \circ \overline{\varphi}_k(C)$ is a quasi-cyclic code of length $nl_1l_2\cdots l_k$ with index $l_1l_2\cdots l_k$ over R.*

3. Skew-cyclic and skew-quasi-cyclic codes

We begin this section with definitions of skew-cyclic and skew-quasi-cyclic codes. Let C be a linear code of length n over the ring $\mathcal{R}_k$. Let θ be an automorphism on the ring $\mathcal{R}_k$. Then C is said to be a *θ-cyclic code* or *skew-cyclic code* if for any $\mathbf{c} = (c_0, c_1, \ldots, c_{n-1})$ in C, we have that $T_\theta(\mathbf{c}) := (\theta(c_{n-1}), \theta(c_0), \ldots, \theta(c_{n-2}))$ is also in C. Also, C is said to be a *skew-quasi-cyclic* or *quasi-θ-cyclic code of index d* if for any $\mathbf{c} = (c_0, c_1, \ldots, c_{n-1})$ in C, we have that $T_\theta^d(\mathbf{c}) := (\theta(c_{n-d}), \theta(c_{n-d+1}), \ldots, \theta(c_{n-d-1}))$ is also in C.
Let Δ be a column cyclic-shift operator on $R^{2^k \times n}$. We have the characterizations as given in the following theorem. Note that in the theorem below, Φ_{S_1,S_2} is the map as defined in [14], page 54. Also, $\sum_S$ and Γ_{S_1,S_2} are the bijective maps induced by Θ_S and Φ_{S_1,S_2}, as defined in [14], page 55. We also consider the automorphism θ of $\mathcal{R}_k$ as a composition of Θ_S or Φ_{S_1,S_2} (or both of them).

Theorem 4. *A linear code C of length n over $\mathcal{R}_k$ is a quasi-θ-cyclic code of index d if and only if $\Delta^d \circ \Sigma_S \circ \Gamma_{S_1,S_2}(\overline{\Psi}(C)) \subseteq \overline{\Psi}(C)$, for some $S, S_1, S_2 \subseteq [1,k]_{\mathbb{Z}}$, with $|S_1| = |S_2|$.*

Proof. Let $\mathbf{c}$ be any element in C. We can see that

$$\overline{\Psi}\left(\sigma_d(\mathbf{c})\right) = \Delta^d(\overline{\Psi}\left(\mathbf{c}\right)).$$

Since θ is a composition of Θ_S and Φ_{S_1,S_2}, for some $S, S_1, S_2 \subseteq [1,k]_{\mathbb{Z}}$, then we have that

$$\overline{\Psi}(T_\theta^d(\mathbf{c})) = \Delta^d\left(\Sigma_S\left(\Gamma_{S_1,S_2}\left(\overline{\Psi}(\mathbf{c})\right)\right)\right).$$

Therefore, C is invariant under the action of T_θ^d if and only if $\overline{\Psi}(C)$ is invariant under the action of $\Delta^d \circ \Sigma_S \circ \Gamma_{S_1,S_2}$. □

We have to note that the map $\Sigma_S \circ \Gamma_{S_1,S_2}$ induced a row permutation on $\overline{\Psi}(C)$. By applying Theorem 4 with $d = 1$, we have the immediate corollary below.

Corollary 4. *A linear code C of length n over $\mathcal{R}_k$ is a θ-cyclic code if and only if $\Delta \circ \Sigma_S \circ \Gamma_{S_1,S_2}(\overline{\Psi}(C)) \subseteq \overline{\Psi}(C)$, for some $S, S_1, S_2 \subseteq [1,k]_{\mathbb{Z}}$, with $|S_1| = |S_2|$.*

We also have a more technical characterization as follows. Again, ϕ_{S_1,S_2} here is a map from $[1,k]_{\mathbb{Z}}$ to $[1,k]_{\mathbb{Z}}$ defined as in [14], page 54.

Theorem 5. *A linear code C of length n over $\mathcal{R}_k$ is a quasi-θ-cyclic code with index d if and only if there exist quasi-$\vartheta^{\mathrm{ord}\left(\Theta_S \circ \phi_{S_1,S_2}\right)}$-cyclic codes $C_1, C_2, \ldots, C_{2^k}$ of length n over R with index $d \cdot \mathrm{ord}\left(\Theta_S \circ \phi_{S_1,S_2}\right)$ such that*

$$C = \overline{\Psi}^{-1}(C_1, C_2, \ldots, C_{2^k}),$$

where ϑ is the restriction of θ on R, and $T_\vartheta^d(C_i) \subseteq C_{\Sigma_S \circ \Gamma_{S_1,S_2}(i)}$, for all $i \in [1, 2^k]_{\mathbb{Z}}$.

Proof. ($\Longrightarrow$) Let there exist linear codes over R, say $C_1, C_2, \ldots, C_{2^k}$, such that

$$C = \overline{\Psi}^{-1}(C_1, C_2, \ldots, C_{2^k}).$$

For any $\mathbf{c}_i \in C_i$, let $\mathbf{c}_i = (\alpha_1, \alpha_2, \ldots, \alpha_n)$. If $\mathbf{c} = \overline{\Psi}^{-1}\left((\mathbf{0}, \ldots, \mathbf{0}, \mathbf{c}_i, \mathbf{0}, \ldots, \mathbf{0})^T\right)$, then

$$\mathbf{c} = \left(\alpha_1 \sum_{S \supseteq S_i} (-1)^{|S|-|S_i|} v_S, \alpha_2 \sum_{S \supseteq S_i} (-1)^{|S|-|S_i|} v_S, \ldots, \alpha_n \sum_{S \supseteq S_i} (-1)^{|S|-|S_i|} v_S\right).$$

So, if we consider

$$\overline{\Psi}(T_\theta^d(\mathbf{c})) = (\mathbf{0}, \ldots, \mathbf{0}, T_\vartheta^d(\mathbf{c}_i), \mathbf{0}, \ldots, \mathbf{0})^T,$$

then we have $T_\vartheta^d(\mathbf{c}_i)$ is in $C_{\Sigma_S \circ \Gamma_{S_1,S_2}(i)}$. By continuing this process, we have

$$\left(T_\vartheta^d\right)^{\mathrm{ord}\left(\Theta_S \circ \phi_{S_1,S_2}\right)}(\mathbf{c}_i) \in C_i,$$

which means, C_i is a quasi-$\vartheta^{\mathrm{ord}\left(\Theta_S \circ \phi_{S_1,S_2}\right)}$-cyclic code over R with index $d \cdot \mathrm{ord}\left(\Theta_S \circ \phi_{S_1,S_2}\right)$, for all $i \in [1, 2^k]_{\mathbb{Z}}$.

($\Longleftarrow$) For any $\mathbf{c} \in C$, we can see that $\overline{\Psi}(\mathbf{c}) = (\mathbf{c}_1, \ldots, \mathbf{c}_{2^k})^T \in (C_1, C_2, \ldots, C_{2^k})^T$, where $\mathbf{c}_i \in C_i$, for all $i \in [1, 2^k]_{\mathbb{Z}}$. We also knew that for all $i \in [1, 2^k]$, C_i is a quasi-$\vartheta^{\mathrm{ord}\left(\Theta_S \circ \phi_{S_1,S_2}\right)}$-cyclic code over R with index $d \cdot \mathrm{ord}\left(\Theta_S \circ \phi_{S_1,S_2}\right)$, and $T_\vartheta^d(C_i) \subseteq C_{\Sigma_S \circ \Gamma_{S_1,S_2}(i)}$. Then we have

$$\overline{\Psi}\left(T_\theta^d(\mathbf{c})\right) = \left(T_\vartheta^d\left(\mathbf{c}_{\left(\Sigma_S \circ \Gamma_{S_1,S_2}\right)^{-1}(1)}\right), \ldots, T_\vartheta^d\left(\mathbf{c}_{\left(\Sigma_S \circ \Gamma_{S_1,S_2}\right)^{-1}(2^k)}\right)\right)^T \in \overline{\Psi}(C).$$

Therefore, $T_\theta^d(\mathbf{c}) \in C$. □

By applying Theorem 5 with $d = 1$, we derive the property below.

Corollary 5. *A linear code C over $\mathcal{R}_k$ is a θ-cyclic code of length n if and only if there exist quasi-$\vartheta^{\mathrm{ord}\left(\Theta_S \circ \phi_{S_1,S_2}\right)}$-cyclic codes $C_1, C_2, \ldots, C_{2^k}$ of length n over R with index* $\mathrm{ord}\left(\Theta_S \circ \phi_{S_1,S_2}\right)$, *such that*

$$C = \overline{\Psi}_k^{-1}(C_1, C_2, \ldots, C_{2^k})$$

where ϑ is the restriction of θ on R, and $T_\vartheta(C_i) \subseteq C_{\Sigma_S \circ \Gamma_{S_1,S_2}(i)}$, for all $i \in [1, 2^k]_{\mathbb{Z}}$.

4. Examples

As a direct consequence of Theorem 3.2 in [14], we have that for any code C of length n over $\mathcal{R}_k = \mathbb{Z}_m[v_1, v_2, \ldots, v_k]$, where $v_i^2 = v_i$, for all $i \in [1,k]_{\mathbb{Z}}$, there exist codes $C_1, C_2, \ldots, C_{2^k}$ of length n over $\mathbb{Z}_m$ such that $C = \overline{\Psi}^{-1}(C_1, C_2, \ldots, C_{2^k})$.
In the first subsection, we provide examples of skew-cyclic as well as skew-quasi-cyclic codes over $\mathbb{Z}_m$ obtained by Theorem 5 and Corollary 5. In the last two subsections, we provide some more examples of linear codes over the ring $\mathbb{Z}_4$ with the highest known minimum Lee distances. For the readers' convenience, we recall the definition of Lee weight and Lee distance. For any element $\mathbf{x} = (x_1, \ldots, x_n)$ in $\mathbb{Z}_4^n$, the Lee weight of $\mathbf{x}$, denoted by $w_L(\mathbf{x})$, is defined as

$$w_L(\mathbf{x}) = \sum_{i=1}^{n} \min\{|x_i|, |4 - x_i|\}.$$

Using the above weight, we define the Lee distance $d_L(C)$ of a code C as

$$d_L(C) = \min_{\substack{\mathbf{c} \in C \\ \mathbf{c} \neq \mathbf{0}}} w_L(\mathbf{c}).$$

4.1. Examples of skew-cyclic codes and skew-quasi-cyclic codes over the ring $\mathcal{R}_k$

Example 1. Let $\mathcal{R}_1 = \mathbb{Z}_m + v\mathbb{Z}_m$, where $v^2 = v$, and

$$\begin{array}{rccc} \theta : & \mathcal{R}_1 & \longrightarrow & \mathcal{R}_1 \\ & \alpha + \beta v & \longmapsto & \vartheta(\alpha) + \vartheta(\beta)(1 - v) \end{array}$$

where ϑ is an automorphism in $\mathbb{Z}_m$. We can see that, with $S = S_1 = S_2 = \{1\}$, $\theta = \Theta_S \circ \Phi_{S_1,S_2}$ and $\mathrm{ord}(\Theta_S \circ \phi_{S_1,S_2}) = 2$. Let $C = \langle (v, 1 - v) \rangle$. Consider,

$$\begin{aligned} T_\theta\left((\alpha_0 + \alpha_1 v)(v, 1 - v)\right) &= T_\theta\left(((\alpha_0 + \alpha_1)v, \alpha_0(1 - v))\right) \\ &= \left(\vartheta(\alpha_0)v, \vartheta(\alpha_0 + \alpha_1)(1 - v)\right). \end{aligned}$$

The codeword $(\vartheta(\alpha_0)v, \vartheta(\alpha_0 + \alpha_1)(1 - v))$ is in C because

$$(\vartheta(\alpha_0 + \alpha_1) - \vartheta(\alpha_1)v)(v, 1 - v) = \left(\vartheta(\alpha_0)v, \vartheta(\alpha_0 + \alpha_1)(1 - v)\right).$$

As a consequence, the code C is a θ-cyclic code over $\mathcal{R}_1$ of length 2. Moreover, we have

$$\overline{\Psi}((\alpha_0 + \alpha_1)v, \alpha_0(1 - v)) = \begin{pmatrix} 0 & \alpha_0 \\ \alpha_0 + \alpha_1 & 0 \end{pmatrix}.$$

If $C_1 = \langle (0, 1) \rangle$ and $C_2 = \langle (1, 0) \rangle$, then $\overline{\Psi}(C) = (C_1, C_2)^T$. We can check that

$$\begin{array}{rccc} \Sigma_S \circ \Gamma_{S_1,S_2} : & [1,2]_{\mathbb{Z}} & \longrightarrow & [1,2]_{\mathbb{Z}} \\ & 1 & \longmapsto & 2 \\ & 2 & \longmapsto & 1. \end{array}$$

So, we have

$$T_\vartheta(C_1) = C_2 \quad \text{and} \quad T_\vartheta(C_2) = C_1.$$

Furthermore, C_1 and C_2 are quasi-ϑ^2-cyclic codes of index $\mathrm{ord}(\Theta_S \circ \phi_{S_1,S_2})(=2)$.

Example 2. Let $\mathcal{R}_2 = \mathbb{Z}_m + v_1\mathbb{Z}_m + v_2\mathbb{Z}_m + v_1v_2\mathbb{Z}_m$, where $v_i^2 = v_i$, for all $i = 1, 2$, and $v_1v_2 = v_2v_1$. Also, let $\theta = \Theta_S \circ \Phi_{S_1,S_2}$, with $S = S_1 = S_2 = \{1, 2\}$ and $\Phi_{S_1,S_2}(\alpha v_i) = \alpha v_{\phi_{S_1,S_2}(i)}$, where

$$\begin{array}{rccc} \phi_{S_1,S_2}: & [1,2]_{\mathbb{Z}} & \longrightarrow & [1,2]_{\mathbb{Z}} \\ & 1 & \longmapsto & 2 \\ & 2 & \longmapsto & 1 \end{array}$$

We can check that, $\mathrm{ord}(\Theta_S \circ \phi_{S_1,S-2}) = 2$. Let $C = \langle \mathbf{c}_1, \mathbf{c}_2 \rangle$, where $\mathbf{c}_1 = (v_1 - v_1v_2, v_2 - v_1v_2)$ and $\mathbf{c}_2 = (v_2 - v_1v_2, v_1 - v_1v_2)$. For $\gamma = \gamma_0 + \gamma_1 v_1 + \gamma_2 v_2 + \gamma_3 v_1 v_2$ and $\lambda = \lambda_0 + \lambda_1 v_1 + \lambda_2 v_2 + \lambda_3 v_1 v_2$, consider

$$\begin{aligned} T_\theta(\gamma\mathbf{c}_1 + \lambda\mathbf{c}_2) &= T_\theta\left((\gamma_0+\gamma_1)(v_1 - v_1v_2) + (\lambda_0+\lambda_2)(v_2 - v_1v_2), (\gamma_0+\gamma_2)(v_2 - v_1v_2)\right. \\ &\quad \left. +(\lambda_0+\lambda_1)(v_1 - v_1v_2)\right) \\ &= \left((\gamma_0+\gamma_2)(v_2 - v_1v_2) + (\lambda_0+\lambda_1)(v_1 - v_1v_2), (\gamma_0+\gamma_1)(v_1 - v_1v_2)\right. \\ &\quad \left. +(\lambda_0+\lambda_2)(v_2 - v_1v_2)\right) \\ &= \gamma\mathbf{c}_2 + \lambda\mathbf{c}_1 \in C. \end{aligned}$$

We can see that C is a θ-cyclic code of length 2. Also, we have that

$$\overline{\Psi}(\gamma\mathbf{c}_1 + \lambda\mathbf{c}_2) = \begin{pmatrix} 0 & 0 \\ \gamma_0+\gamma_1 & \lambda_0+\lambda_1 \\ \lambda_0+\lambda_2 & \gamma_0+\gamma_2 \\ 0 & 0 \end{pmatrix}$$

and

$$\overline{\Psi}(T_\theta(\gamma\mathbf{c}_1 + \lambda\mathbf{c}_2)) = \begin{pmatrix} 0 & 0 \\ \lambda_0+\lambda_1 & \gamma_0+\gamma_1 \\ \gamma_0+\gamma_2 & \lambda_0+\lambda_2 \\ 0 & 0 \end{pmatrix}.$$

As a consequence, if $C_1 = C_4 = \{(0,0)\}$ and $C_2 = C_3 = \{(\alpha,\beta) | \alpha, \beta \in \mathbb{Z}_m\}$, then $C = \overline{\Psi}^{-1}(C_1, C_2, C_3, C_4)$. The induced map $\Sigma_S \circ \Gamma_{S_1,S_2}$ is as follows

$$\begin{array}{rccc} \Sigma_S \circ \Gamma_{S_1,S_2}: & [1,4]_{\mathbb{Z}} & \longrightarrow & [1,4]_{\mathbb{Z}} \\ & 1 & \longmapsto & 1 \\ & 2 & \longmapsto & 3 \\ & 3 & \longmapsto & 2 \\ & 4 & \longmapsto & 4 \end{array}$$

and $T(C_i) = C_{\Sigma_S \circ \Gamma_{S_1,S_2}(i)}$, for all $i = 1, 2, 3, 4$. Moreover, C_i is a quasi-cyclic code of index 2, for all $i = 1, 2, 3, 4$.

Example 3. Let $\mathcal{R}_1$ and θ as in Example 1. Also, let $C = \langle (v, 0, 1-v, 0)\rangle$. For $\alpha = \alpha_0 + \alpha_1 v$, consider

$$\begin{aligned} T_\theta^2(\alpha(v,0,1-v,0)) &= T_\theta^2((\alpha_0+\alpha_1)v, 0, \alpha_0(1-v), 0) \\ &= (\vartheta(\alpha_0)v, 0, \vartheta(\alpha_0+\alpha_1)(1-v), 0) \\ &= (\vartheta(\alpha_0+\alpha_1) - \vartheta(\alpha_1)v)\,(v,0,1-v,0) \in C. \end{aligned}$$

So, C is a quasi-θ-cyclic code of index 2. We can check that, if $C_1 = \langle (0,0,1,0)\rangle$ and $C_1 = \langle (1,0,0,0)\rangle$, then $C = \overline{\Psi}^{-1}(C_1, C_2)$. Moreover,

$$T_\vartheta^2(C_i) \subseteq C_{\Sigma_S \circ \Gamma_{S_1,S_2}(i)}$$

for all $i = 1, 2$, where $\Sigma_S \circ \Gamma_{S_1,S_2}$ map is the same as the one in Example 1. The code C_i is a quasi-ϑ^2-cyclic code of index $2 \times \mathrm{ord}(\Theta_S \circ \phi_{S_1,S_2})(= 4)$.

Example 4. Let $\mathcal{R}_2$ and θ as in Example 2. Also, let $C = \langle \mathbf{c}_1, \mathbf{c}_2\rangle$, where $\mathbf{c}_1 = (v_1 - v_1v_2, 0, v_2 - v_1v_2, 0)$ and $\mathbf{c}_2 = (v_2 - v_1v_2, 0, v_1 - v_1v_2, 0)$. For $\gamma = \gamma_0 + \gamma_1 v_1 + \gamma_2 v_2 + \gamma_3 v_1 v_2$ and $\lambda = \lambda_0 + \lambda_1 v_1 + \lambda_2 v_2 + \lambda_3 v_1 v_2$, consider

$$\begin{aligned} T_\theta^2(\gamma\mathbf{c}_1 + \lambda\mathbf{c}_2) &= T_\theta^2\,((\gamma_0+\gamma_1)(v_1 - v_1v_2) \\ &\quad +(\lambda_0+\lambda_2)(v_2 - v_1v_2), 0, (\gamma_0+\gamma_2)(v_2 - v_1v_2) + (\lambda_0+\lambda_1)(v_1 - v_1v_2), 0) \\ &= ((\gamma_0+\gamma_2)(v_2 - v_1v_2) + (\lambda_0+\lambda_1)(v_1 - v_1v_2), 0, (\gamma_0+\gamma_1)(v_1 - v_1v_2) \\ &\quad +(\lambda_0+\lambda_2)(v_2 - v_1v_2), 0) \\ &= \gamma\mathbf{c}_2 + \lambda\mathbf{c}_1 \in C. \end{aligned}$$

So, C is a quasi-θ-cyclic code of index 2. We can check that, if $C_1 = C_4 = \{(0,0,0,0)\}$ and $C_2 = C_3 = \{(\alpha, 0, \beta, 0) | \alpha, \beta \in \mathbb{Z}_m\}$, then $C = \overline{\Psi}^{-1}(C_1, C_2, C_3, C_4)$. Also, we have that

$$T^2(C_i) \subseteq C_{\Sigma_S \circ \Gamma_{S_1,S_2}(i)},$$

for all $i = 1, 2, 3, 4$, where the map $\Sigma_S \circ \Gamma_{S_1,S_2}$ is the same as the one in Example 2. The code C_i is a quasi-cyclic code of index $2 \times \mathrm{ord}(\Theta_S \circ \phi_{S_1,S_2})(= 4)$, for all $i = 1, 2, 3, 4$.

4.2. Other examples using the map Ψ

Example 5. Let $\mathcal{R}_1 = \mathbb{Z}_4[v]$, where $v^2 = v$. Also, let $C = \langle (1\ v\ 1+v\ 3)\rangle$. We can check that

$$\overline{\Psi}((1\ v\ 1+v\ 3)) = \begin{pmatrix} 1 & 0 & 1 & 3 \\ 1 & 1 & 2 & 3 \end{pmatrix}.$$

Then, if we choose $C_1 = \langle (1\ 0\ 1\ 3)\rangle$ and $C_2 = \langle (1\ 1\ 2\ 3)\rangle$, we have $C = \overline{\Psi}^{-1}(C_1, C_2)$.

Moreover, we may have more explicit examples for Hermitian self-dual codes as follow.

Example 6. Let $\mathcal{R}_1 = \mathbb{Z}_4[v]$, where $v^2 = v$. In this ring, $\Theta_1(v) = 1 - v$. Let $C = \langle (v\ v\ v)\rangle$ be a code over $\mathcal{R}_1$. Then C is a Hermitian self-dual code (by Theorem 3.3 in [14]). Since

$$\overline{\Psi}((v\ v\ v)) = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix},$$

we have that $C = \overline{\Psi}^{-1}(C_1, C_2)$, where $C_1 = C_2 = \langle (1\ 1\ 1)\rangle$. We can check that C_1 is an Euclidean self-dual code over $\mathbb{Z}_4$. Therefore, we have $C_2 = C_1^{\perp}$, as stated in Proposition 3.1 and Theorem 3.4 in [14].

Also, we have the following example for Euclidean self-dual codes.

Example 7. Let $\mathcal{R}_1 = \mathbb{Z}_4[v]$, where $v^2 = v$. Take $C = \langle (v\ 1 - v), (1 - v\ v)\rangle$. We can see that C is an Euclidean self-dual code over $\mathcal{R}_1$. Also, we know that

$$\overline{\Psi}((v\ 1 - v)) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \text{and} \qquad \overline{\Psi}((1 - v\ v)) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

If we take $C_1 = C_2 = \langle (1\ 0), (0\ 1)\rangle$, then we have $C = \overline{\Psi}^{-1}(C_1, C_2)$. We can check that C_1 and C_2 are Euclidean self-dual codes over $\mathbb{Z}_4$ also, as stated in Theorem 3.5 in [14].

4.3. New linear codes over $\mathbb{Z}_4$

In this section, we use the map φ_1 to obtained linear codes over $\mathbb{Z}_4$ from codes over $\mathcal{R}_1 = \mathbb{Z}_4 + v\mathbb{Z}_4$, where $v^2 = v$. Again, see [14] page 54 for the definition of this map. First, let us recall Example 5.1 appeared in [14] below.

Example 8. Define a map φ_1 as follows.

$$\begin{array}{rccc} \varphi_1 : & \mathbb{Z}_4 + v\mathbb{Z}_4 & \longrightarrow & \mathbb{Z}_4^2, \\ & \alpha + v\beta & \longmapsto & (\alpha, 2\alpha + \beta). \end{array}$$

Let $C = \langle 1 + v\rangle = \{0, 1 + v, 2 + 2v, 3 + 3v, 2v, 2, 1 + 3v, 3 + v\}$ be a code of length 1 over $\mathcal{R}_1 = \mathbb{Z}_4 + v\mathbb{Z}_4$, where $v^2 = v$. We have,

$$\varphi_1(1 + v) = (1, 3), \quad \varphi_1(2 + 2v) = (2, 2), \quad \varphi_1(3 + 3v) = (3, 1), \quad \varphi_1(2v) = (0, 2),$$

$$\varphi_1(2) = (2, 0), \quad \varphi_1(1 + 3v) = (1, 1), \quad \varphi_1(3 + v) = (3, 3).$$

We can see that $d_L(\varphi_1(C)) = 2$ and $|\varphi_1(C)| = 8$.

The Table 1 gives some examples of linear codes over $\mathbb{Z}_4$ with the highest known minimum Lee distance (see [2], [3]), obtained by a similar way as in Example 8.

Remark 1. We may obtain many other linear codes over $\mathbb{Z}_4$ having good minimum Lee distances by the similar method. Moreover, the first author with H.C. Tang [20] have also introduced several different methods to construct linear codes over $\mathbb{Z}_4$, and by using the methods we succeed to construct many linear codes over $\mathbb{Z}_4$ with good minimum Lee distances (see Tables 1,2, and 3 in [20]).

n	C	φ_1	$d_L(\varphi_1(C))$	$\lvert\varphi_1(C)\rvert$
2	$\langle 1+v\rangle$	$\alpha+v\beta \longmapsto (\alpha, 2\alpha+\beta)$	2	8
2	$\langle 2\rangle$	$\alpha+v\beta \longmapsto (\alpha, \alpha+\beta)$	2	4
3	$\langle 2\rangle$	$\alpha+v\beta \longmapsto (\alpha, \beta, \alpha+\beta)$	4	4
3	$\langle 2+2v\rangle$	$\alpha+v\beta \longmapsto (\alpha, \beta, \alpha+\beta)$	4	2
3	$\langle 2v\rangle$	$\alpha+v\beta \longmapsto (\alpha, \beta, \alpha+\beta)$	4	2
4	$\langle 2v\rangle$	$\alpha+v\beta \longmapsto (\alpha, \beta, \alpha+\beta, \alpha+\beta)$	6	2
5	$\langle 2\rangle$	$\alpha+v\beta \longmapsto (\alpha, \beta, \alpha+\beta, \alpha, \alpha+\beta)$	6	4
6	$\langle 1+v\rangle$	$\alpha+v\beta \longmapsto (\alpha, 2\alpha+\beta, \alpha, 2\alpha+\beta, \alpha, 2\alpha+\beta)$	6	8
6	$\langle 2\rangle$	$\alpha+v\beta \longmapsto (\alpha, \beta, \alpha+\beta, \alpha, \beta, \alpha+\beta)$	8	4
7	$\langle 2v\rangle$	$\alpha+v\beta \longmapsto (\alpha, \beta, \alpha+\beta, \alpha, \beta, \alpha+\beta, \alpha+\beta)$	10	2

Table 1. Some examples of linear codes over $\mathbb{Z}_4$ with the highest known minimum Lee distances.

5. Conclusion

In this paper, we considered cyclic codes and their generalization over a very general ring,

$$\mathcal{R}_k = R[v_1, v_2, \ldots, v_k]/\left\langle v_i^2 - v_i\right\rangle_{i=1}^{k},$$

where R is a finite commutative Frobenius ring. To be precise, we considered cyclic codes, quasi-cyclic codes, skew-cyclic codes, and also quasi-skew-cyclic codes over the ring $\mathcal{R}_k$. We derived a necessary and sufficient condition for the codes over $\mathcal{R}_k$ to be cyclic, quasi-cyclic, skew-cyclic as well as quasi-skew-cyclic. The necessary and sufficient conditions provide a way to construct skew-cyclic as well as quasi-skew-cyclic codes over the ring $\mathcal{R}_k$. As an application, we also give some examples of optimal codes over the ring $\mathbb{Z}_4$ with respect to the Lee distance.

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