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Research Article

On distance induced Seidel matrices for signed graphs

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Abstract: A signed graph $\Sigma = (G, \sigma)$ is a graph G together with a signature function σ which assigns 1 or -1 on the edges of G. Seidel matrix of an unsigned graph is already defined and researchers investigated some of its spectral and other properties. Considering the recently introduced notion of signed distance in signed graphs and that of the distance compatible signed graphs, we define distance induced Seidel matrices for such signed graphs and analyze their spectrum mainly for some classes of unbalanced distance compatible signed graphs, as balanced signed graphs possess the same distance induced Seidel spectrum as that of its underlying graph. We also deal with the distance compatibility issue in the line graph of a distance compatible signed graph and discuss the corresponding distance induced Seidel spectrum in this regard.

Keywords: signed graphs, distance compatibility, adjacency matrix, Seidel matrix, Seidel spectrum, vertex edge orientation, line graph.

AMS Subject classification: 05C22, 05C50

1. Introduction

In this article, we deal with the distance induced Seidel matrices for the distance compatible signed graphs and study some of their properties, especially their spectral characteristics. Formally, a signed graph $\Sigma = (G, \sigma)$ is a graph $G = (V, E)$ together with a signature or signing function $\sigma : E \to \{-1, 1\}$ that assigns -1 or 1 to each of

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the edges. Those edges with the labels 1 are called the positive edges and naturally the edges with -1 on them are the negative edges. A graph G is treated as a signed graph with all its edges as positive. Signed graphs have many applications, mainly to model the signed social networks to analyze how well such a system will work when the friendship or rivalry existing among the objects in the network. Social balance theory is a well established research area, instances of which can be found in [1, 5, 11–13] to name a few, where signed graphs play their vital role in analysing it. Using the concept of signed distance in signed graphs and the associated notion of distance compatibility [4], we deal with the distance induced Seidel matrices and the corresponding spectrum of some unbalanced distance compatible signed graphs as we found that the distance induced Seidel spectrum of balanced signed graphs coincide with the Seidel spectrum of the underlying graph. This paper also deals with the distance compatibility of the line graph of a signed graph and correspondingly takes into account some of their distance induced Seidel spectral properties.

2. Distance induced Seidel matrix

It was van Lint and Seidel who introduced Seidel matrices for (unsigned) graphs in their seminal paper [10]. Seidel matrices for graphs have several applications, examples of some of which can be had from [7–9]. Here we introduce distance induced Seidel matrices for distance compatible signed graphs. Before we define a distance compatible signed graph, we need notions of signed distance in signed graphs recently introduced by Shahul Hameed et al. [4]. We assume from now on that all the underlying graphs are connected, finite and simple. Given a signed graph $\Sigma = (G, \sigma)$, we define the sign $\sigma(P_{(u,v)})$ of a path $P_{(u,v)} : uv_1v_2 \cdots v_{n-1}v$ with the initial vertex $u = v_0$ and the end vertex $v = v_n$ as $\sigma(P_{(u,v)}) = \prod_{i=0}^{n-1} \sigma(v_iv_{i+1})$. i.e., it is the product of the signs on the edges in the path. Now we have an array of necessary definitions that follows, all of which are taken from [4].

There may be more than one shortest path between two vertices u and v in a given signed graph. We denote the set of all the shortest paths $P_{(u,v)}$ as $\mathcal{P}_{(u,v)}$ and define $\sigma_{\max}(u, v) = 1$, if the sign of at least one shortest path joining u and v is positive; otherwise it is -1 . Similarly, $\sigma_{\min}(u, v) = -1$, if the sign of at least one shortest path joining u and v is negative; otherwise it is 1. When the sign of all shortest paths joining u and v is the same, we say that the vertices u and v are distance compatible vertices. Moreover, when all pairs of vertices in a signed graph are distance compatible, we call such a signed graph a distance compatible signed graph. Simply put, a distance compatible signed graph is the one in which all the shortest paths between any two pairs of vertices have the same sign; either they are all positive paths or all of them are negative paths. Note that the geodesic graphs, i.e., graphs having a unique shortest path between any pairs of vertices, are distance compatible.

Now, after leaving the above discussion there for a while, let us delve into another important application of signed graphs in social psychology and in other fields, namely

balance theory and the associated ideas, the details of which can be had from [13]. In a signed graph $\Sigma = (G, \sigma)$, the sign $\sigma(C)$ of a cycle C is the product of the sign of the edges in that cycle. Σ is called a balanced signed graph if every cycle C in it has the sign $\sigma(C) = 1$. Also a signed graph $\Sigma = (G, \sigma)$ is said to be anti-balanced if its negative $-\Sigma = (G, -\sigma)$ is balanced. There are various characterizations to verify whether a signed graph is balanced or not. It is shown in [4] that every balanced signed graph is distance compatible, but the converse need not always be true. For instance, an odd unbalanced cycle, being geodesic, is distance compatible. If the underlying graph is bipartite, then the signed graph built on such a graph will be balanced if and only if it is distance compatible [4]. Now we need a few more definitions before we deal with the distance induced Seidel matrix for a distance compatible signed graph using which we analyse the state of balance and unbalance in such signed graphs.
Consider a distance compatible signed graph $\Sigma = (G, \sigma)$ having order n. For this distance compatible signed graph we denote the common value of $\sigma_{\max}(u, v) = \sigma_{\min}(u, v)$ by $\sigma(u, v)$. If u and v are adjacent, we denote the same by $\sigma(uv)$ or $\sigma(u, v)$. Define the complete signed graph $K_n(\Sigma)$ associated with Σ as the signed graph with the adjacency matrix $A(K_n(\Sigma)) = (\sigma(u, v))_{u,v \in V(G)}$. With the help of this $K_n(\Sigma)$, we define the distance induced Seidel matrix denoted by $\mathbb{S}(\Sigma)$ as:

$$\mathbb{S}(\Sigma) = A(K_n(\Sigma)) - 2A(\Sigma) \tag{2.1}$$

Note that in the case of an unsigned graph G, the Seidel matrix will be nothing but $J - I - 2A(G)$, the usual Seidel matrix $\mathbb{S}(G)$ of G.
To move on, we need a very important operation in signed graphs called switching which is defined as follows. By switching a signed graph $\Sigma = (G, \sigma)$, we mean that it gives rise to a signed graph $\Sigma^{\zeta} = (G, \sigma^{\zeta})$ where $\zeta : V \to \{-1, 1\}$ is called the switching function and the signature function $\sigma^{\zeta}(uv)$ satisfies the equation $\sigma^{\zeta}(uv) = \zeta(u)\sigma(uv)\zeta(v)$. Note that the sign of a path in the switched signed graph satisfy $\sigma^{\zeta}(P_{(u,v)}) = \zeta(u)\sigma(P_{(u,v)})\zeta(v)$. A switching function ζ provides a self-invertible diagonal matrix $T = \text{diag}(\zeta(u))_{u \in V(\Sigma)}$. Let us call it the switching matrix associated with ζ. Two signed graphs are said to be switching equivalent if one of them can be obtained from the other by a switching.
Consider the following two results, which we recall very frequently for the discussion that follows.

Lemma 1 ([11]). *A signed graph is balanced if and only if it is switching equivalent to the underlying graph.*

By adjacency spectrum or simply spectrum of a signed graph we mean the multiset of eigenvalues of its adjacency matrix counting multiplicities. We use the notation of a two-row matrix form to denote the spectrum where the top row indicates the spectral values and the second row gives the corresponding multiplicities. Two signed graphs are said to be cospectral if they have the same spectrum with respect to their adjacency matrices. For convenience, let us call the distance-induced Seidel matrix of a distance-compatible signed graph the $\mathbb{S}$-matrix and the corresponding spectrum the $\mathbb{S}$-spectrum. When two signed graphs have the same spectrum corresponding to their $\mathbb{S}$-matrices, we call them $\mathbb{S}$-cospectral. The following is a significant theorem [1] that deals with the cospectrality of a signed graph and its underlying graph, of course with respect to their adjacency matrices.

Theorem 1 ([1]). *A signed graph $\Sigma = (G, \sigma)$ is balanced if and only if Σ and G are cospectral.*

It is interesting to see that the $\mathbb{S}$-matrix, $\mathbb{S}(\Sigma) = A(K_n(\Sigma)) - 2A(\Sigma)$ in Equation (2.1) is actually the adjacency matrix of a signed complete graph. We denote this complete signed graph by $K_n^*(\Sigma)$ or simply by K_n^*, if no confusion arises. i.e., $\mathbb{S}(\Sigma) = A(K_n^*(\Sigma))$. We deal with this $K_n^*(\Sigma)$ in some detail in Section 4. Indeed,

$$\mathbb{S}(\Sigma)(i,j) = \begin{cases} 0 & if\ u_i = u_j \\ \sigma(u_i, u_j) & if\ u_i \nsim u_j\ in\ \Sigma \\ -\sigma(u_i u_j) & if\ u_i \sim u_j\ in\ \Sigma. \end{cases}$$

Now we provide two results regarding the $\mathbb{S}$-matrices of distance compatible signed graphs, where the latter one deals with balanced signed graphs.

Lemma 2. *The $\mathbb{S}$-matrices of two switching equivalent distance compatible signed graphs are similar. i.e., a switching preserves $\mathbb{S}$-spectrum.*

Proof. Let Σ_1 and Σ_2 be two switching equivalent distance compatible signed graphs. Then using the switching matrix T associated with the switching function, $A(\Sigma_2) = TA(\Sigma_1)T^{-1}$ and since $\sigma^{\zeta}(P_{(u,v)}) = \zeta(u)\sigma(P_{(u,v)})\zeta(v)$, $A(K_n(\Sigma_2)) = TA(K_n(\Sigma_1))T^{-1}$. Therefore,

$$\begin{aligned} \mathbb{S}(\Sigma_2) &= A(K_n(\Sigma_2)) - 2A(\Sigma_2) \\ &= T\big(A(K_n(\Sigma_1)) - 2A(\Sigma_1)\big)T^{-1} \\ &= T\mathbb{S}(\Sigma_1)T^{-1}. \end{aligned}$$ □

Theorem 2. *If a distance compatible signed graph $\Sigma = (G, \sigma)$ is balanced then Σ and G are $\mathbb{S}$-cospectral.*

Proof. This follows easily from Lemma 1 and Lemma 2. □

The converse need not be true since the unbalanced odd cycle C_7^-, for example, has the same $\mathbb{S}$-spectrum as that of C_7 (see the formula in Theorem 3).

3. $\mathbb{S}$-spectrum of some unbalanced signed graphs

In this section, first we explicitly provide a formula for the computation of eigenvalues of the $\mathbb{S}$-matrix of an unbalanced odd cycle. We take $n = 2k+1 \geq 3$ and denote the unbalanced cycle of this order as C_n^-. Since a switching preserves the $\mathbb{S}$-spectrum, we take the unbalanced cycle to have only one negative sign such that $\sigma(v_n v_1) = -1$ and all the other edges are assigned as positive when the cycle C_n is represented as $C_n : v_1 v_2 \cdots v_n v_1$. We use the following lemma for computational purposes in the theorem that follows.

Lemma 3 ([6]). $\sum\limits_{r=1}^{k} \cos(r\theta) = \frac{1}{2}\Big(\frac{\sin\big((2k+1)\theta/2\big)}{\sin(\theta/2)} - 1\Big).$

The next theorem provides a formula for computing the $\mathbb{S}$-spectrum of the unbalanced odd cycle C_n^-.

Theorem 3. *For an odd unbalanced cycle C_n^- of order $n = 2k+1 \geq 3$, $\mathbb{S}$-spectrum is given by*

$$\begin{pmatrix} 3+(-1)^k & (-1)^j \csc\big((2j+1)\frac{\pi}{2n}\big) - 4\cos\big((2j+1)\frac{\pi}{n}\big) - 1 \\ 1 & 2 \quad (j = 0, 1, 2, \ldots, k-1) \end{pmatrix}. \tag{3.1}$$

Proof. The $\mathbb{S}$-matrix here is $\mathbb{S}(C_n^-) = A(K_n(C_n^-)) - 2A(C_n^-)$ of the distance compatible odd signed cycle C_n^- which is a real symmetric matrix that simplifies to

$$\mathbb{S}(C_n^-) = \begin{pmatrix} 0 & -1 & 1 & \cdots 1 & 1 & -1 \cdots & -1 & 1 \\ -1 & 0 & -1 & 1 \cdots 1 & 1 & \cdots & \cdots & -1 \\ \cdots & \cdots & \cdots & \cdots\cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots\cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & -1 & \cdots & \cdots & \cdots & \cdots & -1 & 0 \end{pmatrix}$$

To find the eigenvalues of this matrix which are real numbers, we proceed as follows. We choose $\rho = e^{i\theta}$ and the eigenvector corresponding to the eigenvalue λ as $X = [\rho, \rho^2, \ldots, \rho^n]^T$. Thus, from equation $\mathbb{S}(\Sigma)X = \lambda X$, $\lambda = \sum_{r=1}^{k} c_i \rho^r - \sum_{r=1}^{k} c_i \rho^{n-r}$, where $c_1 = -1$ and $c_i = 1$ for i = 2, 3, ..., k. Choosing now $\rho^n = -1$ so that $\rho = e^{i\theta_j} = e^{\frac{(2j+1)i\pi}{n}}$ for $j = 0, 1, \ldots, n-1$, and hence,

$$\begin{aligned} \lambda_j &= \sum_{r=1}^{k} c_i \big[\rho^r + \rho^{-r}\big] \\ &= 2\sum_{r=1}^{k} c_i \cos(r\theta_j) \\ &= 2\sum_{r=1}^{k} \cos(r\theta_j) - 4\cos(\theta_j) \end{aligned}$$

$$= 2\sum_{r=1}^{k}\cos(r\theta_j) - 4\cos(\theta_j)$$
$$= \frac{\sin\big((2k+1)\theta_j/2\big)}{\sin(\theta_j/2)} - 1 - 4\cos(\theta_j)$$

Now for $j = k, \theta_k = \pi$ and $\lambda_k = 3 + (-1)^k$. The remaining eigenvalues are paired as $\lambda_{2k} = \lambda_0, \lambda_{2k-1} = \lambda_1, \cdots, \lambda_{k+1} = \lambda_{k-1}$. Thus for $j = 0, 1, \ldots k-1$,

$$\lambda_j = \frac{\sin\big((2k+1)\theta_j/2\big)}{\sin(\theta_j/2)} - 1 - 4\cos(\theta_j)$$
$$= (-1)^j \csc\big((2j+1)\frac{\pi}{2n}\big) - 4\cos\big((2j+1)\frac{\pi}{n}\big) - 1. \qquad \square$$

The above formula brings out an important corollary, given below.

Corollary 1. *Among all odd unbalanced signed cycles C_n^-, only C_3^- and C_5^- have the $\mathbb{S}$-spectrum coinciding with the adjacency spectrum of the corresponding all-positive complete graph K_n.*

Proof. It is well known that the spectrum of all positive complete graph K_n is $\begin{pmatrix} n-1 & -1 \\ 1 & n-1 \end{pmatrix}$. As such using the formula in (3.1), the value $3 + (-1)^k = 3 + (-1)^{(n-1)/2}$ coincides with $n-1$ only when $n = 3$ and $n = 5$, which completes the proof. $\square$

Next, as an important example, we compute the $\mathbb{S}$-spectrum of an unbalanced signed wheel $(W_{n+1}, \sigma) = (C_n \vee K_1, \sigma)$ where $n = 2k+1$ is odd, and the signature σ is such that $\sigma(e) = -1$ if $e \in E(C_n)$ and 1 otherwise. We deal with the line graph of this wheel and its $\mathbb{S}$-spectrum in Section 5.

Theorem 4. *The $\mathbb{S}$-spectrum of the signed wheel $(W_{n+1}, \sigma) = (C_n \vee K_1, \sigma)$, $n = 2k+1$, is*

$$\begin{pmatrix} n & -1 \\ 1 & n \end{pmatrix}.$$

Proof. Here $\mathbb{S}(W_{n+1}, \sigma) = \mathbb{S}$, say for convenience, is the block matrix given by

$$\mathbb{S} = \begin{bmatrix} 0 & -J_{1\times n} \\ -J_{n\times 1} & (J-I)_{n\times n} \end{bmatrix}_{(n+1)}$$

The characteristic polynomial,

$$\det(xI - \mathbb{S}) = \det\begin{bmatrix} x & J_{1\times n} \\ J_{n\times 1} & (x+1)I_n - J_{n\times n} \end{bmatrix}$$

Without loss of generality, we assume $((x+1)I_n - J_{n\times n})^{-1}$ exist. Then by Schur's complement method of determinant we can write it as

$$\det\Big((x+1)I_n - J_{n\times n}\Big)\det\Big(x - J_{1\times n}\ \big((x+1)I_n - J_{n\times n}\big)^{-1}\ J_{n\times 1}\Big)$$

$= \det\Big((x+1)I_n - J_{n\times n}\Big)\Big(x - \dfrac{n}{x-n+1}\Big) = (x+1)^n\ (x-n)$, which completes the proof. □

As pointed out in Corollary 1, only C_5^- and C_3^- have the $\mathbb{S}$-spectrum coinciding with the adjacency spectrum of the all positive complete graph K_n. But in the case of signed wheels of special types which are considered in the above theorem, irrespective of the order, all such signed wheels do exhibit this peculiar phenomenon. Note that the signed wheels under discussion are all anti-balanced. Thus the above two examples open up a few more problems for further exploration which we do in the next section.

4. Signed graphs for which K_n^* are balanced

In the case of a signed graph $\Sigma = (G, \sigma)$, $K_n^*(\Sigma)$ has $-\Sigma = (G, -\sigma)$ as its subgraph. Therefore, when $K_n^*(\Sigma)$ is balanced, $-\Sigma$ must be balanced or in other words, Σ must be anti-balanced. But the anti-balance in Σ alone need not ensure the balance in $K_n^*(\Sigma)$. In fact, Corollary 1 provides many counterexamples in this regard in the form of unbalanced signed odd cycles C_n^- when $n \neq 3$ and $n \neq 5$. Now we provide a characterization of a signed graph Σ for which K_n^* is balanced. We begin with a simple lemma for which the proof is omitted since it is straight forward from the notion of anti-balance.

Lemma 4. *For every anti-balanced complete signed graph Σ, $K_n^*(\Sigma)$ is balanced.*

Theorem 5. *Let $\Sigma = (G, \sigma)$ be a distance compatible (non complete) signed graph. Then K_n^* is balanced if and only if Σ is anti-balanced and G is of diameter two.*

Proof. First we assume that K_n^* is balanced. This means that its subgraph $-\Sigma$ is balanced, i.e., Σ is anti-balanced. Let us denote the signature of an edge uv by $\sigma^*(uv)$. Then,

$$\sigma^*(uv) = \begin{cases} -\sigma(uv) & if\ u \sim v\ in\ \Sigma \\ \sigma(u,v) & if\ u \nsim v\ in\ \Sigma \end{cases}$$

Let ζ be the switching function that switches the K_n^* to the all complete graph K_n using its balance. Then $\zeta(u)\sigma^*(uv)\zeta(v) = 1$ for all edges uv in K_n^*. If possible suppose that the diameter of G is $k > 2$ and take two vertices u and v in G such that, the distance, $d(u,v) = k$. Let $P(u,v) : uu_1u_2\cdots u_{k-1}v$ be a shortest path joining u and

v in Σ. Then

$$\begin{aligned}1=\zeta(u)\sigma^*(uv)\zeta(v) &= \zeta(u)\sigma(uu_1)\zeta(u_1)\sigma(u_1u_2)\cdots\sigma(u_{k-1}v)\zeta(v)\\ &= \big[\zeta(u)\sigma(uu_1)\zeta(u_1)\big]\big[\zeta(u_1)\sigma(u_1u_2)\zeta(u_2)\big]\cdots\times\\ &\quad \big[\zeta(u_{k-1})\sigma(u_{k-1}v)\zeta(v)\big]\\ &= (-1)^k\big[\zeta(u)\sigma^*(uu_1))\zeta(u_1)\big]\big[\zeta(u_1)\sigma^*(u_1u_2)\zeta(u_2)\big]\cdots\times\\ &\quad \big[\zeta(u_{k-1})\sigma^*(u_{k-1}v)\zeta(v)\big]\\ &= (-1)^k,\end{aligned}$$

since the product after $(-1)^k$ amounts to 1 due to the assumption of the switching equivalency of K_n^*. Now $(-1)^k = 1$ holds only if k is an even number. Keeping this in mind, if we assume that $k > 2$, $P(u, , u_{k-1})$ being a shortest path of length $k-1$, proceeding as above,

$$\zeta(u)\sigma^*(uu_{k-1})\zeta(u_{k-1}) = (-1)^{k-1} = -1, \text{ a contradiction.}$$

This implies $k \leq 2$. But Σ being built on a non-complete underlying graph, this implies that $k = 2$.
Conversely, assume that Σ is anti-balanced and G is of diameter 2. The signature of every edge uv in $-\Sigma$ is $-\sigma(uv)$. The graph $-\Sigma$ is balanced by assumption. Thus there exists a switching function ζ that switches $-\Sigma$ to G; that is, $\zeta(u)\big(-\sigma(uv)\big)\zeta(v) = 1$ for every $u \sim v$ in Σ. Let uv be any edge in K_n^* but not in $-\Sigma$. Then $u \nsim v$ in Σ leading to the existence of a vertex u' such that $P_{(u,v)} : uu'v$ has distance $d(u, v) = 2$. Hence $\sigma^*(uv) = \sigma(uu')\sigma(u'v)$. As such,

$$\begin{aligned}\zeta(u)\sigma^*(u,v)\zeta(v) &= \zeta(u)\big[\sigma(uu')\sigma(u'v)\big]\zeta(v)\\ &= \big[\zeta(u)\big(-\sigma(uu')\big)\zeta(u')\big]\big[\zeta(u')\big(-\sigma(u'v)\big)\zeta(v)\big]\\ &= \big[\zeta(u)\sigma^*(uu')\zeta(u')\big]\big[\zeta(u')\sigma^*(u'v)\zeta(v)\big]\\ &= 1.\end{aligned}$$

This proves that ζ switches K_n^* to K_n and hence the balance. □

5. Distance compatibility in the line graph of a signed graph

There are various ways of defining a line graph of a signed graph, the details of which are given in [2]. We follow the definition given below, taken from [2].

Definition 1. Let $\Sigma = (G, \sigma)$ be a signed graph. A vertex-edge orientation, or simply orientation, on Σ is a function $\eta : V(G) \times E(G) \to \{1, 0, -1\}$ satisfying the conditions

i. $\eta(u, e) = 0$ when u is not an end vertex of the edge e;

ii. $\eta(u, e) = 1$ or -1 if u is an end vertex of the edge e;

iii. $\eta(u,e)\eta(v,e) = -\sigma(e)$ if $e = uv$.

The vertex-edge incidence matrix of Σ corresponding to the orientation η is denoted by B_η such that (u,e) entry is equal to $\eta(u,e)$. The adjacency matrix of the line graph $\Lambda(\Sigma)$ of Σ, denoted by $A(\Lambda(\Sigma))$, will be then equal to $B_\eta^T B_\eta - 2I$, irrespective of the orientation chosen. We remark that the signature function σ^L on $\Lambda(\Sigma)$ is defined by using the orientation η on Σ is $\sigma^L(e_i, e_j) = \eta(u, e_i)\eta(u, e_j)$, where u is the vertex common to e_i and e_j and zero otherwise. The following figure Fig 1 contains a signed graph Σ, $K_n^*(\Sigma)$, and the line graph $\Lambda(\Sigma)$ for illustrating these concepts.

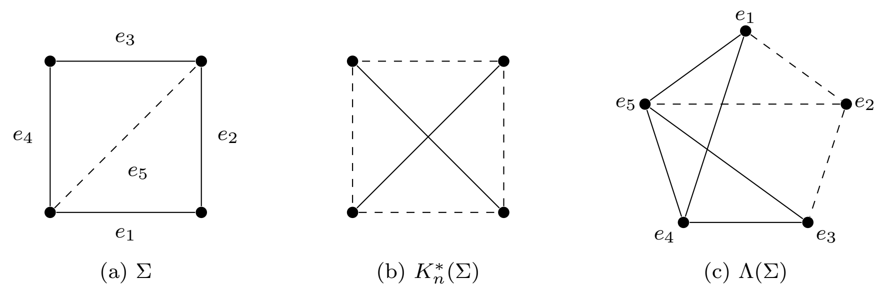

Figure 1. **An illustration for the line graph of a signed graph**

We recall the following facts about the line graphs:

i. The line graph of a cycle graph is a cycle grpah;

ii. The line graph of a star graph is a complete graph;

iii. The line graph of a tree is a block graph.

Lemma 5. *Line graph of a signed tree is always distance compatible.*

Proof. Line graph of a signed tree is a a signed block graph and we have that every signed block graph is distance compatible. □

Lemma 6. *Let $\Lambda(G)$ be the line graph of a connected graph G. Let $e \nsim f$ in $\Lambda(G)$ and $P_{(e,f)} : e_0e_1 \dots e_k$ be any shortest path joining e and f, where we denote $e_0 = e, e_k = f$ and $e_i = u_iu_{i+1}$ for some vertices u_i in $V(G), i = 0, 1, 2, \cdots, k$. Then the path $u_1u_2 \dots u_k$ is a shortest path joining u_1 and u_k in G.*

Proof. We prove by contradiction method. If possible suppose $u_1u_2 \dots u_k$ is not a shortest path joining u_1 and u_k in G, then there is another a path $u_1'u_2'u_3' \dots u_m'$, where $u_1' = u_1$ and $u_m' = u_k$ with $m < k$. Let $e_i' = u_i'u_{i+1}'$ for $i = 1, 2, \cdots, m-1$. This gives another a path joining e and $f, Q_{(e,f)} : e_0e_1'e_2' \dots e_{m-1}'e_k$ of length smaller than the given one and hence a contradiction. □

Theorem 6. *If $\Sigma = (G, \sigma)$ is a distance compatible signed graph, then so is the line graph $\Lambda(\Sigma) = (\Lambda(G), \sigma^L)$.*

Proof. Let e, f be any two non adjacent vertices in $\Lambda(\Sigma)$. If there exist only one shortest path joining them, there is nothing to prove. Otherwise, we arbitrarily pick two shortest paths $P_{(e,f)} : e = u_0u_1, u_1u_2, u_2u_3, \ldots, u_{k-1}u_k, u_ku_{k+1} = f$ and $Q_{(e,f)} : e = u_0u_1, u_1u_2', u_2'u_3', \ldots, u_{k-1}'u_k, u_ku_{k+1} = f$. We need to show that $\sigma^L(P_{(e,f)}) = \sigma^L(Q_{(e,f)})$.

$$\begin{aligned}
\text{Now, } \sigma^L(P_{(e,f)}) &= \sigma^L(u_0u_1, u_1u_2)\sigma^L(u_1u_2, u_2u_3)\cdots\sigma^L(u_{k-1}u_k, u_ku_{k+1}) \\
&= \big[\eta(u_1, u_0u_1)\eta(u_1, u_1u_2)\big]\big[\eta(u_2, u_1u_2)\eta(u_2, u_2u_3)\big]\cdots\times \\
&\quad \eta(u_{k-1}, u_{k-1}u_k)\big[\eta(u_k, u_{k-1}u_k)\eta(u_k, u_ku_{k+1})\big] \\
&= \eta(u_1, u_0u_1)\big[\eta(u_1, u_1u_2)\eta(u_2, u_1u_2)\big]\big[\eta(u_2, u_2u_3)\eta(u_3, u_2u_3)\big]\cdots\times \\
&\quad \big[\eta(u_{k-1}, u_{k-1}u_k)\eta(u_k, u_{k-1}u_k)\big]\eta(u_k, u_ku_{k+1}) \\
&= \eta(u_1, u_0u_1)\big[-\sigma(u_1u_2)\big]\big[-\sigma(u_2u_3)\big]\cdots\times \\
&\quad \big[-\sigma(u_{k-1}u_k)\big]\eta(u_k, u_ku_{k+1})
\end{aligned}$$

Thus,
$\sigma^L(P_{(e,f)}) = (-1)^{(k-1)}\big[\sigma(u_1u_2)\sigma(u_2u_3)\cdots\sigma(u_{k-1}u_k)\big]\big[\eta(u_1, u_0u_1)\eta(u_k, u_ku_{k+1})\big]$.
Similarly,
$\sigma^L(Q_{(e,f)}) = (-1)^{(k-1)}\big[\sigma(u_1u_2')\sigma(u_2'u_3')\cdots\sigma(u_{k-1}'u_k)\big]\big[\eta(u_1, u_0u_1)\eta(u_k, u_ku_{k+1})\big]$.
Since Σ is distance compatible, by Lemma 6,

$$\sigma(u_1u_2)\sigma(u_2u_3)\ldots\sigma(u_{k-1}u_k) = \sigma(u_1u_2')\sigma(u_2'u_3')\ldots\sigma(u_{k-1}'u_k)$$

leading to the equality $\sigma^L(P_{(e,f)}) = \sigma^L(Q_{(e,f)})$. Hence, $\Lambda(\Sigma)$ is also distance compatible. □

Lemma 7. *Let $\Sigma = (G, \sigma)$ be any given signed graph. Let η_1, η_2 be two orientations for Σ, then the line graph $(\Lambda(G), \sigma_1)$, $(\Lambda(G), \sigma_2)$ with respect to the two orientations η_1, η_2 are switching equivalent.*

Proof. Define ζ for any edge $uu', \zeta(uu') = \eta_1(u, uu')\sigma(uu')\eta_2(u', uu')$. Then, for any two adjacent edges uu' and $u'v$, it is easy to compute $\zeta(uu')\sigma_1(uu', u'v)\zeta(u'v) = \sigma_2(uu', u'v)$. For,

$$\begin{aligned}
\zeta(uu')\sigma_1(uu', u'v)\zeta(u'v) &= \big[\eta_1(u, uu')\sigma(uu')\eta_2(u', uu')\big]\sigma_1(uu', u'v)\times \\
&\quad \big[\eta_1(u', u'v)\sigma(u'v)\eta_2(v, u'v)\big] \\
&= \big[\eta_1(u, uu')\sigma(uu')\eta_2(u', uu')\big]\big[\eta_1(u', uu')\eta_1(u', u'v)\big]\times \\
&\quad \big[\eta_1(u', u'v)\sigma(u'v)\eta_2(v, u'v)\big] \\
&= \big[\eta_1(u, uu')\eta_1(u', uu')\big]\big[\eta_2(u', uu')\eta_2(v, u'v)\big]\times \\
&\quad \big[\sigma(uu')\sigma(u'v)\big]
\end{aligned}$$

$$
\begin{aligned}
&= \big[-\sigma(uu')\big]\big[\sigma(uu')\sigma(u'v)\big]\times \\
&\quad \big[\eta_2(u',uu')\eta_2(u',u'v)\eta_2(u',u'v)\eta_2(v,u'v)\big] \\
&= -\sigma(u'v)\big[-\sigma_2(uu',u'v)\sigma(u'v)\big] \\
&= \sigma_2(uu',u'v).
\end{aligned}
$$
□

The following lemma is a restatement of the results in [2].

Lemma 8. *Let $\Sigma = (G, \sigma)$ be any given connected signed graph. Then*

i. All triangles that arise from star of Σ are positive in $\Lambda(\Sigma)$.

ii. Every even cycle of Σ keeps its signature in $\Lambda(\Sigma)$ and every odd cycle of Σ reverses its signature in $\Lambda(\Sigma)$.

iii. $\Lambda(\Sigma)$ is anti-balanced if and only if Σ is a positive cycle.

iv. $\Lambda(\Sigma)$ is balanced if and only if Σ is anti-balanced.

We use the notation $K_m^*(\Lambda(\Sigma))$ for the complete graph determined by the $\mathbb{S}$-matrix of the line graph $\Lambda(\Sigma)$, where m is the number of edges in $\Lambda(\Sigma)$.

Theorem 7. *Let Σ be a distance compatible signed graph of order n and size m. Then $K_m^*(\Lambda(\Sigma))$ is switching equivalent to K_m if and only if Σ is switching equivalent to C_n, for some $n \leq 5$.*

Proof. By Theorem 5 and Lemma 4, $K_m^*(\Lambda(\Sigma))$ is switching equivalent to K_m if and only if $\Lambda(\Sigma)$ is anti-balanced and of diameter less than or equal to 2. But by Lemma 8, $\Lambda(\Sigma)$ is anti-balanced if and only if Σ is a positive cycle. The positive cycle of diameter less than or equal to 2 is switching equivalent to C_n, for some $n \leq 5$. □

5.1. $\mathbb{S}$-spectrum of the line graph of a signed graph

In this section we deal with some results involving $\mathbb{S}$-spectrum of the line graph of a distance compatible signed graph.

Theorem 8. *Let $\Sigma = (G, \sigma)$ be a distance compatible, anti-balanced signed graph. Then $\Lambda(\Sigma)$ and $\Lambda(G)$ are $\mathbb{S}$-cospectral.*

Proof. By Lemma 8, if Σ is anti-balanced then $\Lambda(\Sigma)$ is balanced. By Theorem 2, as $\Lambda(\Sigma)$ is balanced, $\Lambda(G)$ and $\Lambda(\Sigma)$ are $\mathbb{S}$-cospectral. □

As the proof of the results in the following lemma are simple, we omit them by stating the results only.

Lemma 9. *Let I be the identity matrix and J be the all one matrix of order n, a and b are constants, then* *i.* $det\big[(aI+bJ)\big] = a^{n-1}(a+nb)$.

ii. $(aI+bJ)^{-1} = \frac{1}{a(a+nb)}[(a+nb)I - bJ]$.

iii. $J_{k\times m}(xI-J)_{m\times m}J_{m\times k} = m(x-m)J_{k\times k}$.

Theorem 9. *The $\mathbb{S}$-spectrum of the line graph $\Lambda(\Sigma)$ of every signed star $\Sigma = (K_{1,n},\sigma)$ is*

$$\begin{pmatrix} 1 & -(n-1) \\ n-1 & 1 \end{pmatrix}.$$

Proof. Let Σ be the given signed star having m number of positive edges and k number of negative edges, where $m+k=n$. We arrange the edges as $e_1, e_2, \cdots, e_m, e_{m+1}, \cdots, e_n$ for which the first m edges are positive and the last k edges are negative. Also consider an orientation of each edge in Σ towards the center vertex. Then $\sigma^L(e_i, e_j)$ is positive when $\sigma(e_i) = \sigma(e_j)$ and negative otherwise. Then the line graph $\Lambda(\Sigma)$ of the signed star $\Sigma = (K_{1,n},\sigma)$ is the complete graph whose adjacency matrix is the block matrix

$$A(\Lambda(\Sigma)) = \begin{bmatrix} (J-I)_{m\times m} & -J_{m\times k} \\ -J_{k\times m} & (J-I)_{k\times k} \end{bmatrix}_{n\times n}.$$

Then the Seidel matrix is, $\mathbb{S}(\Lambda(\Sigma)) = -A(\Lambda(\Sigma))$ and by Schur's complement method of determinant,

$$\begin{aligned}
\det(xI-\mathbb{S}) &= det\begin{bmatrix} \big((x-1)I+J\big)_{m\times m} & -J_{m\times k} \\ -J_{k\times m} & \big((x-1)I+J\big)_{k\times k} \end{bmatrix}_{n\times n} \\
&= \det\big[(x-1)I+J\big]_{m\times m}\times \\
&\quad \det\Big[\big((x-1)I+J\big)_{k\times k} - J_{k\times m}\big((x-1)I+J\big)^{-1}_{m\times m}J_{m\times k}\Big] \\
&= (x-1)^{m-1}(x+m-1)det\Big[\frac{(x-1)(x+m-1)I+(x-1)J}{(x+m-1)}\Big] \\
&= \frac{(x-1)^{m-1}}{(x+m-1)^{k-1}}det\big[(x-1)(x+m-1)I+(x-1)J\big]_{k\times k} \\
&= \frac{(x-1)^{m-1}}{(x+m-1)^{k-1}}(x-1)^k(x+m-1)^{k-1}(x+m+k-1) \\
&= (x-1)^{m+k-1}(x+m+k-1) \\
&= (x-1)^{n-1}(x+n-1).
\end{aligned}$$

Hence, the spectral values are 1 (n-1 times) and $-(n-1)$. □

Next we consider the signed wheel $W^{\sigma}_{n+1} = (C_n \vee K_1, \sigma)$, $\sigma(e) = -1$ if $e \in E(C_n)$ and 1 otherwise and find the $\mathbb{S}$-spectrum of its line graph. An example of a signed wheel is given in Fig. 2. Before that we go through some basic results on circulant matrices which are needed for the computation of the spectrum.

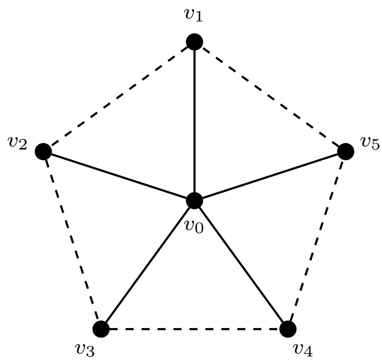

Figure 2. **A signed wheel W_6^σ**

Lemma 10 ([3]). *(i). Sum, product and inverse(if it exist) of circulant matrices is circulant.*

(ii). Let $A = circ(x_1, x_2, \cdots, x_n)$, then $A^T = circ(x_1, x_n, x_{n-1}, \cdots, x_2)$.

(iii). The eigenvalues of the circulant matrix $circ(c_1, c_2, \cdots, c_n)$ is $\{c_1 + c_2\omega_j + c_3\omega_j^2 + \cdots + c_n\omega_j^{n-1}, \omega_j = e^{\frac{2\pi i j}{n}}, j = 1, 2, \ldots, n \text{ and } i = \sqrt{-1}\ \}$.

(iv). $1 + \omega_j + \omega_j^2 + \cdots + \omega_j^{n-1} = 0, j \neq n$.

(v). $\det\ circ(c_1, c_2, \cdots, c_n) = \prod_{j=1}^{n} f(\omega_j)$, where $f(x) = c_1 + c_2x + c_3x^2 + \cdots + c_nx^{n-1}$.

(vi). $\det\ circ(ac_1, ac_2, \cdots, ac_n) = a^n \det\ circ(c_1, c_2, \cdots, c_n)$.

Lemma 11. *Let $A = circ(x_1, x_2, \cdots, x_n)$ and $B = circ(y_1, y_2, \cdots, y_n)$, then $AB = circ(z_1, z_2, \cdots, z_n)$, where $z_k = \sum_{j=1}^{k} x_j y_{(k+1-j)} + \sum_{j=k+1}^{n} x_j y_{(n+k+1-j)}$.*

Lemma 12. *The adjacency matrix of the line graph of the signed wheel is*

$$A\big(\Lambda(W_{n+1}^\sigma\big) = \begin{bmatrix} A & B \\ B^T & C \end{bmatrix}_{2n},$$

where A, B and C are $n \times n$ circulant matrices $circ(0, 1, 1, \cdots 1)$, $circ(1, 0, 0, \cdots, 0, 1)$ and $circ(0, 1, 0, 0, \cdots, 0, 1)$ respectively.

Proof. Let u_0 be the center vertex and $u_1u_2u_3\cdots u_nu_1$ be the cycle of the wheel W_{n+1}^σ. Let $u_0u_j = e_j, j = 1, 2, \cdots, n$ and $u_ju_{j+1} = e_{n+j}, j = 1, 2, \cdots, n-1$ and $u_nu_1 = e_{2n}$. Then, by the definition of the wheel $\sigma(e_j) = 1$, $\sigma(e_{n+j}) = -1, j = 1, 2, \cdots, n$. Choose the orientation μ, $\mu(u_j, e_j) = 1 = \mu(u_j, e_{n+j}) = 1, j = 1, 2, \cdots, n$. Now, $\sigma(e_i, e_j) = \mu(u, e_i)\mu(u, e_j) = 1$ if $e_i \sim e_j$ with common vertex u, otherwise it is zero. Then it is clear that signature function on the line graph of the wheel, if $e_i \sim e_j$, σ^L is $\sigma^L(e_ie_j) = 1$ for $1 \leq i,j \leq n, i \neq j$. Also, $\sigma^L(e_je_{n+j}) = 1$ for $1 \leq j \leq n$ and $\sigma^L(e_je_{j-n}) = 1 = \sigma^L(e_je_{j-n+1}) = \sigma^L(e_{2n}, e_1)$ for $n+1 \leq j \leq 2n-1$ $\sigma^L(e_je_{j+1}) = 1 = \sigma^L(e_{2n}, e_{n+1})$ for $n+1 \leq j \leq 2n-1$ and hence the adjacency matrix is the given block matrix. □

Lemma 13. *The $\mathbb{S}$-matrix of the line graph of the signed wheel is*

$$\mathbb{S}(\Lambda(W_{n+1}^{\sigma})) = \begin{bmatrix} C_1 & C_2 \\ C_2^T & C_3 \end{bmatrix}_{2n},$$

where C_1, C_2 and C_3 are $n \times n$ circulant matrices $circ(0, -1, -1, \cdots - 1)$, $circ(-1, 1, 1, \cdots, 1, -1)$ and $circ(0, -1, 1, 1, \cdots, 1, -1)$ respectively.

Proof. The line graph of the wheel graph is a positive (unsigned) graph, as we noticed and hence its associated complete graph is K_{2n}. Thus the $\mathbb{S}$-matrix of the line graph is $J - I - 2A(\Lambda(W_{n+1}^{\sigma}))$. If we use the form of $A(\Lambda(W_{n+1}))$ in Lemma 12, then this will lead to the given block matrix. □

Theorem 10. *The characteristic polynomial of the $\mathbb{S}$-matrix of the line graph of the signed wheel W_{n+1}^{σ} is $f(x) = (x-3)^{n-1}\big(x+2-a_n\big)\big(x+2+a_n\big)\prod_{j=1}^{n-1}\big(x+3+4\cos(\frac{2\pi j}{n})\big)$, where $a_n = \sqrt{2n^2 - 14n + 25}$.*

Proof. The characteristic polynomial of the $\mathbb{S}$-matrix of the line graph of the special signed wheel is

$$\det(xI - \mathbb{S}) = \begin{bmatrix} xI - C_1 & -C_2 \\ -C_2^T & xI - C_3 \end{bmatrix}_{(2n)} = \det(xI - C_1)\ \det\big(xI - (C_3 + C_2^T(xI - C_1)^{-1}C_2)\big).$$

We note that all the blocks are circulant matrices of order n and using Lemma 9 and Lemma 11 it is easy to compute

$\det(xI - C_1) = (x-1)^{n-1}(x+n-1)$ and
$\big(xI - (C_3 + C_2^T(xI - C_1)^{-1}C_2)\big) = circ\big(r_1(x), r_2(x), r_3(x), \cdots, r_3(x), r_2(x)\big)$, where
$r_1(x) = \dfrac{x^3 + (n-2)x^2 + (1-2n)x + (16-7n)}{(x-1)(x+n-1)}$, $r_2(x) = \dfrac{x^2 + 2x - 4n + 13}{(x-1)(x+n-1)}$ and $r_3(x)$
$= \dfrac{-x^2 - 2(n-5)x + (2n+7)}{(x-1)(x+n-1)}$. By Lemma 10, $\det\big(xI - (C_3 + C_2^T(xI - C_1)^{-1}C_2)\big)$

$$= \prod_{j=1}^{n}\big(r_1 + r_2\omega_j + r_3\omega_j^2 + r_3\omega_j^3 + \cdots + r_3\omega_j^{n-2} + r_2\omega_j^{n-1}\big)$$

$$= \prod_{j=1}^{n}\big(r_1 + r_2(\omega_j + \omega_j^{n-1}) + r_3(\omega_j^2 + \omega_j^3 + \cdots + \omega_j^{n-2})\big)$$

$$= \big(r_1 + 2r_2 + (n-3)r_3\big) \prod_{j=1}^{n-1} \big(r_1 + 2r_2\cos(\frac{2\pi j}{n}) - r_3(1 + 2\cos(\frac{2\pi j}{n}))\big)$$

$$= \frac{(x^2 + 4x + 14n - 2n^2 - 21)}{(x + n - 1)} \prod_{j=1}^{n-1} \big((r_1 - r_3) + 2(r_2 - r_3)\cos(\frac{2\pi j}{n})\big)$$

$$= \frac{(x^2 + 4x + 14n - 2n^2 - 21)}{(x + n - 1)} \prod_{j=1}^{n-1} \big(\frac{(x+3)(x-3)}{(x-1)} + 4\frac{(x-3)}{(x-1)}\cos(\frac{2\pi j}{n})\big)$$

$$= \frac{(x^2 + 4x + 14n - 2n^2 - 21)(x-3)^{n-1}}{(x + n - 1)(x-1)^{n-1}} \prod_{j=1}^{n-1} \big((x+3) + 4\cos(\frac{2\pi j}{n})\big)$$

$$= \frac{(x + 2 + a_n)(x + 2 - a_n)(x-3)^{n-1}}{(x + n - 1)(x-1)^{n-1}} \prod_{j=1}^{n-1} \big(x + 3 + 4\cos(\frac{2\pi j}{n})\big)$$

Thus $f(x) = \det(xI - C_1)\det\big(xI - (C_3 + C_2^T(xI - C_1)^{-1}C_2)\big)$

$$= (x-1)^{n-1}(x + n - 1)\frac{(x + 2 + a_n)(x + 2 - a_n)(x-3)^{n-1}}{(x + n - 1)(x-1)^{n-1}} \prod_{j=1}^{n-1} \big(x + 3 + 4\cos(\frac{2\pi j}{n})\big)$$

$$= (x-3)^{n-1}\big(x + 2 - a_n\big)\big(x + 2 + a_n\big) \prod_{j=1}^{n-1} \big((x + 3 + 4\cos(\frac{2\pi j}{n})\big). \qquad \square$$

Remark 1. $\mathbb{S}$-spectrum of the line graph of the signed wheel W_{n+1}^{σ} is $\big(3, -2 \pm a_n, -(3 + 4\cos(\frac{2\pi j}{n})), j = 1, 2, \cdots, n-1\big)$ with multiplicity of the eigenvalues depending on the value of n.

6. Conclusion

In this introductory paper, we discussed at some length about the distance induced Seidel matrices of certain signed graphs. We propose to have a separate paper dealing with the properties of signed graphs for which K_n^* are anti-balanced and, of course, studying the properties of $K_n^*(G)$ for an unsigned connected graph G itself will be interesting. We postpone this analysis to a later stage.

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