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*Research Article*

# Optimality conditions for mathematical programming problem with equilibrium constraints in terms of tangential subdifferentiable

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**Abstract:** The aim of this article is to develop necessary and sufficient optimality conditions for nonsmooth mathematical programs with equilibrium constraints ($\mathcal{MPEC}$). We introduce a nonsmooth variant of the standard $\partial^T$-Abadie constraint qualification ($\partial^T$-$ACQ(\mathfrak{B}_1, \mathfrak{B}_2)$) and propose $\partial^T$-generalized alternatively stationary conditions using the tangential subdifferential framework. Building on these new conditions, we derive first-order optimality criteria under $\partial^T$-$ACQ(\mathfrak{B}_1, \mathfrak{B}_2)$. Additionally, we establish sufficient optimality conditions within a framework of generalized convexity assumptions. The effectiveness and applicability of these conditions are demonstrated through several examples.

**Keywords:** tangential subdifferentials, equilibrium constraints, constraint qualification, optimality conditions.

**AMS Subject classification:** 90C26, 90C30, 90C33, 90C46

## 1. Introduction

Mathematical programs with equilibrium constraints ($\mathcal{MPEC}s$) have been the subject of deep research due to their rich applications in areas such as economics, multilevel games, engineering design, and transportation planning, among others. For insights

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into these applications and the latest developments in theory and algorithms, see [2, 16, 24]. A serious drawback of $\mathcal{MPEC}s$ is that their general structure is overly complicated to guarantee the satisfaction of standard constraint qualifications (CQs), like the Mangasarian-Fromovitz CQ or Slater CQ, at feasible points. To address this challenge, researchers have proposed various modifications to existing CQ, which has led to the introduction of new stationarity concepts tailored to $\mathcal{MPEC}s$. Interest in this area has continued to grow, as reflected in a steady rise in related research contributions [1, 4–8, 14, 22, 33].

Recent advancements in the fields of variational analysis, optimal control, and nonsmooth optimization have inspired the development of more generalized models and solution concepts, particularly in settings involving vector inequalities, multiple cost functions, and interval-valued frameworks. Various contributions have established foundational results and efficiency criteria by leveraging structural properties of the underlying functionals and generalized convexity notions, often motivated by applications in physics and engineering systems [11, 26–29]. These developments provide valuable insights and tools that can potentially be extended to more complex mathematical structures, such as those involving equilibrium constraints or nonsmooth variational inequalities.

This investigation focuses on the subsequent mathematical program with equilibrium constraints ($\mathcal{MPEC}$) of the form:

$$(\mathcal{MPEC}): \quad \begin{cases} Minimize \qquad \Gamma(\mathfrak{z}) \\ s.\ t. \begin{cases} c(\mathfrak{z}) \leq 0, \quad d(\mathfrak{z}) = 0, \\ \mathcal{T}(\mathfrak{z}) \geq 0, \ \zeta(\mathfrak{z}) \geq 0, \\ \mathcal{T}(\mathfrak{z})^T \zeta(\mathfrak{z}) = 0, \end{cases} \end{cases}$$

where $\Gamma : \mathbb{R}^n \longrightarrow \mathbb{R}$, $c : \mathbb{R}^n \longrightarrow \mathbb{R}^l$, $d : \mathbb{R}^n \longrightarrow \mathbb{R}^m$, $\mathcal{T} : \mathbb{R}^n \longrightarrow \mathbb{R}^p$ and $\zeta : \mathbb{R}^n \longrightarrow \mathbb{R}^p$ are given functions, $n, l, m, p \in \mathbb{N}$.

The tangential subdifferential [18–20, 32], a generalized derivative concept that encompasses both the Gâteaux derivative and the convex subdifferential, was first introduced and applied to derive optimality conditions for nonlinear programming in [23]. Subsequent studies have expanded its applications in various optimization problems. For instance, Tung [31] utilized the tangential subdifferential to establish strong KKT optimality conditions for Pareto efficient and weakly efficient solutions in semi-infinite multiobjective programming. Later, Jennane and Kalmoun [12] derived optimality conditions for multiobjective semi-infinite programming with switching constraints using tangential subdifferentials. Gadhi and Odha [9] developed necessary optimality conditions by leveraging tangential subdifferentials in conjunction with optimal value reformulation and the partial calmness property. Since the tangential subdifferential generalizes both the Clarke subdifferential [3] and the Michel-Penot subdifferential [21], optimality conditions formulated in terms of tangential subdifferentials yield broader results than those based on more restrictive subdifferentials. In particular, the findings obtained via tangential subdifferentials extend and unify previous results derived using Clarke subdifferentials by Luu and Hang [17] and Michel-Penot subdifferentials by Khanh and Tung [13].

In this work, our objective is to establish necessary and sufficient optimality results for nonsmooth $\mathcal{MPEC}s$, using the concept of tangential subdifferentials. To achieve this, we have introduced nonsmooth versions of the constraint qualification $\partial^T$-$ACQ(\mathfrak{B}_1, \mathfrak{B}_2)$ for a surrogate problem, utilizing the tangential subdifferential framework. Additionally, we have proposed a $\partial^T$-generalized alternatively stationary concept in terms of tangential subdifferentials and demonstrated that it serves as a first-order necessary optimality condition under $\partial^T$-$ACQ(\mathfrak{B}_1, \mathfrak{B}_2)$. Furthermore, we focus on sufficient optimality conditions, showing that $\partial^T$-generalized alternatively stationary can also serve as a global sufficient optimality condition under certain generalized convexity assumptions. To clarify our findings, we provide various examples.

This study was motivated by the apparent lack of research on mathematical programs with equilibrium constraints ($\mathcal{MPEC}s$) employing the tangential subdifferential, a gap we identified after an extensive literature review. Previous works on $\mathcal{MPEC}s$ primarily focus on cases where the involved functions are either continuously differentiable or locally Lipschitz. To the best of our knowledge, this is the first investigation to establish optimality conditions for $\mathcal{MPEC}s$ using the tangential subdifferential. The feasible set of an $\mathcal{MPEC}$ is not necessarily convex, even when the underlying functions are convex, rendering classical convex analysis techniques inadequate. Moreover, the local Lipschitz continuity or differentiability of these functions is not always assured. Consequently, we explore $\mathcal{MPEC}s$ for a broader class of functions, specifically tangentially convex functions, addressing a notable gap in existing literature. The findings of this study provide new theoretical insights and represent a significant contribution to the field.

The article is structured as follows: In Section 2, we provide some preliminaries and review key definitions. Section 3 introduces the generalized alternatively stationary concept and demonstrates that it is a necessary optimality condition for $\mathcal{MPEC}s$. In Section 4, we establish sufficient optimality conditions under assumptions of pseudoconvexity and quasiconvexity. Finally, we conclude our work in Section 5.

## 2. Definitions and preliminaries

In this section, we state a few definitions, notations and results, which we will refer to later in the article. In what follows throughout this work $\mathbb{R}^n$ denotes the standard $n$-dimensional Euclidean space. We write the inner product as $\langle \cdot, \cdot \rangle$. Let $\mathfrak{E}$ be a nonempty subset of $\mathbb{R}^n$. We define the convex hull of $\mathfrak{E}$ to be $co\,\mathfrak{E}$, interior of $\mathfrak{E}$ to be $int\,\mathfrak{E}$, convex cone (including the origin) of $\mathfrak{E}$ to be $pos\,\mathfrak{E}$, cone of $\mathfrak{E}$ to be $cone\,\mathfrak{E}$ and the closure of $\mathfrak{E}$ to be $cl\,\mathfrak{E}$.

Let $\mathfrak{E} \subseteq \mathbb{R}^n$ and $\bar{\mathfrak{z}} \in cl\,\mathfrak{E}$. The contingent cone and the negative polar cone of $\mathfrak{E}$ at $\bar{\mathfrak{z}}$ are defined, respectively, as follows

- $\mathfrak{T}(\bar{\mathfrak{z}}, \mathfrak{C}) := \left\{ \delta \in \mathbb{R}^n : \exists\ u_n \downarrow 0,\ \exists\ \{\delta_n\} \subseteq \mathbb{R}^n,\ \delta_n \to \delta,\ \bar{\mathfrak{z}} + u_n \delta_n \in \mathfrak{C} \right\}.$

- $\mathfrak{C}^\circ := \{\delta \in \mathbb{R}^n : \langle \delta, \mathfrak{z} \rangle \leq 0, \ \forall\, \mathfrak{z} \in \mathfrak{C}\}$.

The Fréchet normal cone $\mathcal{N}(\bar{\mathfrak{z}}, \mathfrak{C})$ to a set $\mathfrak{C}$ at a point $\bar{\mathfrak{z}}$ is defined as

$$\mathcal{N}(\bar{\mathfrak{z}}, \mathfrak{C}) = \big(\mathfrak{T}(\bar{\mathfrak{z}}, \mathfrak{C})\big)^\circ = \{\mathfrak{z}^* \in \mathbb{R}^n : \langle \mathfrak{z}^*, \delta \rangle \leq 0, \quad \forall\, \delta \in \mathfrak{T}(\bar{\mathfrak{z}}, \mathfrak{C})\}.$$

Note that while the tangent cone $\mathfrak{T}(\bar{\mathfrak{z}}, \mathfrak{C})$ is always closed, it is not necessarily convex. However, if $\mathfrak{C}$ is a convex set, then $\mathfrak{T}(\bar{\mathfrak{z}}, \mathfrak{C})$ is convex, and

$$\mathcal{N}(\bar{\mathfrak{z}}, \mathfrak{C}) = \{\mathfrak{z}^* \in \mathbb{R}^n : \langle \mathfrak{z}^*, \mathfrak{z} - \bar{\mathfrak{z}} \rangle \leq 0, \quad \forall\, \mathfrak{z} \in \mathfrak{C}\}.$$

Let $\varphi : \mathbb{R}^n \to \mathbb{R}$ and $\bar{\mathfrak{z}} \in \mathbb{R}^n$ be given. The directional derivative of $\varphi$ at $\bar{\mathfrak{z}}$ in the direction $\delta \in \mathbb{R}^n$ is defined by

$$\varphi'(\bar{\mathfrak{z}}, \delta) = \lim_{t \downarrow 0} \frac{\varphi(\bar{\mathfrak{z}} + t\delta) - \varphi(\bar{\mathfrak{z}})}{t}.$$

In the next definition, the class of functions referred to as "tangentially convex" is defined, which was first proposed by Pshenichnyi [23] and termed by Lemaréchal [15].

**Definition 1.** ([15, 23]): A function $\varphi : \mathbb{R}^n \to \mathbb{R}$ is tangentially convex at $\bar{\mathfrak{z}} \in \varphi^{-1}(\mathbb{R})$ if its directional derivative $\varphi'(\bar{\mathfrak{z}}, \delta)$ exists, is finite for all directions $\delta \in \mathbb{R}^n$ and the function $\delta \mapsto \varphi'(\bar{\mathfrak{z}}, \delta)$ is convex.

Based on the definition of a tangentially convex function, Pshenichnyi [23] introduced the concept of its associated subdifferential, known as the tangential subdifferential (see also [18]).

**Definition 2.** ([18, 23]): Let $\varphi : \mathbb{R}^n \to \mathbb{R}$ be a tangentially convex function at $\bar{\mathfrak{z}} \in \varphi^{-1}(\mathbb{R})$. It is said that the nonempty compact convex set $\partial^T \varphi(\bar{\mathfrak{z}}) \subset \mathbb{R}^n$ is the tangential subdifferential of $\varphi$ at $\bar{\mathfrak{z}}$, if, for every $\delta \in \mathbb{R}^n$, one has

$$\varphi'(\bar{\mathfrak{z}}; \delta) = \max_{\mathfrak{z} \in \partial^T \varphi(\bar{\mathfrak{z}})} \langle \mathfrak{z}, \delta \rangle, \tag{2.1}$$

which is equivalent to

$$\partial^T \varphi(\bar{\mathfrak{z}}) = \{\mathfrak{z} \in \mathbb{R}^n : \langle \mathfrak{z}, \delta \rangle \leq \varphi'(\bar{\mathfrak{z}}; \delta), \quad \forall \delta \in \mathbb{R}^n\}.$$

**Remark 1.** Note that the definition of the tangential subdifferential is also equivalent to $\partial^T \varphi(\bar{\mathfrak{z}}) = \partial \varphi'(\bar{\mathfrak{z}}, \cdot)(0)$, where $\partial$ denotes the subdifferential of a convex function in convex analysis. Additionally, according to Definition 2, if $\varphi$ is tangentially convex at $\bar{\mathfrak{z}} \in \varphi^{-1}(\mathbb{R})$, then its tangential subdifferential at $\bar{\mathfrak{z}}$ is nonempty, as follows from the sublinearity of $\varphi'(\bar{\mathfrak{z}}; \cdot)$. Furthermore, from equation (2.1), $\varphi'(\bar{\mathfrak{z}}; \cdot)$ acts as the support functional of the tangential subdifferential $\partial^T \varphi(\bar{\mathfrak{z}})$ of $\varphi$ at $\bar{\mathfrak{z}}$.

It can be noted that, among the various calculus rules applicable to tangential subdifferentials of tangentially convex functions, additivity is one of them. Specifically, if $\varphi_1$ and $\varphi_2$ are tangentially convex functions at a common point $\bar{\mathfrak{z}}$, then the following holds

$$\begin{aligned}\partial^T\big(\varphi_1+\varphi_2\big)(\bar{\mathfrak{z}}) &= \partial\big(\varphi_1+\varphi_2\big)'(\bar{\mathfrak{z}},\cdot)(0) = \partial\big(\varphi_1'(\bar{\mathfrak{z}},\cdot)+\varphi_2'(\bar{\mathfrak{z}},\cdot)\big)(0)\\ &= \partial\varphi_1'(\bar{\mathfrak{z}},\cdot)(0)+\partial\varphi_2'(\bar{\mathfrak{z}},\cdot)(0) = \partial^T\varphi_1(\bar{\mathfrak{z}})+\partial^T\varphi_2(\bar{\mathfrak{z}})\end{aligned}$$

demonstrating that the tangential subdifferential of the sum of two tangentially convex functions is the sum of their individual tangential subdifferentials at the common point $\bar{\mathfrak{z}}$.

Now, we recall the definitions of Dini-convexity, Dini-pseudoconvexity and Dini-quasiconvexity of a function formulated in terms of tangential subdifferential. The aforesaid definitions have been derived by Tung [31].

**Definition 3.** [31] Let $\mathcal{S}\subset\mathbb{R}^n$ be a nonempty convex set and $\bar{q}\in\mathcal{S}$ be given. Further, assume that $\varphi:\mathcal{S}\to\mathbb{R}$ is a tangentially convex at $\bar{q}$. Then:

- $\varphi$ is Dini-convex at $\bar{q}$ on $\mathcal{S}$ if the relation

$$\varphi(q)-\varphi(\bar{q})\geq\big\langle\vartheta,\ q-\bar{q}\big\rangle,\quad\forall\,\vartheta\in\partial^T\varphi(\bar{q})$$

  hold for all $q\in\mathcal{S}$.

- $\varphi$ is Dini-pseudoconvex at $\bar{q}$ on $\mathcal{S}$ if the relation

$$\varphi(q)<\varphi(\bar{q})\Longrightarrow\big\langle\vartheta,\ q-\bar{q}\big\rangle<0,\quad\forall\,\vartheta\in\partial^T\varphi(\bar{q})$$

  hold for all $q\in\mathcal{S}$.

- $\varphi$ is Dini-quasiconvex at $\bar{q}$ on $\mathcal{S}$ if the relation

$$\varphi(q)\leq\varphi(\bar{q})\Longrightarrow\big\langle\vartheta,\ q-\bar{q}\big\rangle\leq0,\quad\forall\,\vartheta\in\partial^T\varphi(\bar{q})$$

  hold for all $q\in\mathcal{S}$.

## 3. Necessary optimality condition

To derive the necessary optimality conditions for $\mathcal{MPEC}$, we introduce new concepts of stationary points using tangential subdifferential. To do this, let $\mathcal{G}$ represent the collection of feasible points in $\mathcal{MPEC}$.

$$\mathcal{G}:=\Big\{\mathfrak{z}\in\mathbb{R}^n:c(\mathfrak{z})\leq0,d(\mathfrak{z})=0,\mathcal{T}(\mathfrak{z})\geq0,\ \zeta(\mathfrak{z})\geq0,\mathcal{T}(\mathfrak{z})^T\zeta(\mathfrak{z})=0\Big\}.$$

Let $\bar{\mathfrak{z}} \in \mathcal{G}$, and let

$$\mathcal{P} := \{1, \ldots, l\},\ \mathcal{Q} := \{1, \ldots, m\},\ \mathcal{R} := \{1, \ldots, p\}$$

and

$$\mathcal{I}_c(\bar{\mathfrak{z}}) := \{i \in \mathcal{P} : c_i(\bar{\mathfrak{z}}) = 0\}.$$

Consider the sets

$$\begin{aligned} A &:= \{i \in \mathcal{R} : \mathcal{T}_i(\bar{\mathfrak{z}}) = 0,\ \zeta_i(\bar{\mathfrak{z}}) > 0\}, \\ \mathfrak{B} &:= \{i \in \mathcal{R} : \mathcal{T}_i(\bar{\mathfrak{z}}) = 0,\ \zeta_i(\bar{\mathfrak{z}}) = 0\}, \\ \mathfrak{D} &:= \{i \in \mathcal{R} : \mathcal{T}_i(\bar{\mathfrak{z}}) > 0,\ \zeta_i(\bar{\mathfrak{z}}) = 0\}. \end{aligned}$$

The set $\mathfrak{B}$ is referred to as the degenerate set. When $\mathfrak{B}$ is empty, the vector $\bar{\mathfrak{z}}$ is said to satisfy strict complementarity condition [33]. In this section, we assume that $\mathfrak{B}$ is a nonempty set and we define $\mathcal{P}(\mathfrak{B})$ as the collection of all disjoint bipartitions of $\mathfrak{B}$; that is

$$\mathcal{P}(\mathfrak{B}) := \Big\{(\mathfrak{B}_1, \mathfrak{B}_2) : \mathfrak{B}_1 \cup \mathfrak{B}_2 = \mathfrak{B},\ \mathfrak{B}_1 \cap \mathfrak{B}_2 = \emptyset\Big\}.$$

We now refer to the nonlinear program $\mathcal{MPEC}(\mathfrak{B}_1, \mathfrak{B}_2)$ as defined by Ye [33], concerning the partition $(\mathfrak{B}_1, \mathfrak{B}_2)$ of $\mathfrak{B}$.

$$\mathcal{MPEC}(\mathfrak{B}_1, \mathfrak{B}_2) : \begin{cases} \quad Minimize\ \ \Gamma(\mathfrak{z}) \\ s.\ t. \begin{cases} c(\mathfrak{z}) \leq 0, \quad d(\mathfrak{z}) = 0, \\ \mathcal{T}_i(\mathfrak{z}) \geq 0,\ i \in \mathfrak{B}_1,\ \zeta_i(\mathfrak{z}) \geq 0, i \in \mathfrak{B}_2, \\ \mathcal{T}_i(\mathfrak{z}) = 0,\ i \in A \cup \mathfrak{B}_2,\ \zeta_i(\mathfrak{z}) = 0,\ i \in \mathfrak{D} \cup \mathfrak{B}_1. \end{cases} \end{cases}$$

It is clear that $\bar{\mathfrak{z}} \in \mathcal{G}$ is a local optimal solution of the $\mathcal{MPEC}$ if and only if it is a local optimal solution of the $\mathcal{MPEC}$ for every partition $(\mathfrak{B}_1, \mathfrak{B}_2)$ in $\mathcal{P}(\mathfrak{B})$.
The following lemma is instrumental in proving one of the main results presented in this article.

**Lemma 1.** *Let $\mathcal{A}_1$ be a non-empty convex cone and $\mathcal{A}_2$ be a non-empty, convex and compact set. If*

$$\max_{v \in \mathcal{A}_2} \langle v,\ \delta\rangle \geq 0, \qquad \forall\ \delta \in \mathcal{A}_1^\circ,$$

*then $0 \in cl\, \mathcal{A}_1 + \mathcal{A}_2$.*

*Proof.* On the contrary, suppose that $0 \notin cl\,\mathcal{A}_1 + \mathcal{A}_2$. Hence, the sets $cl\,\mathcal{A}_1$ and $\mathcal{A}_2$ are disjoint. Using the separation theorem (Theorem 6.10) [10], there exist a non-zero vector $\mathfrak{h}^* \in \mathbb{R}^n$ and a real number $\alpha \in \mathbb{R}$ such that

$$\langle \mathfrak{h}^*,\ \beta\rangle < \alpha \leq \langle \mathfrak{h}^*,\ \gamma\rangle \qquad \forall\ \beta \in \mathcal{A}_2,\ \gamma \in -cl\,\mathcal{A}_1.$$

Thus,

$$\langle \mathfrak{h}^*,\ \beta\rangle < \alpha \leq \langle \mathfrak{h}^*,\ \gamma\rangle \qquad \forall\ \beta \in \mathcal{A}_2,\ \gamma \in -\mathcal{A}_1. \tag{3.1}$$

Since $-\mathcal{A}_1$ is a cone, we can set $\gamma = 0$ to get

$$\langle \mathfrak{h}^*,\ \beta\rangle < 0 \qquad \forall\ \beta \in \mathcal{A}_2.$$

By the compactness assumption of $\mathcal{A}_2$, we get

$$\max_{\beta \in \mathcal{A}_2} \langle \mathfrak{h}^*,\ \beta\rangle < 0.$$

Since $\mathcal{A}_1$ is cone and if we let $\gamma = -\gamma',\ \gamma' \in \mathcal{A}_1$, therefore, we have

$$\rho\gamma \in -\mathcal{A}_1, \qquad \forall\ \rho \in \mathbb{N}\backslash\{0\}.$$

By (3.1), it follows that

$$\frac{\alpha}{\rho} \leq \langle \mathfrak{h}^*,\ \gamma\rangle.$$

Letting $\rho \to \infty$, we obtain

$$0 \leq \langle \mathfrak{h}^*,\ \gamma\rangle.$$

Then, for all $\gamma' \in \mathcal{A}_1$, one has

$$\langle \mathfrak{h}^*,\ \gamma'\rangle \leq 0, \Rightarrow \mathfrak{h}^* \in \mathcal{A}_1^\circ.$$

Since $\mathfrak{h}^* \in \mathcal{A}_1^\circ$, this leads to a contradiction and completes the proof of this lemma. □

Now, we defne the $\partial^T$-Abadie constraint qualifcation ($\partial^T$-$ACQ(\mathfrak{B}_1, \mathfrak{B}_2)$) for $\mathcal{MPEC}$.

**Definition 4.** Let $\bar{\mathfrak{z}} \in \mathcal{G}$ and $(\mathfrak{B}_1, \mathfrak{B}_2)$ be a partition of $\mathfrak{B} \neq \emptyset$. Assume that $c_i, i \in \mathcal{I}_c(\bar{\mathfrak{z}}), d_i, -d_i, i \in \mathcal{Q}, \mathcal{T}_i, -\mathcal{T}_i, i \in A \cup \mathfrak{B}, \zeta_i, -\zeta_i, i \in \mathfrak{D} \cup \mathfrak{B}$ are tangentially convex at $\bar{\mathfrak{z}}$. We say that $\partial^T$-$ACQ(\mathfrak{B}_1, \mathfrak{B}_2)$ holds at $\bar{\mathfrak{z}} \in \mathcal{G}$ if

$$\Theta(\bar{\mathfrak{z}})^\circ \subseteq \mathfrak{T}(\bar{\mathfrak{z}}, \mathcal{G}),$$

where

$$\Theta(\bar{\mathfrak{z}}) := \left( \bigcup_{i \in \mathcal{I}_c(\bar{\mathfrak{z}})} \partial^T c_i(\bar{\mathfrak{z}}) \right) \cup \left( \bigcup_{i \in \mathcal{Q}} \partial^T d_i(\bar{\mathfrak{z}}) \right) \cup \left( \bigcup_{i \in \mathcal{Q}} \partial^T (-d_i)(\bar{\mathfrak{z}}) \right)$$
$$\cup \left( \bigcup_{i \in A \cup \mathfrak{B}_2} \left( \partial^T \mathcal{T}_i(\bar{\mathfrak{z}}) \cup \partial^T (-\mathcal{T}_i)(\bar{\mathfrak{z}}) \right) \right) \cup \left( \bigcup_{i \in \mathfrak{D} \cup \mathfrak{B}_1} \left( \partial^T \zeta_i(\bar{\mathfrak{z}}) \cup \partial^T (-\zeta_i)(\bar{\mathfrak{z}}) \right) \right)$$
$$\cup \left( \bigcup_{i \in \mathfrak{B}_1} \partial^T (-\mathcal{T}_i)(\bar{\mathfrak{z}}) \right) \cup \left( \bigcup_{i \in \mathfrak{B}_2} \partial^T (-\zeta_i)(\bar{\mathfrak{z}}) \right).$$

In the following definition, we introduce a generalized concept of alternative stationarity expressed in terms of the tangential subdifferential.

**Definition 5.** A feasible point $\bar{\mathfrak{z}}$ of $\mathcal{MPEC}$ is called a $\partial^T$-generalized alternatively stationary ($\partial^T$-$\mathcal{GA}$-stationary) point if there exists a vectors $\lambda = (\lambda^c, \lambda^d, \lambda^{\mathcal{T}}, \lambda^{\zeta}) \in \mathbb{R}^l \times \mathbb{R}^m \times \mathbb{R}^{2p}$ and $\mu = (\mu^d, \mu^{\mathcal{T}}, \mu^{\zeta}) \in \mathbb{R}^m \times \mathbb{R}^{2p}$ such that the following conditions are satisfied:

$$0 \in \partial^T \Gamma(\bar{\mathfrak{z}}) + \sum_{i=1}^{l} \lambda_i^c \, \partial^T c_i(\bar{\mathfrak{z}}) + \sum_{i \in \mathcal{Q}} \mu_i^d \, \partial^T d_i(\bar{\mathfrak{z}}) + \sum_{i \in \mathcal{Q}} \lambda_i^d \, \partial^T (-d_i)(\bar{\mathfrak{z}}) + \sum_{i=1}^{p} \lambda_i^{\mathcal{T}} \, \partial^T (-\mathcal{T}_i)(\bar{\mathfrak{z}})$$
$$+ \sum_{i=1}^{p} \lambda_i^{\zeta} \, \partial^T (-\zeta_i)(\bar{\mathfrak{z}}) + \sum_{i=1}^{p} \mu_i^{\mathcal{T}} \, \partial^T \mathcal{T}_i(\bar{\mathfrak{z}}) + \sum_{i=1}^{p} \mu_i^{\zeta} \, \partial^T \zeta_i(\bar{\mathfrak{z}}), \quad (3.2)$$

with

$$\lambda_i^c \, c_i(\bar{\mathfrak{z}}) = 0, \ \forall \ i \in \mathcal{P} \quad (3.3)$$

and

$$\begin{cases} \lambda_i^c \geq 0, \ \forall \, i \in \mathcal{P}, \ \lambda_i^d \geq 0, \ \mu_i^d \geq 0, \ \forall \, i \in \mathcal{Q}, \\ \lambda_i^{\mathcal{T}} = 0, \ \mu_i^{\mathcal{T}} = 0, \ \ \forall \, i \in \mathfrak{D}, \\ \lambda_i^{\zeta} = 0, \ \mu_i^{\zeta} = 0, \ \ \forall \, i \in A, \\ \lambda_i^{\mathcal{T}}, \lambda_i^{\zeta}, \mu_i^{\mathcal{T}}, \mu_i^{\zeta} \geq 0, \ \ \forall \, i \in \mathcal{R}, \\ \mu_i^{\mathcal{T}} = 0 \text{ or } \mu_i^{\zeta} = 0, \ \forall \, i \in \mathfrak{B}. \end{cases} \quad (3.4)$$

We are now prepared to demonstrate the primary result of this article, which establishes that $\partial^T$-generalized alternatively stationary is a necessary condition for optimality.

**Theorem 1.** *Let $\bar{\mathfrak{z}} \in \mathcal{G}$ be a local optimal solution of $\mathcal{MPEC}$. Assume that $\Gamma$ is locally Lipschitz and tangentially convex at $\bar{\mathfrak{z}}$, that $c_i, i \in \mathcal{I}_c(\bar{\mathfrak{z}})$, $d_i, -d_i, i \in \mathcal{Q}, \mathcal{T}_i, -\mathcal{T}_i, i \in A \cup \mathfrak{B}, \zeta_i, -\zeta_i, i \in \mathfrak{D} \cup \mathfrak{B}$ are tangentially convex at $\bar{\mathfrak{z}}$. Additionally, suppose that the set pos $\Theta(\bar{\mathfrak{z}})$ is closed and that there exists a partition $(\mathfrak{B}_1, \mathfrak{B}_2)$ of $\mathfrak{B}$ such that $\partial^T$-ACQ$(\mathfrak{B}_1, \mathfrak{B}_2)$ is satisfied at $\bar{\mathfrak{z}}$. Then, $\bar{\mathfrak{z}}$ is a $\partial^T$-generalized alternatively stationary point.*

*Proof.* Since $\bar{\mathfrak{z}} \in \mathcal{G}$ is a local optimal solution of $\mathcal{MPEC}$, there exists a neighborhood $U$ of $\bar{\mathfrak{z}}$ such that

$$\Gamma(\bar{\mathfrak{z}}) \leq \Gamma(\mathfrak{z}), \quad \forall\, \mathfrak{z} \in \mathcal{G} \cap U.$$

Let $\delta \in \mathfrak{T}(\bar{\mathfrak{z}}, \mathcal{G})$ be arbitrary. By definition there exist $u_n \to 0^+$ and $\delta_n \to \delta$ such that $\bar{\mathfrak{z}} + u_n\delta_n \in \mathcal{G}$ for all $n$. For $n$ large enough, $\bar{\mathfrak{z}} + u_n\delta_n \in U$. Since $\bar{\mathfrak{z}}$ is a local optimal solution of $\Gamma$ over $\mathcal{G}$, therefore, we have

$$\frac{\Gamma(\bar{\mathfrak{z}} + u_n\delta_n) - \Gamma(\bar{\mathfrak{z}})}{u_n} \geq 0.$$

Note that

$$\frac{\Gamma(\bar{\mathfrak{z}} + u_n\delta_n) - \Gamma(\bar{\mathfrak{z}})}{u_n} = \frac{\Gamma(\bar{\mathfrak{z}} + u_n\delta_n) - \Gamma(\bar{\mathfrak{z}} + u_n\delta)}{u_n} + \frac{\Gamma(\bar{\mathfrak{z}} + u_n\delta) - \Gamma(\bar{\mathfrak{z}})}{u_n}.$$

Since $\Gamma$ is locally Lipschitz, therefore, as $n \to \infty$

$$\frac{\Gamma(\bar{\mathfrak{z}} + u_n\delta_n) - \Gamma(\bar{\mathfrak{z}} + u_n\delta)}{u_n} \to 0$$

we get

$$\begin{aligned}
\Gamma'(\bar{\mathfrak{z}}, \delta) &= \lim_{n\to\infty} \frac{\Gamma(\bar{\mathfrak{z}} + u_n\delta) - \Gamma(\bar{\mathfrak{z}})}{u_n} \\
&= \lim_{n\to\infty} \frac{\Gamma(\bar{\mathfrak{z}} + u_n\delta_n) - \Gamma(\bar{\mathfrak{z}})}{u_n} \geq 0 \\
&\geq 0.
\end{aligned}$$

Thus, we have shown that $\Gamma'(\bar{\mathfrak{z}}, \delta) \geq 0$, for all $\delta \in \mathfrak{T}(\bar{\mathfrak{z}}, \mathcal{G})$. Hence, by the Definition 2 of tangential subdifferential

$$\max_{\xi \in \partial^T \Gamma(\bar{\mathfrak{z}})} \langle \mathfrak{z}, \delta \rangle \geq 0, \qquad \text{for all } \delta \in \mathfrak{T}(\bar{\mathfrak{z}}, \mathcal{G}).$$

Here $\partial^T \Gamma(\bar{\mathfrak{z}})$ is a tangential subdifferential of $\Gamma$ at $\bar{\mathfrak{z}}$.

Since $\partial^T$-$ACQ(\mathfrak{B}_1, \mathfrak{B}_2)$ holds at $\bar{\mathfrak{z}} \in \mathcal{G}$, we have

$$\max_{\xi \in \partial^T \Gamma(\bar{\mathfrak{z}})} \langle \mathfrak{z}, \delta \rangle \geq 0, \qquad \text{for all } \delta \in \Theta(\bar{\mathfrak{z}})^\circ.$$

Since $\Theta(\bar{\mathfrak{z}}) \subseteq pos\, \Theta(\bar{\mathfrak{z}})$, we get

$$\max_{\xi \in \partial^T \Gamma(\bar{\mathfrak{z}})} \langle \mathfrak{z}, \delta \rangle \geq 0, \qquad \text{for all } \delta \in \big(pos\, \Theta(\bar{\mathfrak{z}})\big)^\circ.$$

Since $\partial^T \Gamma(\bar{\mathfrak{z}})$ is compact, we can conclude, as stated in Lemma 1, that

$$0 \in \partial^T \Gamma(\bar{\mathfrak{z}}) + cl\, pos\, \Theta(\bar{\mathfrak{z}}).$$

The closedness of $pos\, \Theta(\bar{\mathfrak{z}})$ implies

$$0 \in \partial^T \Gamma(\bar{\mathfrak{z}}) + pos\, \Theta(\bar{\mathfrak{z}}).$$

Thus,

$$\begin{aligned} 0 \in \partial^T \Gamma(\bar{\mathfrak{z}}) &+ \sum_{i \in \mathcal{I}_c(\bar{\mathfrak{z}})} pos\, \partial^T c_i(\bar{\mathfrak{z}}) + \sum_{i \in \mathcal{Q}} pos\, \partial^T d_i(\bar{\mathfrak{z}}) + \sum_{i \in \mathcal{Q}} pos\, \partial^T (-d_i)(\bar{\mathfrak{z}}) \\ &+ \sum_{i \in A \cup \mathfrak{B}_2} pos\, \partial^T \mathcal{T}_i(\bar{\mathfrak{z}}) + \sum_{i \in A \cup \mathfrak{B}} pos\, \partial^T (-\mathcal{T}_i)(\bar{\mathfrak{z}}) \\ &+ \sum_{i \in \mathfrak{D} \cup \mathfrak{B}_1} pos\, \partial^T \zeta_i(\bar{\mathfrak{z}}) + \sum_{i \in \mathfrak{D} \cup \mathfrak{B}} pos\, \partial^T (-\zeta_i)(\bar{\mathfrak{z}}). \end{aligned}$$

Consequently, we can find scalars $\lambda_i^c \geq 0$, $i \in \mathcal{I}_c(\bar{\mathfrak{z}})$, $\mu_i^d \geq 0$, $\lambda_i^d \geq 0$, $i \in \mathcal{Q}$, $\mu_i^{\mathcal{T}} \geq 0$, $i \in A \cup \mathfrak{B}_2$, $\lambda_i^{\mathcal{T}} \geq 0$, $i \in A \cup \mathfrak{B}$, $\mu_i^{\zeta} \geq 0$, $i \in \mathfrak{D} \cup \mathfrak{B}_1$ and $\lambda_i^{\zeta} \geq 0$, $i \in \mathfrak{D} \cup \mathfrak{B}$, such that

$$\begin{aligned} 0 \in \partial^T \Gamma(\bar{\mathfrak{z}}) &+ \sum_{i \in \mathcal{I}_c(\bar{\mathfrak{z}})} \lambda_i^c\, \partial^T c_i(\bar{\mathfrak{z}}) + \sum_{i \in \mathcal{Q}} \mu_i^d\, \partial^T d_i(\bar{\mathfrak{z}}) + \sum_{i \in \mathcal{Q}} \lambda_i^d\, \partial^T (-d_i)(\bar{\mathfrak{z}}) \\ &+ \sum_{i \in A \cup \mathfrak{B}_2} \mu_i^{\mathcal{T}}\, \partial^T \mathcal{T}_i(\bar{\mathfrak{z}}) + \sum_{i \in A \cup \mathfrak{B}} \lambda_i^{\mathcal{T}}\, \partial^T (-\mathcal{T}_i)(\bar{\mathfrak{z}}) \\ &+ \sum_{i \in \mathfrak{D} \cup \mathfrak{B}_1} \mu_i^{\zeta}\, \partial^T \zeta_i(\bar{\mathfrak{z}}) + \sum_{i \in \mathfrak{D} \cup \mathfrak{B}} \lambda_i^{\zeta}\, \partial^T (-\zeta_i)(\bar{\mathfrak{z}}). \end{aligned}$$

Setting

$$\lambda_i^{\mathcal{T}} = 0, \quad \forall\, i \in \mathfrak{D},\ \lambda_i^{\zeta} = 0, \quad \forall\, i \in A,$$

$$\mu_i^{\mathcal{T}} = 0, \quad \forall\, i \in \mathfrak{D} \cup \mathfrak{B}_1,\ \mu_i^{\zeta} = 0, \quad \forall\, i \in A \cup \mathfrak{B}_2.$$

Thus, $\bar{\mathfrak{z}}$ is a $\partial^T$-$\mathcal{GA}$-stationary point, and the proof is complete. □

To demonstrate the necessary optimality conditions derived in Theorem 1, we provide an example of $\mathcal{MPEC}$.

**Example 1.** Consider the following problem

$$(\mathcal{EXMPEC}): \begin{cases} Minimize & \Gamma(\mathfrak{z}_1, \mathfrak{z}_2) \\ s.t. & \begin{cases} c(\mathfrak{z}_1, \mathfrak{z}_2) \leq 0, \quad d(\mathfrak{z}_1, \mathfrak{z}_2) = 0, \\ \mathcal{T}(\mathfrak{z}_1, \mathfrak{z}_2) \geq 0, \ \zeta(\mathfrak{z}_1, \mathfrak{z}_2) \geq 0, \\ \mathcal{T}(\mathfrak{z}_1, \mathfrak{z}_2)^T \zeta(\mathfrak{z}_1, \mathfrak{z}_2) = 0, \end{cases} \end{cases}$$

where $\Gamma(\mathfrak{z}_1,\mathfrak{z}_2) = \mathfrak{z}_1 + |\mathfrak{z}_1| - 2\max(\mathfrak{z}_2,\mathfrak{z}_2^3)$, $c(\mathfrak{z}_1,\mathfrak{z}_2) = |\mathfrak{z}_2|$, $d(\mathfrak{z}_1,\mathfrak{z}_2) = 0$

$$\mathcal{T}(\mathfrak{z}_1,\mathfrak{z}_2) = \begin{cases} -\frac{1}{2}, & \mathfrak{z}_1 \geq 0, \mathfrak{z}_2 < 0 \\ -\frac{1}{4}, & \mathfrak{z}_1 < 0, \mathfrak{z}_2 < 0 \\ \mathfrak{z}_1, & \mathfrak{z}_1 \in \mathbb{R},\, \mathfrak{z}_2 \geq 0, \end{cases} \qquad \zeta(\mathfrak{z}_1,\mathfrak{z}_2) = \begin{cases} -\frac{1}{4}, & \mathfrak{z}_1 < 0,\, \mathfrak{z}_2 \geq 0 \\ -\frac{1}{2}, & \mathfrak{z}_1 < 0,\, \mathfrak{z}_2 < 0 \\ \mathfrak{z}_2, & \mathfrak{z}_1 \geq 0,\, \mathfrak{z}_2 \in \mathbb{R}. \end{cases}$$

We have $\mathcal{G} = \big\{(\mathfrak{z}_1,\mathfrak{z}_2) : \mathfrak{z}_1 \in \mathbb{R}^+,\, \mathfrak{z}_2 = 0\big\}$, $A = \mathfrak{D} = \emptyset$, $\mathfrak{B} = \{1\}$, $\mathcal{P} = \{1\}$, $\mathcal{Q} = \{1\}$ and $\bar{\mathfrak{z}} = (0,\,0) \in \mathcal{G}$ is an optimal solution of $(\mathcal{EXMPEC})$.
Observe that,

$$\partial^T\Gamma(0,0) = \big[0,\,2\big] \times \big[-2,\,0\big],\ \partial^T c(0,0) = \{0\} \times \big[-1,1\big],\ \partial^T d(0,0) = \big\{(0,\,0)\big\}$$

$$\partial^T(\mathcal{T})(0,0) = \big\{(1,0)\big\} \text{ and } \partial^T(\zeta)(0,0) = \big\{(0,1)\big\}$$

are the tangential subdifferentials of $\Gamma$, $c$, $d$, $\mathcal{T}$, $\zeta$ at $(0,\,0)$.
In choosing $\mathfrak{B}_1 = \{1\}$, $\mathfrak{B}_2 = \emptyset$, we have

$$\Theta(0,0) = \big(\{0\} \times \big[-1,1\big]\big) \cup \big\{(0,1),\,(0,-1),(-1,0)\big\}.$$

Then,

$$\Theta(0,0)^\circ = \mathbb{R}^+ \times \{0\}.$$

Since $\mathfrak{T}\big((0,0),\,\mathcal{G}\big) = \mathbb{R}^+ \times \{0\}$, we deduce that $\Theta(0,0)^\circ \subseteq \mathfrak{T}\big((0,0),\,\mathcal{G}\big)$.
The set $pos\,\Theta(0,0)$ is closed. In fact, a straightforward calculation demonstrates this

$$pos\,\Theta(0,0) = \mathbb{R}^- \times \mathbb{R}.$$

Taking $\lambda^c = \frac{5}{2}$, $\mu^d = \lambda^d = \mu^\zeta = 0$, $\mu^{\mathcal{T}} = \frac{1}{4}$, $\lambda^{\mathcal{T}} = \frac{7}{4}$ and $\lambda^\zeta = \frac{1}{2}$, since $(\frac{3}{2},-2) \in \partial^T\Gamma(0,0)$, we get

$$\begin{aligned} 0 \in \partial^T\Gamma(0,0) + \lambda^c\,\partial^T c(0,0) + \mu^d\,\partial^T d(0,0) + \lambda^d\,\partial^T(-d)(0,0) + \mu^{\mathcal{T}}\,\partial^T\mathcal{T}(0,0) \\ + \lambda^{\mathcal{T}}\,\partial^T(-\mathcal{T})(0,0) + \mu^\zeta\,\partial^T\zeta(0,0) + \lambda^\zeta\,\partial^T(-\zeta)(0,0). \end{aligned}$$

## 4. Sufficient optimality conditions

In this section, we analyze and prove the sufficiency of the necessary optimality conditions of Karush-Kuhn-Tucker type established in the previous section. Since we establish the aforementioned sufficient optimality conditions for the considered $\mathcal{MPEC}$ problem under generalized convexity, which is formulated in terms of tangential subdifferentials of the involved functions.
Let $\bar{\mathfrak{z}} \in \mathcal{G}$ be a feasible point that satisfies the $\partial^T$-$\mathcal{GA}$-stationary condition. Define the set $\Omega$ as follows:

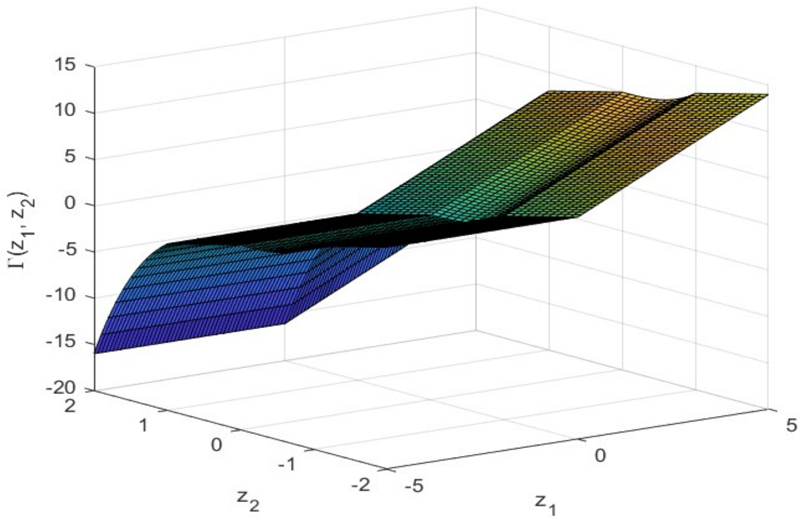


Figure 1. The graph of the objective function $\Gamma(\mathfrak{z}_1, \mathfrak{z}_2) = \mathfrak{z}_1 + |\mathfrak{z}_1| - 2\max(\mathfrak{z}_2, \mathfrak{z}_2^3)$ of $(\mathcal{EXMPEC})$ considered in Example 1.

$$\Omega := A^+ \cup \mathfrak{D}^+ \cup \mathfrak{B}^+ \cup \mathfrak{B}^+_{\mathcal{T}} \cup \mathfrak{B}^+_{\zeta},$$

where

$$A^+ := \{i \in A : \mu_i^{\mathcal{T}} > 0\},\ \mathfrak{D}^+ := \{i \in \mathfrak{D} : \mu_i^{\zeta} > 0\},$$

$$\mathfrak{B}^+ := \{i \in \mathfrak{B} : \mu_i^{\mathcal{T}} > 0 \text{ and } \mu_i^{\zeta} > 0\},\ \mathfrak{B}^+_{\mathcal{T}} := \{i \in \mathfrak{B} : \mu_i^{\mathcal{T}} = 0 \text{ and } \mu_i^{\zeta} > 0\},$$

$$\text{and } \mathfrak{B}^+_{\zeta} := \{i \in \mathfrak{B} : \mu_i^{\mathcal{T}} > 0 \text{ and } \mu_i^{\zeta} = 0\}.$$

Here, $\mu^{\mathcal{T}}$ and $\mu^{\zeta}$ are the multipliers associated with the point $\bar{\mathfrak{z}}$ that satisfies the $\partial^T$-generalized alternatively stationary condition.
Now, we derive and prove the sufficient conditions for a feasible solution to be globally optimal in the considered optimization problem $\mathcal{MPEC}$ under the Dini generalized convexity assumptions.

**Theorem 2.** *Let $\bar{\mathfrak{z}} \in \mathcal{G}$ be a feasible solution of $\mathcal{MPEC}$ and $\partial^T$-generalized alternatively stationary condition holds at $\bar{\mathfrak{z}}$. Suppose that $\Gamma$ is Dini-pseudoconvex at $\bar{\mathfrak{z}}$ on $\mathcal{G}$, that $c_i$, $i \in \mathcal{I}_c(\bar{\mathfrak{z}})$, $\pm d_i$, $i \in \mathcal{Q}, -\mathcal{T}_i$, $i \in A \cup \mathfrak{B}$ and $-\zeta_i$, $i \in \mathfrak{D} \cup \mathfrak{B}$ are Dini-quasiconvex at $\bar{\mathfrak{z}}$ on $\mathcal{G}$. If $\Omega$ is empty, then $\bar{\mathfrak{z}}$ is a global optimal solution of $\mathcal{MPEC}$.*

*Proof.* By contrary, suppose that $\bar{\mathfrak{z}} \in \mathcal{G}$ is not a global optimal solution of $\mathcal{MPEC}$. Hence, by Definition, there exists $\mathfrak{z}_0 \in \mathcal{G}$ such that

$$\Gamma(\mathfrak{z}_0) - \Gamma(\bar{\mathfrak{z}}) < 0.$$

By (3.2), we get $\eta \in \partial^T \Gamma(\mathfrak{z})$, $\varrho_i \in \partial^T c_i(\bar{\mathfrak{z}})$, $\sigma_i \in \partial^T d_i(\bar{\mathfrak{z}})$, $\theta_i \in \partial^T(-d_i)(\bar{\mathfrak{z}})$, $\mathfrak{x}_i \in \partial^T(-\mathcal{T}_i)(\bar{\mathfrak{z}})$, $\mathfrak{x}_i^* \in \partial^T \mathcal{T}_i(\bar{\mathfrak{z}})$, $\mathfrak{y}_i \in \partial^T(-\zeta_i)(\bar{\mathfrak{z}})$ and $\mathfrak{y}_i^* \in \partial^T \zeta_i(\bar{\mathfrak{z}})$ such that

$$0 = \eta + \sum_{i=1}^{l} \lambda_i^c \varrho_i + \sum_{i \in \mathcal{Q}} \mu_i^d \sigma_i + \sum_{i \in \mathcal{Q}} \lambda_i^d \theta_i + \sum_{i=1}^{p} \lambda_i^{\mathcal{T}} \mathfrak{x}_i + \sum_{i=1}^{p} \lambda_i^{\zeta} \mathfrak{y}_i + \sum_{i=1}^{p} \mu_i^{\mathcal{T}} \mathfrak{x}_i^* + \sum_{i=1}^{p} \mu_i^{\zeta} \mathfrak{y}_i^*.$$

Then,

$$\begin{aligned} 0 = \left\langle \eta, \mathfrak{z}_0 - \bar{\mathfrak{z}} \right\rangle &+ \sum_{i=1}^{l} \lambda_i^c \left\langle \varrho_i, \mathfrak{z}_0 - \bar{\mathfrak{z}} \right\rangle + \sum_{i \in \mathcal{Q}} \mu_i^d \left\langle \sigma_i, \mathfrak{z}_0 - \bar{\mathfrak{z}} \right\rangle \\ &+ \sum_{i \in \mathcal{Q}} \lambda_i^d \left\langle \theta_i, \mathfrak{z}_0 - \bar{\mathfrak{z}} \right\rangle + \sum_{i=1}^{p} \lambda_i^{\mathcal{T}} \left\langle \mathfrak{x}_i, \mathfrak{z}_0 - \bar{\mathfrak{z}} \right\rangle + \sum_{i=1}^{p} \lambda_i^{\zeta} \left\langle \mathfrak{y}_i, \mathfrak{z}_0 - \bar{\mathfrak{z}} \right\rangle \\ &+ \sum_{i=1}^{p} \mu_i^{\mathcal{T}} \left\langle \mathfrak{x}_i^*, \mathfrak{z}_0 - \bar{\mathfrak{z}} \right\rangle + \sum_{i=1}^{p} \mu_i^{\zeta} \left\langle \mathfrak{y}_i^*, \mathfrak{z}_0 - \bar{\mathfrak{z}} \right\rangle. \end{aligned} \tag{4.1}$$

Since $\Gamma$ is Dini-pseudoconvex at $\bar{\mathfrak{z}}$ on $\mathcal{G}$, we get

$$\Big\langle \eta, \mathfrak{z}_0 - \bar{\mathfrak{z}} \Big\rangle < 0, \ \forall\, \eta \in \partial^T \Gamma(\mathfrak{z}).$$

Consequently,

$$\begin{aligned} &\sum_{i=1}^{l} \lambda_i^c \left\langle \varrho_i, \mathfrak{z}_0 - \bar{\mathfrak{z}} \right\rangle + \sum_{i \in \mathcal{Q}} \mu_i^d \left\langle \sigma_i, \mathfrak{z}_0 - \bar{\mathfrak{z}} \right\rangle + \sum_{i \in \mathcal{Q}} \lambda_i^d \left\langle \theta_i, \mathfrak{z}_0 - \bar{\mathfrak{z}} \right\rangle + \sum_{i=1}^{p} \lambda_i^{\mathcal{T}} \left\langle \mathfrak{x}_i, \mathfrak{z}_0 - \bar{\mathfrak{z}} \right\rangle \\ &\quad + \sum_{i=1}^{p} \lambda_i^{\zeta} \left\langle \mathfrak{y}_i, \mathfrak{z}_0 - \bar{\mathfrak{z}} \right\rangle + \sum_{i=1}^{p} \mu_i^{\mathcal{T}} \left\langle \mathfrak{x}_i^*, \mathfrak{z}_0 - \bar{\mathfrak{z}} \right\rangle + \sum_{i=1}^{p} \mu_i^{\zeta} \left\langle \mathfrak{y}_i^*, \mathfrak{z}_0 - \bar{\mathfrak{z}} \right\rangle > 0. \end{aligned} \tag{4.2}$$

Since $\mathfrak{z}_0 \in \mathcal{G}$, we have

$$c_i(\mathfrak{z}_0) \le 0 = c_i(\bar{\mathfrak{z}}), \quad i \in \mathcal{I}_c(\bar{\mathfrak{z}}),$$

$$d_i(\mathfrak{z}_0) = 0 = d_i(\bar{\mathfrak{z}}), \quad i \in \mathcal{Q},$$

$$(-\mathcal{T}_i)(\mathfrak{z}_0) \le 0 = (-\mathcal{T}_i)(\bar{\mathfrak{z}}), \quad i \in A \cup \mathfrak{B},$$

$$(-\zeta_i)(\mathfrak{z}_0) \le 0 = (-\zeta_i)(\bar{\mathfrak{z}}), \quad i \in \mathfrak{D} \cup \mathfrak{B}.$$

By the Dini-quasiconvexity of $c_i$, $i \in \mathcal{I}_c(\bar{\mathfrak{z}})$, $-\mathcal{T}_i$, $i \in A \cup \mathfrak{B}$ and $-\zeta_i$, $i \in \mathfrak{D} \cup \mathfrak{B}$ at $\bar{\mathfrak{z}}$ on $\mathcal{G}$, we obtain

$$\left\langle \varrho_i, \mathfrak{z}_0 - \bar{\mathfrak{z}} \right\rangle \le 0, \quad i \in \mathcal{I}_c(\bar{\mathfrak{z}}),$$

$$\langle \mathfrak{x}_i, \mathfrak{z}_0 - \bar{\mathfrak{z}} \rangle \leq 0, \quad i \in A \cup \mathfrak{B},$$

$$\langle \mathfrak{y}_i, \mathfrak{z}_0 - \bar{\mathfrak{z}} \rangle \leq 0, \quad i \in \mathfrak{D} \cup \mathfrak{B}.$$

Using the Dini-quasiconvexity of $\pm d_i$, $i \in \mathcal{Q}$, we obtain

$$\langle \sigma_i, \mathfrak{z}_0 - \bar{\mathfrak{z}} \rangle \leq 0, \quad i \in \mathcal{Q},$$

$$\langle \theta_i, \mathfrak{z}_0 - \bar{\mathfrak{z}} \rangle \leq 0, \quad i \in \mathcal{Q}.$$

Then,

$$\left\langle \sum_{i \in \mathcal{I}_c(\bar{\mathfrak{z}})} \lambda_i^c \, \varrho_i, \mathfrak{z}_0 - \bar{\mathfrak{z}} \right\rangle \leq 0, \quad \text{as } \lambda_i^c \geq 0, \ i \in \mathcal{I}_c(\bar{\mathfrak{z}}),$$

$$\left\langle \sum_{i \in A \cup \mathfrak{B}} \lambda_i^{\mathcal{T}} \mathfrak{x}_i, \mathfrak{z}_0 - \bar{\mathfrak{z}} \right\rangle \leq 0, \quad \text{as } \lambda_i^{\mathcal{T}} \geq 0, \ i \in A \cup \mathfrak{B},$$

$$\left\langle \sum_{i \in \mathfrak{D} \cup \mathfrak{B}} \lambda_i^{\zeta} \mathfrak{y}_i, \mathfrak{z}_0 - \bar{\mathfrak{z}} \right\rangle \leq 0, \quad \text{as } \lambda_i^{\zeta} \geq 0, \ i \in \mathfrak{D} \cup \mathfrak{B}.$$

$$\left\langle \sum_{i \in \mathcal{Q}} \mu_i^d \, \sigma_i, \mathfrak{z}_0 - \bar{\mathfrak{z}} \right\rangle \leq 0, \quad \text{as } \mu_i^d \geq 0, \ i \in \mathcal{Q}, \tag{4.3}$$

$$\left\langle \sum_{i \in \mathcal{Q}} \lambda_i^d \, \theta_i, \mathfrak{z}_0 - \bar{\mathfrak{z}} \right\rangle \leq 0, \quad \text{as } \lambda_i^d \geq 0, \ i \in \mathcal{Q}. \tag{4.4}$$

- By (3.3), we have $\lambda_i^c = 0, \ \forall \, i \notin \mathcal{I}_c(\bar{\mathfrak{z}})$. Consequently,

$$\left\langle \sum_{i=1}^{l} \lambda_i^c \, \varrho_i, \mathfrak{z}_0 - \bar{\mathfrak{z}} \right\rangle \leq 0. \tag{4.5}$$

- By (3.4), we have $\lambda_i^{\mathcal{T}} = 0, \ \forall \, i \in \mathfrak{D}$ and $\zeta_i = 0, \ \forall \, i \in A$. Consequently,

$$\left\langle \sum_{i=1}^{p} \lambda_i^{\mathcal{T}} \mathfrak{x}_i, \mathfrak{z}_0 - \bar{\mathfrak{z}} \right\rangle \leq 0, \tag{4.6}$$

$$\left\langle \sum_{i=1}^{p} \lambda_i^{\zeta} \mathfrak{y}_i, \mathfrak{z}_0 - \bar{\mathfrak{z}} \right\rangle \leq 0. \tag{4.7}$$

- Since $\Omega$ is empty, we deduce that $\mu_i^{\mathcal{T}} = 0$ and $\mu_i^{\zeta} = 0, \ \forall \, i \in \mathcal{R}$. Then

$$\sum_{i=1}^{p} \mu_i^{\mathcal{T}} \langle \mathfrak{x}_i^*, \mathfrak{z}_0 - \bar{\mathfrak{z}} \rangle + \sum_{i=1}^{p} \mu_i^{\zeta} \langle \mathfrak{y}_i^*, \mathfrak{z}_0 - \bar{\mathfrak{z}} \rangle = 0. \tag{4.8}$$

Combining (4.3), (4.4), (4.5), (4.6), (4.7) and (4.8), we obtain

$$\sum_{i=1}^{l} \lambda_i^c \left\langle \varrho_i,\, \mathfrak{z}_0 - \bar{\mathfrak{z}} \right\rangle + \sum_{i\in \mathcal{Q}} \mu_i^d \left\langle \sigma_i,\, \mathfrak{z}_0 - \bar{\mathfrak{z}} \right\rangle + \sum_{i\in \mathcal{Q}} \lambda_i^d \left\langle \theta_i,\, \mathfrak{z}_0 - \bar{\mathfrak{z}} \right\rangle + \sum_{i=1}^{p} \lambda_i^{\mathcal{T}} \left\langle \mathfrak{x}_i,\, \mathfrak{z}_0 - \bar{\mathfrak{z}} \right\rangle$$
$$+ \sum_{i=1}^{p} \lambda_i^{\zeta} \left\langle \mathfrak{y}_i,\, \mathfrak{z}_0 - \bar{\mathfrak{z}} \right\rangle + \sum_{i=1}^{p} \mu_i^{\mathcal{T}} \left\langle \mathfrak{x}_i^*,\, \mathfrak{z}_0 - \bar{\mathfrak{z}} \right\rangle + \sum_{i=1}^{p} \mu_i^{\zeta} \left\langle \mathfrak{y}_i^*,\, \mathfrak{z}_0 - \bar{\mathfrak{z}} \right\rangle \leq 0.$$

We reach a contradiction with (4.2), thereby concluding the proof. □

In Theorem 2, we demonstrated that the $\partial^T$-generalized alternatively stationary condition, under the assumption of generalized convexity, also serves as a global sufficient optimality condition when the set $\Omega = \emptyset$. It is important to note that this condition, in its current form, was initially explored in [33] Theorem 2.3 for smooth mathematical programs with equilibrium constraints, and was later extended to the nonsmooth context by Ansari et al. [1].
We now present the example of an nonsmooth $\mathcal{MPEC}$ to demonstrate the sufficient optimality conditions established in this section.

**Example 2.** Consider the following problem:

$$(\mathcal{EXMPEC}): \quad \begin{cases} Minimize & \Gamma(\mathfrak{z}_1, \mathfrak{z}_2) \\ s.t. \begin{cases} c(\mathfrak{z}_1, \mathfrak{z}_2) \leq 0, \;\; d(\mathfrak{z}_1, \mathfrak{z}_2) = 0, \\ \mathcal{T}(\mathfrak{z}_1, \mathfrak{z}_2) \geq 0, \;\; \zeta(\mathfrak{z}_1, \mathfrak{z}_2) \geq 0, \\ \mathcal{T}(\mathfrak{z}_1, \mathfrak{z}_2)^T \zeta(\mathfrak{z}_1, \mathfrak{z}_2) = 0, \end{cases} \end{cases}$$

where

$$\Gamma(\mathfrak{z}_1, \mathfrak{z}_2) = \begin{cases} \mathfrak{z}_1 - \mathfrak{z}_2 & \text{if } \; \mathfrak{z}_1 \geq 0, \; \mathfrak{z}_2 \leq 0, \\ \frac{-\mathfrak{z}_1}{\mathfrak{z}_2^2+1} + \sqrt{-\mathfrak{z}_1} + \frac{1}{2} & \text{if } \; \mathfrak{z}_1 < 0, \; \mathfrak{z}_2 \in \mathbb{R}, \\ \mathfrak{z}_1^2 + \mathfrak{z}_2^2 + 1 & \text{if } \; \mathfrak{z}_1 \geq 0, \; \mathfrak{z}_2 > 0, \end{cases}$$

$$c(\mathfrak{z}_1, \mathfrak{z}_2) = \mathfrak{z}_2^2 + \mathfrak{z}_2, \;\; d(\mathfrak{z}_1, \mathfrak{z}_2) = 0, \;\; \zeta(\mathfrak{z}_1, \mathfrak{z}_2) = \mathfrak{z}_2,$$

$$\mathcal{T}(\mathfrak{z}_1, \mathfrak{z}_2) = \begin{cases} \mathfrak{z}_1 & \text{if } \; \mathfrak{z}_2 \geq 0, \\ \mathfrak{z}_2 + 1 & \text{otherwise.} \end{cases}$$

We have $\mathcal{G} = \{(\mathfrak{z}_1, \mathfrak{z}_2) : \mathfrak{z}_1 \in \mathbb{R}^+, \, \mathfrak{z}_2 = 0\}$, $A = \mathfrak{D} = \emptyset$, $\mathfrak{B} = \{1\}$, $\mathcal{P} = \{1\}$, $\mathcal{Q} = \{1\}$ and $\bar{\mathfrak{z}} = (0,\, 0) \in \mathcal{G}$ is a feasible point of $(\mathcal{EXMPEC})$.
Observe that,

$$\partial^T \Gamma(0,0) = \{(1,\, -1)\}, \, \partial^T c(0,0) = \{(0,\, 1)\}, \, \partial^T d(0,0) = \{(0,\, 0)\}$$

$$\partial^T (\mathcal{T})(0,0) = \{(1,0)\} \text{ and } \partial^T (\zeta)(0,0) = \{(0,\, 1)\}$$

are the tangential subdifferentials of $\Gamma$, $c$, $d$, $\mathcal{T}$, $\zeta$ at $(0,\, 0)$.

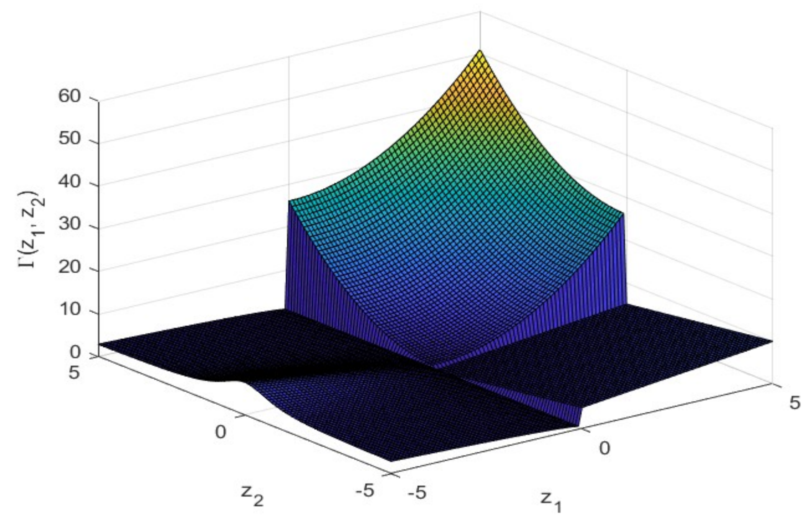


Figure 2. The graph of the objective function $\Gamma(\mathfrak{z}_1, \mathfrak{z}_2)$ of $(\mathcal{EXMPEC})$ considered in Example 2.

- Note that $\Gamma$ is Dini-pseudoconvex at (0, 0), that $c$, $\pm d$, $-\mathcal{T}$ and $-\zeta$ are Dini-quasiconvex at (0, 0).
- For $\lambda^c = \frac{4}{3}$, $\lambda^{\mathcal{T}} = 1$, $\mu^d = \lambda^d = 0$ and $\lambda^\zeta = \frac{1}{3}$, we have

$$0 \in \partial^T \Gamma(0,0) + \lambda^c \, \partial^T c(0,0) + \mu^d \, \partial^T d(0,0) + \lambda^d \, \partial^T(-d)(0,0) \\ + \lambda^{\mathcal{T}} \, \partial^T(-\mathcal{T})(0,0) + \lambda^\zeta \, \partial^T(-\zeta)(0,0).$$

- Moreover, since $\mu_i^{\mathcal{T}} = 0$ for all $i \in A$, $\mu_i^\zeta = 0$ for all $i \in \mathfrak{D}$, and $\mu_i^\zeta = \mu_i^{\mathcal{T}} = 0$ for all $i \in \mathfrak{B}$, it follows that $\Omega = \emptyset$. Therefore, $(0,0)$ is a $\partial^T$-generalized alternatively stationary point. Consequently, $(0,0)$ is an optimal solution to the problem.

## 5. Conclusion

In this article, we have investigated the necessary and sufficient optimality conditions for nonsmooth mathematical programs with equilibrium constraints ($\mathcal{MPEC}s$) using the concept of tangential subdifferentials. Our approach introduces nonsmooth versions of the constraint qualification $\partial^T$-$ACQ(\mathfrak{B}_1, \mathfrak{B}_2)$ for a surrogate problem, grounded in tangential subdifferentials, and establishes a $\partial^T$-generalized alternatively stationary ($\partial^T$-$\mathcal{GA}$-stationary) concept as a first-order necessary optimality condition. Through this framework, we have shown that $\partial^T$-$\mathcal{GA}$-stationarity can also serve as a global sufficient optimality condition under certain generalized convexity assumptions. As far as we know, no existing research has explored $\mathcal{MPEC}s$ using the framework of tangential subdifferentials. This gap in the literature makes our findings particularly significant, the results we present are entirely new contributions to the field. By approaching the problem from this previously unexplored perspective, we provide fresh insights that deepen the understanding of $\mathcal{MPEC}s$ and lay the groundwork for future developments.

This study lays the groundwork for several promising directions of future research across various areas of mathematics. The key research questions, current limitations, and possible extensions are outlined below:

- The sufficiency results depend on generalized convexity assumptions which, although broader than classical convexity, may still not cover all nonsmooth or nonconvex real-world settings.
- This study is confined to finite-dimensional spaces. Extending the analysis to infinite-dimensional settings, such as Banach spaces, would require a different approach due to the complexities of nonsmooth analysis in such environments.
- Explore duality theory for $\mathcal{MPEC}s$ in the tangential subdifferential framework. In particular, formulating Mond-Weir-type and Wolfe-type dual problems could provide a deeper understanding of the primal-dual relationship in nonsmooth equilibrium-constrained settings. Establishing weak, strong, and converse duality results under generalized Dini-convexity and tangential subdifferential conditions would be a valuable direction for further advancement.

**Data availability** Data sharing is not applicable to this article as no data sets were generated or analyzed during the current study.

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## Appendix

**(A1):** A feasible point $\bar{\mathfrak{z}}$ is said to be a local optimal solution of $\mathcal{MPEC}$, if there exists a neighborhood $U$ of $\bar{\mathfrak{z}}$ such that

$$\Gamma(\bar{\mathfrak{z}}) \leq \Gamma(\mathfrak{z}_0), \quad \forall\ \mathfrak{z}_0 \in \mathcal{G} \cap U.$$

**(A2):** A feasible point $\bar{\mathfrak{z}}$ is said to be a global optimal solution of $\mathcal{MPEC}$, if there exists $\mathfrak{z}_0 \in \mathcal{G}$ such that

$$\Gamma(\mathfrak{z}_0) - \Gamma(\bar{\mathfrak{z}}) < 0.$$

In optimization problems, a local optimal solution is a point where the objective function reaches its best value in a small surrounding region. On the other hand, a global optimal solution is the absolute best point across the entire feasible range. While every global optimum is also a local one, the reverse is not always true, especially in non-convex problems, where multiple local optima can exist without all being globally optimal.

**(A3):** The class of tangentially convex functions is quite broad. It includes all convex functions with open domains, as well as any function that is Gateaux differentiable at a point $\mathfrak{z}$, since in that case, the directional derivative $\varphi'(\bar{\mathfrak{z}}, \cdot)$ becomes linear. In fact, a function with an open domain that is Gateaux differentiable everywhere is tangentially convex at every point in its domain, even if it is not convex itself. Interestingly, the set of tangentially convex functions at a given point forms a real vector space. This means, for example, that adding a convex function to a differentiable one gives us a tangentially convex function—which is usually neither convex nor differentiable. Another example is the product of two nonnegative tangentially convex functions.

**(A4):** It can be noted that, among the various calculus rules applicable to tangential subdifferentials of tangentially convex functions.

- [25] If $\varphi$ is tangentially convex function at $\bar{\mathfrak{z}}$, then the following holds

$$\partial^T\big(\alpha\,\varphi\big)(\bar{\mathfrak{z}}) = \alpha\,\partial^T\varphi(\bar{\mathfrak{z}}),\ \forall\,\alpha \in \mathbb{R}_+.$$

- [30] If $\varphi_1$ and $\varphi_2$ are tangentially convex functions at a common point $\bar{\mathfrak{z}}$ with $\varphi_1(\bar{\mathfrak{z}}) \geq 0, \varphi_2(\bar{\mathfrak{z}}) \geq 0$, then the following holds

$$\partial^T\big((\varphi_1\,\varphi_2)(\bar{\mathfrak{z}})\big) = \varphi_1(\bar{\mathfrak{z}})\,\partial^T\varphi_2(\bar{\mathfrak{z}}) + \partial^T\varphi_1(\bar{\mathfrak{z}})\,\varphi_2(\bar{\mathfrak{z}})$$

  demonstrating that the tangential subdifferential of the product of two tangentially convex functions (evaluated at a point where both are non-negative) is given by a Leibniz-type rule involving their individual tangential subdifferentials.

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**(A5): Algorithm 1** An algorithm for finding the $\partial^T$-generalized alternatively stationary point of the problem $\mathcal{MPEC}$

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**Step 1. Provide Problem Data**

Start by supplying the input data for the given $\mathcal{MPEC}$ problem:

- Input $\Gamma$, $c_i$, $i \in \mathcal{P}$, $d_i$, $i \in \mathcal{Q}$, $\mathcal{T}_i$, $i \in \mathcal{R}$ and $\zeta_i$, $i \in \mathcal{R}$.

**Step 2. Identify the Feasible Set**

- Construct the feasible region as follows:

$$\mathcal{G} := \Big\{\mathfrak{z} \in \mathbb{R}^n : c(\mathfrak{z}) \leq 0, d(\mathfrak{z}) = 0, \mathcal{T}(\mathfrak{z}) \geq 0,\ \zeta(\mathfrak{z}) \geq 0, \mathcal{T}(\mathfrak{z})^T\zeta(\mathfrak{z}) = 0\Big\}.$$

**Step 3. Select a Feasible Point**

- If the feasible set $\mathcal{G}$ is empty, terminate the algorithm.
- Otherwise, choose any point $\bar{\mathfrak{z}} \in \mathcal{G}$, and update the feasible set by removing this point: $\mathcal{G} = \mathcal{G} \setminus \{\bar{\mathfrak{z}}\}$.

**Step 4. Check tangential convexity of the functions**

- Verify whether each functions $\Gamma$, $c_i, i \in \mathcal{I}_c(\bar{\mathfrak{z}})$, $d_i, -d_i, i \in \mathcal{Q}, \mathcal{T}_i, -\mathcal{T}_i, i \in A \cup \mathfrak{B}, \zeta_i, -\zeta_i, i \in \mathfrak{D} \cup \mathfrak{B}$ are tangentially convex at the point $\bar{\mathfrak{z}}$.
- If all functions are tangentially convex at $\bar{\mathfrak{z}}$, proceed to Step 5.
- If any function fails this condition, return to Step 3.

**Step 5. Verify tangential subdifferentiability**

- Compute the tangential subdifferential of each function at $\bar{\mathfrak{z}}$.

**Step 6. Check the $\partial^T$-$ACQ(\mathfrak{B}_1, \mathfrak{B}_2)$ Condition**

- If the $\partial^T$-Abadie constraint qualifcation ($\partial^T$-$ACQ(\mathfrak{B}_1, \mathfrak{B}_2)$) holds at $\bar{\mathfrak{z}}$, proceed to the next step.

- If not, return to Step 3.

**Step 7. Test $\partial^T$-generalized alternatively stationary Conditions**
Determine if there exist multipliers Choose the multipliers $\lambda_i^c \geq 0, i \in \mathcal{I}_c(\bar{\mathfrak{z}})$, $\mu_i^d \geq 0$, $\lambda_i^d \geq 0$, $i \in \mathcal{Q}$, $\mu_i^{\mathcal{T}} \geq 0$, $i \in A \cup \mathfrak{B}_2$, $\lambda_i^{\mathcal{T}} \geq 0$, $i \in A \cup \mathfrak{B}$, $\mu_i^{\zeta} \geq 0$, $i \in \mathfrak{D} \cup \mathfrak{B}_1$ and $\lambda_i^{\zeta} \geq 0$, $i \in \mathfrak{D} \cup \mathfrak{B}$ such that condition (3.2) is satisfied.

- If such multipliers can be found, then $\bar{\mathfrak{z}}$ is $\partial^T$-generalized alternatively stationary point of $\mathcal{MPEC}$.
- If not, return to Step 3.

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**Algorithm 2** An algorithm for finding global optimal solution of the problem $\mathcal{MPEC}$

**(A6):** **Step 1. Provide Problem Data**
Start by supplying the input data for the given $\mathcal{MPEC}$ problem:

- Input $\Gamma$, $c_i$, $i \in \mathcal{P}$, $d_i$, $i \in \mathcal{Q}$, $\mathcal{T}_i$, $i \in \mathcal{R}$ and $\zeta_i$, $i \in \mathcal{R}$.

**Step 2. Identify the Feasible Set**

- Construct the feasible region as follows:

$$\mathcal{G} := \Big\{\mathfrak{z} \in \mathbb{R}^n : c(\mathfrak{z}) \leq 0, d(\mathfrak{z}) = 0, \mathcal{T}(\mathfrak{z}) \geq 0,\ \zeta(\mathfrak{z}) \geq 0, \mathcal{T}(\mathfrak{z})^T \zeta(\mathfrak{z}) = 0\Big\}.$$

**Step 3. Select a Feasible Point**

- If the feasible set $\mathcal{G}$ is empty, terminate the algorithm.
- Otherwise, choose any point $\bar{\mathfrak{z}} \in \mathcal{G}$, and update the feasible set by removing this point: $\mathcal{G} = \mathcal{G} \setminus \{\bar{\mathfrak{z}}\}$.

---

**Step 4. Check tangential convexity of the functions**

- Verify whether each functions $\Gamma$, $c_i, i \in \mathcal{I}_c(\bar{\mathfrak{z}})$, $d_i, -d_i, i \in \mathcal{Q}, \mathcal{T}_i, -\mathcal{T}_i, i \in A \cup \mathfrak{B}, \zeta_i, -\zeta_i, i \in \mathfrak{D} \cup \mathfrak{B}$ are tangentially convex at the point $\bar{\mathfrak{z}}$.
- If all functions are tangentially convex at $\bar{\mathfrak{z}}$, proceed to Step 5.
- If any function fails this condition, return to Step 3.

**Step 5. Verify tangential subdifferentiability**

- Compute the tangential subdifferential of each function at $\bar{\mathfrak{z}}$.

**Step 6. Check Index Set Emptiness**
Confirm that the following combined index set is empty:

- $\Omega := A^+ \cup \mathfrak{D}^+ \cup \mathfrak{B}^+ \cup \mathfrak{B}^+_{\mathcal{T}} \cup \mathfrak{B}^+_{\zeta} = \emptyset$.

**Step 7. Test $\partial^T$-generalized alternatively stationary Conditions**

- Verify whether the $\partial^T$-generalized alternatively stationary condition holds at the point $\bar{\mathfrak{z}}$.

**Step 8. Verify Generalized Convexity Conditions**
At the point $\bar{\mathfrak{z}}$, confirm the following:

- The objective function $\Gamma$ is Dini-pseudoconvex .
- The constraint functions $c_i$, $i \in \mathcal{I}_c(\bar{\mathfrak{z}})$, $\pm d_i$, $i \in \mathcal{Q}, -\mathcal{T}_i$, $i \in A \cup \mathfrak{B}$ and $-\zeta_i$, $i \in \mathfrak{D} \cup \mathfrak{B}$ are Dini-quasiconvex.

If these conditions are not met, the problem cannot be solved using the current framework-return to Step 3.

**Step 9. Output the Solution**
The point $\bar{\mathfrak{z}}$ obtained through this process is a global optimal solution of the problem $\mathcal{MPEC}$.